On the Inverse Problem of Constrained Data Reconstruction

by

Moody T. Chu

joined with

Robert J. Plemmons

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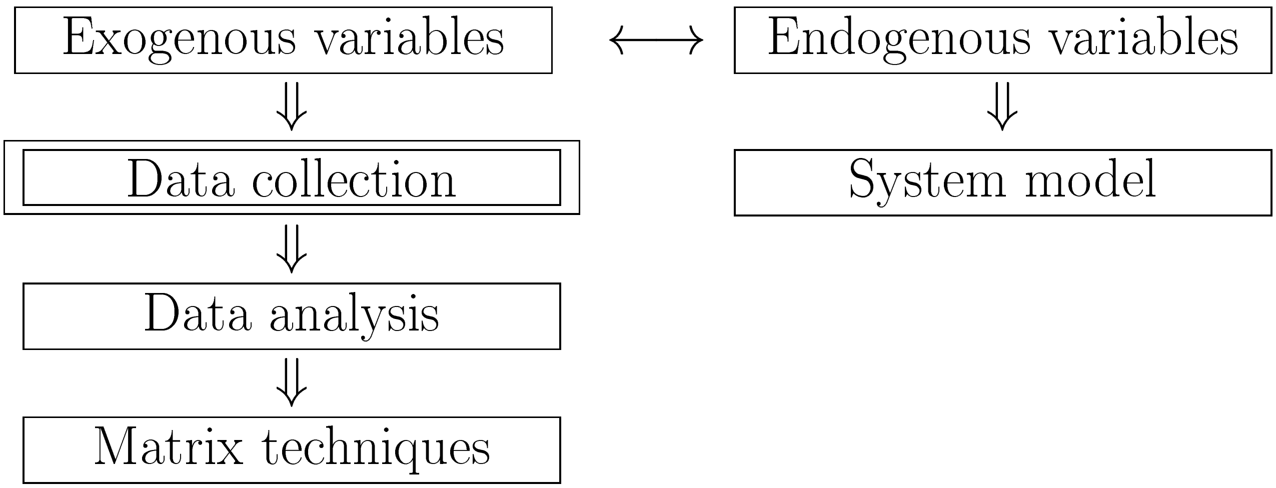
Warning

The following presentation contains no essential materials. It raises more questions than answers, and will cause more anxieties than satisfactions. If you are here looking for a definitive result, then you are in the wrong room.

Outline

- Background
  - ◇ Forward Problem
  - ◇ Inverse Problem
  - ◇ Constraints
- A Few Case Studies
  - ◇ Low Rank Approximation Problem
  - ◇ Inverse Eigenvalue Problem
  - ◇ Matrix Completion Problem
  - ◇ Inverse Testing Problem
- Fundamental Questions
- Conclusion

Forward Problem



- Solving linear equations.
- Spectral analysis.
- SVD
- TLS
- . . .

Inverse Problem

- Information gathering devices have only finite bandwidth.
- Data collected are not exact.
  - ◇ Instrumental noises in signals received by antenna arrays.
  - ◇ Atmospheric turbulence in astronomical images received by telescopes.
  - ◇ Blurring effect in image or signal process.
  - ◇ Intrinsic physical constraints are not satisfied even in laboratory setting.
- Need to reconstruct data so that
  - ◇ Inexactness is reduced.
  - ◇ A certain feasibility conditions are satisfied.

Some Typical Constraints

- Feasibility constraints:
  - ◇ Bounds on data.
  - ◇ Non-negativity on data.
  - ◇ Fixed data on non-free standing components.
- Structural constraints:
  - ◇ Banded matrices.
  - ◇ Toeplitz or block Toeplitz matrices.
  - ◇ Hankel matrices.
  - ◇ Toeplitz plus Hankel matrices.
  - ◇ Circulant or block circulant matrices.
- Spectral constraints:
  - ◇ Prescribed information about eigenvalues/vectors.
  - ◇ Prescribed singular value information.
- Other types of constraints:
  - ◇ Rank condition.
  - ◇ ...

Structured Low Rank Approximation

- Given
  - ◇ A target matrix $A \in R^{n \times n}$,
  - ◇ An integer k, $1 \leq k < \text{rank}(A)$,
  - ◇ A class of matrices Ω with linear structure,
  - ◇ a fixed matrix norm $\|\cdot\|$;

  Find a matrix $\hat{B} \in \Omega$ of rank k such that

$$\|A - \hat{B}\| = \min_{B \in \Omega,\, \text{rank}(B)=k} \|A - B\|.$$

Applications

- Noise removal in signal/image processing with Toeplitz structure. (Classical)
  - ◇ rank = noise level where SNR is high.
- Model reduction problem in speech encoding and filter design with Hankel structure. (Cadzow'90, Park'97)
  - ◇ rank = # of sinusoidal components in the signal.
- GCD approximation for multivariate polynomials with Sylvester structure. (Corless'95)
  - ◇ rank = degree of GCD.
- Molecular structure modeling for protein folding with nonnegative matrices. (Hayden'90)
  - ◇ rank ≤ 5.
- LSI application.
  - ◇ rank = # of factors capturing the random nature of the indexing matrix but structure = ?
- Preconditioning/regularization of ill-posed inverse problems. (Nagy'97)

Inverse Eigenvalue Problem

- Given
  - ⋄ A set of scalars $\lambda_1, \ldots, \lambda_n$,
  - ⋄ A class of matrices Ω with linear structure,

  Find a matrix $X \in \Omega$ such that

$$\sigma(X) = \{\lambda_1, \ldots, \lambda_n\}.$$

- Given
  - ⋄ A set of scalars $\lambda_1, \ldots, \lambda_k$,
  - ⋄ A set of vectors $v_1, \ldots, v_k$,
  - ⋄ A class of matrices Ω with linear structure,

  Find a matrix $X \in \Omega$ such that

$$Xv_i = \lambda_i v_i, \quad i = 1, \ldots k.$$

Applications

- Construct a mass-spring system with specific type and prescribed natural frequency/mode. (Classical)
  - ◇ Given A, find $X \in \Omega$ such that
  $$\sigma(A + X) = \{\lambda_1, \ldots, \lambda_n\}.$$
- Control the vibration of a string by placing weights at designated points. (Tire balancing)
- State-feedback/output-feedback pole assignment problem in control. (Classical)
  - ◇ Given A, B, C, find X such that
  $$\sigma(A + BXC) = \{\lambda_1, \ldots, \lambda_n\}.$$
- Preconditioning or stabilizing process. (Classical)
  - ◇ Given A and a domain $\mathcal{D} \in C$, find $X \in \Omega$ such that
  $$\sigma(XA) \subset \mathcal{D}.$$
- Construct a symmetric Toeplitz matrix with prescribed spectrum. (Lauri'88, Laundau'92)
- Construct a row-stochastic matrix with specific transit structure and prescribed spectrum. (Chu&Guo'98)

Matrix Completion Problem

- Given
  - ⋄ A set of scalars $\lambda_1, \ldots, \lambda_n$,
  - ⋄ A class of matrices Ω with partially fixed entries,

  Find (complete) a matrix $X \in \Omega$ such that

$$\sigma(X) = \{\lambda_1, \ldots, \lambda_n\}.$$

Applications

- Schur-Horn Theorem:
  - ◇ Given vectors $a, \lambda \in R^n$ such that

$$\begin{array}{rcl} a_{j_1} & \leq \ldots & \leq a_{j_n}, \\ \lambda_{m_1} & \leq \ldots & \leq \lambda_{m_n}, \end{array}$$

  - ◇ Then a Hermitian matrix H with eigenvalues λ and diagonal entries a exists if and only if a *majorizes* λ, i.e.,

$$\begin{array}{rcl} \sum_{i=1}^{k} \lambda_{m_i} & \leq & \sum_{i=1}^{k} a_{j_i}, \quad \text{for } k = 1, \ldots n, \\ \sum_{i=1}^{n} \lambda_{m_i} & = & \sum_{i=1}^{n} a_{j_i}. \end{array}$$

  - ◇ Construct such a Hermitian matrix with given diagonals and eigenvalues. (Chu'95, Zha&Zhang'95)

- Sing-Thompson Theorem:
  - ◇ Given vectors $d, s \in R^n$ such that

$$\begin{array}{rcl} s_1 & \geq s_2 & \geq \ldots s_n, \\ |d_1| & \geq |d_2| & \geq \ldots |d_n|. \end{array}$$

  - ◇ Then a real matrix with singular values s and main diagonal entries d (possibly in different order) exists if and only if

$$\begin{array}{rcl} \sum_{i=1}^{k} |d_i| & \leq & \sum_{i=1}^{k} s_i, \quad \text{for } k = 1, \ldots, n, \\ \left(\sum_{i=1}^{n-1} |d_i|\right) - |d_n| & \leq & \left(\sum_{i=1}^{n-1} s_i\right) - s_n. \end{array}$$

  - ◇ Construct such a square matrix with given diagonals and singular values. (Chu'99)

Inverse Testing Problem

- Given
  - ◇ A statement of property $\mathcal{P}$,

  Find a matrix X with property $\mathcal{P}$.
- The property $\mathcal{P}$ can be any of those previously mentioned.
- The property $\mathcal{P}$ can be any more general conditions.

Applications

- Weyl-Horn Theorem.
  - ⋄ Given vectors $\lambda \in C^n$ and $\alpha \in R^n$ such that

$$\begin{aligned} |\lambda_1| &\geq \ldots \geq |\lambda_n|, \\ \alpha_1 &\geq \ldots \geq \alpha_n, \end{aligned}$$

  - ⋄ Then a matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ and singular values $\alpha_1, \ldots, \alpha_n$ exists if and only if

$$\begin{aligned} \prod_{j=1}^{k} |\lambda_j| &\leq \prod_{j=1}^{k} \alpha_j, \quad k = 1, \ldots, n-1, \\ \prod_{j=1}^{n} |\lambda_j| &= \prod_{j=1}^{n} \alpha_j. \end{aligned}$$

  - ⋄ Construct such a matrix with prescribed singular values and eigenvalues. (Chu'00, Li&Mathias'00)
- Educational testing problem: (Fletcher'85)
  - ⋄ Given S symmetric and positive definite, find the largest trace of a non-negative diagonal matrix D such that $S - D$ is positive semi-definite.
- Nearest correlation matrix approximation. (Higham'00)

Fundamental Questions

- Solvability: Determine a necessary or a sufficient condition under which an inverse data reconstruction problem has a solution.
- Computability: Develop a procedure by which, knowing a priori that the given inverse problem is solvable, an approximate data matrix can be constructed numerically.
- Stability: Determine how sensitive a reconstructed matrix is subject to perturbation of the given inexact data.

Many pieces of this puzzle are missing!

Conclusion

- The inverse problem of matrix construction arises in many areas of important applications.
- Matrices under construction are supposed to satisfy certain specific constraints.
- The constraints could be inherited intrinsically from the physical feasibility of a certain mechanical structure or could be driven extrinsically by the desirable property of a certain design parameter.
- We have studied a few cases where such an inverse problem of data reconstruction is needed.
- We did not talk about any technical details on how such a problem could be solved.
- Indeed, theories and numerical methods are far from being complete. Many open questions need to be answered.