Minimal Variance Estimator of Reconstructor in Adaptive Optics Systems

by

Moody T. Chu (NC State)

joint with

Robert J. Plemmons (Wake Forest)

Xiaobai Sun (Duke)

Victor P. Pauca (Duke)

January 9, 1999

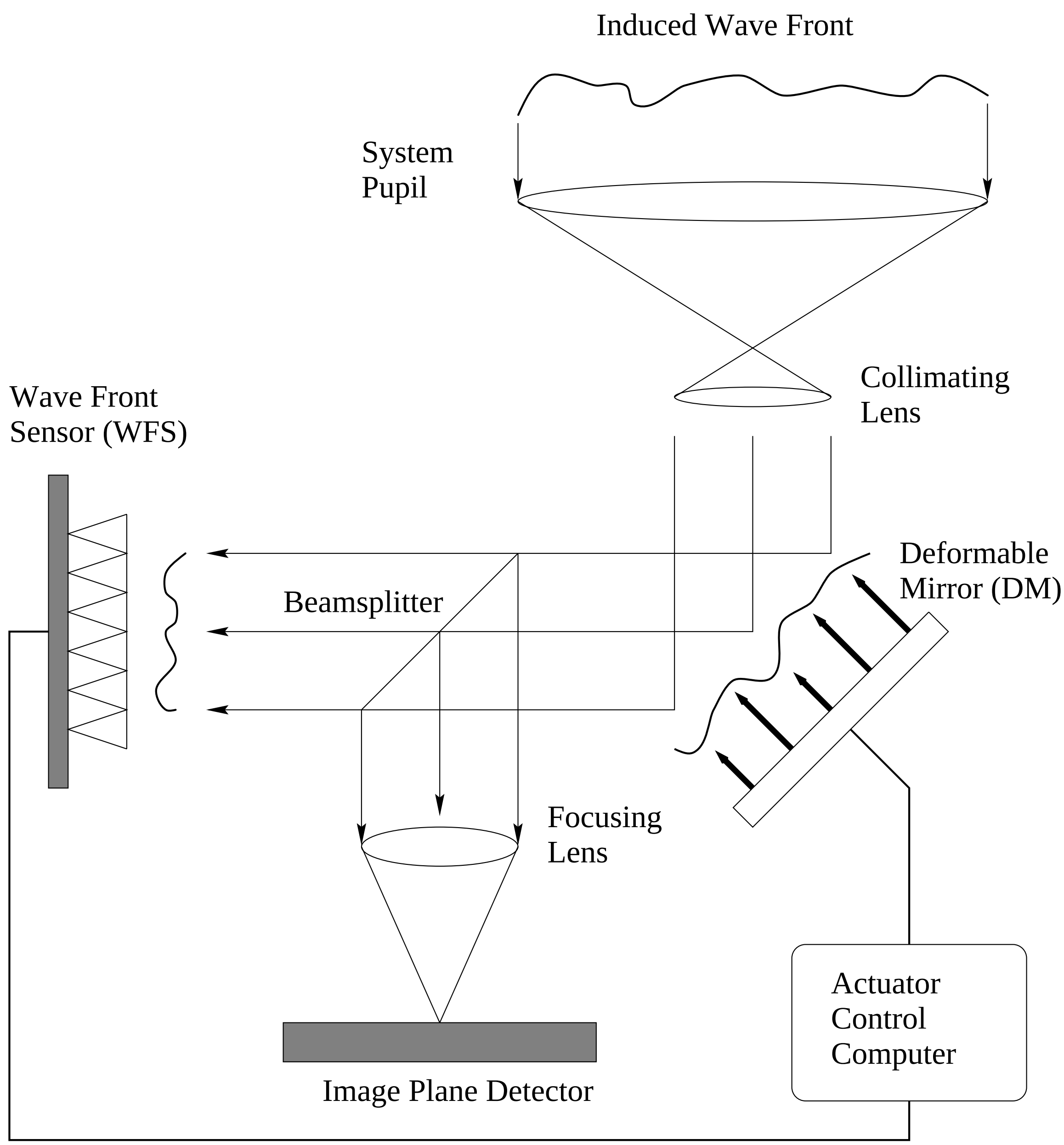
Outline

- Imaging through the Atmosphere
- Closed-loop Adaptive Optics Model
- Adaptive Optics Control
  - ◇ Estimating the Reconstructor
  - ◇ Controlling the Deformable Mirror
- Numerical Challenges

Atmospheric Imaging Computation

- Purpose:
  - ◇ To compensate for the degradation of astronomical image quality caused by the effects of atmospheric turbulence.
- Two stages of approach:
  - ◇ Partially nullify optical distortions by a deformable mirror (DM) operated from a closed-loop adaptive optics (AO) system.
  - ◇ Minimize noise or blur via off-line post-processing deconvolution techniques (not this talk).
- Challenges:
  - ◇ Atmospheric turbulence can only be measured adaptively.
  - ◇ Need theory to pass atmospheric measurements to knowledge of actuating the DM.
  - ◇ Require fast performance of large-scale data processing and computations.

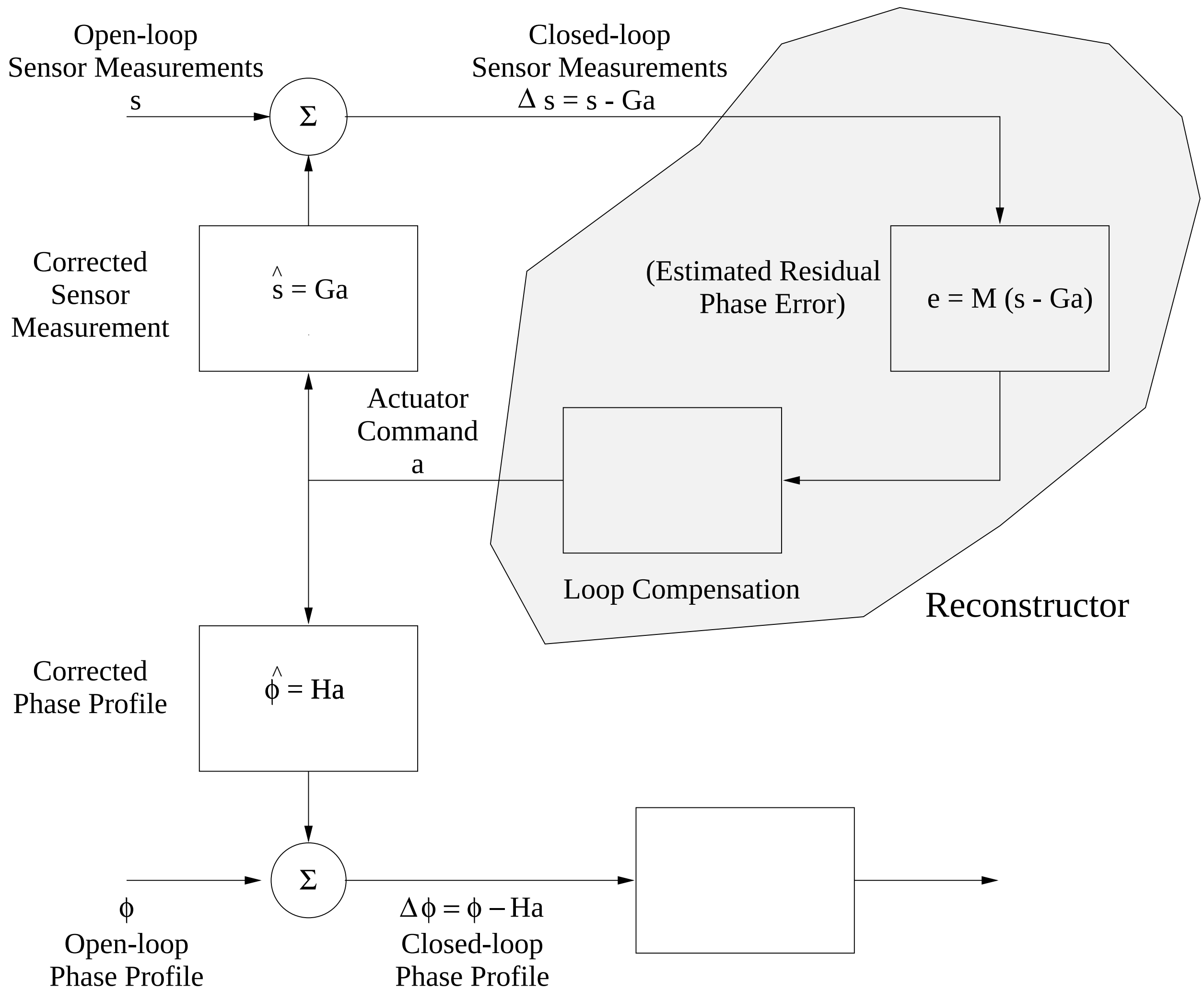
A Simplified AO System



Basic Notation

- Three quantities:
  - ◇ $\phi(t)$ = turbulence-induced phase profile at time t.
  - ◇ $a(t)$ = deformable mirror (DM) actuator command at time t.
  - ◇ $s(t)$ = wavefront slope sensor (WFS) measurement at time t and with no correction.
- Two transformations:
  - ◇ H := transformation from actuator commands to resulting phase profile adjustments.
  - ◇ G := transformation from actuator commands to slope sensor measurement adjustments.

A Close-loop AO Control Model



From Actuator to DM Surface

- H is used to describe the DM surface change due to the application of actuators.
- $r_i(\vec{x})$ = influence function on the DM surface at position $\vec{x}$ with an unit adjustment to the ith actuator.
- Assuming m actuators and linear response of actuators to the command, model the DM surface by

$$\hat{\phi}(\vec{x}, t) = \sum_{i=1}^{m} a_i(t) r_i(\vec{x}).$$

  - ◇ Sampled at n DM surface positions, can write

$$\hat{\phi}(t) = Ha(t)$$

    - ▷ $H = (r_i(\vec{x}_j)) \in R^{n \times m}$.
    - ▷ $\hat{\phi}(t) = [\hat{\phi}(\vec{x}_1, t), \ldots, \hat{\phi}(\vec{x}_n, t)]^T \in R^n$ = discrete corrected phase profile at time t.

From Actuator to WFS Measurement

- G is used to describe the WFS slope measurement associated with the actuator command a.
- Consider the H-WFS model where

$$s_j(t) := -\int d\vec{x}(\nabla W_{sj}(\vec{x}) \cdot \vec{d}_j)\phi(\vec{x},t), \quad j = 1,\ldots,\ell.$$

  ◇ $W_{sj}, \vec{d}_j$ = given specifications of jth subaperture.

- The measurement corresponding to $\hat{\phi}(\vec{x},t)$ would be

$$\hat{s}_j(t) = \sum_{i=1}^{m} \underbrace{\left(-\int d\vec{x}(\nabla W_{sj}(\vec{x}) \cdot \vec{d}_j)r_i(\vec{x})\right)}_{G_{ji}} a_i(t).$$

  ◇ Can write

$$\hat{s}(t) = Ga(t)$$

  where $G = [G_{ij}] \in R^{\ell \times m}$.

  ◇ The DM actuators are *not* capable of producing the exact wavefront phase $\phi(\vec{x},t)$ due to its finiteness of degrees of freedom. So $\hat{s} = Ga$ is not an exact measurement.

What is Available?

- Two residuals that are available in a *closed-loop* AO system:
  - ◇ $\Delta\phi(t) := \phi(t) - Ha(t)$
    - ▷ Represents the residual phase error remaining after the AO correction.
    - ▷ Also means instantaneous closed-loop wavefront distortion at time t.
  - ◇ $\Delta s(t) := s(t) - Ga(t)$
    - ▷ Represents feedback applied to $s(t)$ by DM actuator adjustment.
    - ▷ Also means *observable* wavefront sensor measurement at time t.
- In practice, there is a servo lag or delay in time Δt, i.e., it is likely
  - ◇ $\Delta\phi(t) := \phi(t) - Ha(t - \Delta t)$.
  - ◇ $\Delta s(t) := s(t) - Ga(t - \Delta t)$.

  Thus the data collected are not perfect.

Open-loop Model

- Assume a linear relationship between open-loop WFS measurement s and turbulence-induced phase profile ϕ:

$$s = W\phi + \epsilon \tag{1}$$

  - $\diamond$ ϵ = measurement noise with mean zero.
  - $\diamond$ In the H-WFS model, W represents a quadrature of the integral operator evaluated at designated positions $\vec{x}_j$, $j = 1, \ldots n$.

- Want to estimate ϕ using $\tilde{\phi}$ from the model

$$\tilde{\phi} = M_{open} s$$

so that the variance

$$\mathcal{E}[\|\phi - \tilde{\phi}\|^2]$$

is minimized.

  - $\diamond$ The wave front reconstruction matrix M_{open} is given by

$$M_{open} = \mathcal{E}[\phi s^T](\mathcal{E}[ss^T])^{-1}.$$

  - $\diamond$ For unbiased estimation, need to enforce the condition that $M_{open}W = I$.

Closed-loop Model

- For the H-WFS model, it is reasonable to assume the relationship
$$WH = G. \tag{2}$$
- Then
$$\begin{aligned} s &= W\phi + \epsilon \\ &= W(Ha + \Delta\phi) + \epsilon \\ &= WHa + (W\Delta\phi + \epsilon). \end{aligned}$$
It follows that
$$\Delta s = W\Delta\phi + \epsilon. \tag{3}$$
  - ◇ The closed-loop relationship (3) is identical to the open-loop relationship (1).
- Can estimate the residual phase error $\Delta\phi(t)$ using $e(t)$ from the model
$$e = M_{closed}\Delta s$$
  - ◇ M_{closed} = wavefront reconstruction matrix.
  - ◇ For unbiased estimation, it requires that $M_{closed}W = I$. Hence
$$M_{closed}G = M_{closed}(WH) = H.$$

Estimating the Reconstructor and the Actuator Command

A. Compute M based on the control law $a = Ms$ so that

$$\mathcal{E}[\|\Delta s\|^2] = \mathcal{E}[\|s - Ga\|^2]$$

is minimized.

B. Compute M subject to the servo-loop compensator so that

$$\mathcal{E}[\langle \Delta\phi, \Delta\phi \rangle]$$

is minimized, and then determine the DM actuator command A from the finite temporal response loop model

$$\frac{da}{dt} = k\Delta s = kM(s - Ga). \tag{4}$$

C. Compute M based on open-loop measurement so that

$$\mathcal{E}[\|\Delta\phi - M\Delta s\|^2]$$

is minimized.

D. Find a such that

$$\mathcal{E}[\|\Delta\phi\|^2] = \mathcal{E}[\|\phi - Ha\|^2]$$

is minimized subject to

$$Ha = M_{open}s$$

- ◇ This is equivalent to the idea case when both the minimum variance approximation $\tilde{\phi} = M_{open}s$ and the DM surface $\hat{\phi} = Ha$ is exactly equal to the induced wave front ϕ.

Idea A:
Minimize $\mathcal{E}[\|\Delta s\|^2]$

- Consider the model

$$s = Ga + \Delta s.$$

Want to determine M and the estimated command $\hat{a}$ of the form

$$\hat{a} = Ms$$

so that

$$\mathcal{E}[\|s - G\hat{a}\|^2]$$

is minimized.

  - ◇ The issue is not to minimize $\mathcal{E}[\|Ms - a\|^2]$.

- The optimal solution is given by

$$M = \left(G^T\left(\mathcal{E}[\Delta s(\Delta s)^T]\right)^{-1}G + (\mathcal{E}[aa^T])^{-1}\right)^{-1} G^T\left(\mathcal{E}[\Delta s(\Delta s)^T]\right)^{-1}.$$

  - ◇ If the noise variance matrix $\mathcal{E}[\Delta s(\Delta s)^T] = \sigma^2 I$, then

$$M = \left(G^TG + \sigma^2(\mathcal{E}[aa^T])^{-1}\right)^{-1} G^T$$

which is reduced to the standard least squares solution if noise variance in Δs decreases to zero.

Idea B:
Minimize $\mathcal{E}[\langle \Delta\phi, \Delta\phi \rangle]$ with Loop Compensation

---

- Assume $MG \equiv H$ and $HM = H$. The steady-state solution is given by

$$\begin{aligned} a(t) &= \int_0^\infty e^{-kMG\tau} kMs(t-\tau)\,d\tau \\ &= M \underbrace{\left(\int_0^\infty e^{-k\tau} ks(t-\tau)\,d\tau\right)}_{y(t)}. \end{aligned}$$

  - ◇ $y(t)$ means temporally filtered version of the instantaneous slope $s(t)$.

- To minimize $\mathcal{E}[\langle \phi, \phi \rangle]$, M must be given by

$$M = H\left(BS^{-1} + (I - BS^{-1}G)(G^T S^{-1} G)^{-1} G^T S^{-1}\right).$$

  where

$$\begin{aligned} B_{ij} &:= \mathcal{E}[\langle \phi, h_i \rangle y_j] \\ S_{ij} &:= \mathcal{E}[y_i y_j] \\ H &= [h_1, \ldots, h_n]. \end{aligned}$$

Idea C:

Minimize $\mathcal{E}[\|\Delta\phi - M\Delta s\|^2]$

- Consider the closed-loop model

$$\Delta s = W\Delta\phi + \epsilon.$$

and the relationship

$$\begin{aligned} \Delta\phi - M\Delta s &= (\phi - Ha) - M(s - Ga) \\ &= (\phi - Ms) + (MG - H)a. \end{aligned}$$

- One could minimize the closed-loop system $\mathcal{E}[\|\Delta\phi - M\Delta s\|^2]$ via minimizing the open-loop system

$$\begin{array}{rc} \text{minimize} & \mathcal{E}[\|\phi - Ms\|^2] \\ \text{subject to} & MG = H. \end{array}$$

Numerical Challenges

- Need to compute M fast enough.
- Every formulation involves calculating the inverse of some covariance matrices or sum of nested matrices.
  - ◇ Noise covariance matrix $(\mathcal{E}[\Delta s(\Delta s)^T])^{-1}$.
  - ◇ Control covariance matrix $(\mathcal{E}[aa^T])^{-1}$.
  - ◇ Nested matrix $\left(G^T\left(\mathcal{E}[\Delta s(\Delta s)^T]\right)^{-1}G+(\mathcal{E}[aa^T])^{-1}\right)^{-1}$.
  - ◇ Open-loop estimator $M_{open} = \mathcal{E}[\phi s^T](\mathcal{E}[ss^T])^{-1}$.
- Statistical information about ϕ, s, Δs and a varies in time and is available only adaptively.
- Could the constructor be estimated adaptively from the optimization problem itself, instead of the closed-form formulation?