Research Profile of Moody T. Chu

I was formally trained as a numerical analyst under the traditional atmosphere of pure mathematics at Michigan State University. My 1982 dissertation was on the subject of solving stiff ordinary differential equations by nonlinear multi-step methods. Very quickly, however, I became interested in the application of differential equation techniques to problems arising from numerical analysis. Generally speaking, my research can be categorized as *the study of inverse problems by realization processes*. The realm of my research ranges from theorization of dynamics for classical iterative schemes to development of algorithms for challenging inverse problems, and from algebraic abstraction of existence questions to practical implementation for engineering applications. The techniques I use in research involve computer experiments as well as mathematical analysis. In the computer experiments, I need to do extensive numerical and symbolic computations. The computer experiments often will provide insights for later verification by more conventional mathematical means. In mathematical analysis, I need as a synthesis of tools knowledge from differential geometry, matrix theory, numerical analysis, optimization theory and dynamical systems. The mathematical analysis is likely to affect the design, implementation and our understanding of the solvability issue in general and computability issue in particular. In this profile, I will briefly outline some of my research highlights.

1. General Background. Realization process, in a sense, is a daily tool used to guide our thinking for solving problems. In mathematics, especially for existence questions, a realization process often is described in the form of an iterative procedure or a differential equation. My research has been centering around the investigation of one special realization process, i.e., matrix differential equation with its broad spectrum of applications.

The basic idea in my approach is to "evolve" into the solution of a difficult problem through the trajectory of a differential equation whose initial value is easier to be identified. The differential system may be derived in several different ways: Sometimes an existing discrete numerical method may be extended directly into a continuous model; sometimes the equation arises naturally with a certain physics background; and more often a vector field is constructed with a specific task in mind.

Matrix differential equations have important applications to various fields of disciplines. My own motivation for studying matrix differential equations is attributed mainly to:

1. Areas of applications of continuous realization process is very broad while many well developed classical results are immediately available for studying the dynamics of continuous systems. The study of continuous system could shed critical insights into the understanding of the dynamics of an existing discrete methods.
2. Continuous realization sometimes unifies different discrete methods as special cases of its discretization and often gives rise to the design of new numerical

algorithms.

3. Differential systems results from continuous realization present immediate challenge to the current numerical ODE techniques. Partially this is due to the fact that usually a certain invariant manifold needs to be preserved during the integration. Partially this is due to the fact that matrix differential equations are especially suitable for integration on a massively data-parallel computing system. Thus matrix differential equations may be used as benchmark problems for testing new ODE techniques. Conversely, new ODE techniques may further benefit the numerical computation of matrix differential equations.
4. Many existence problems, seemingly impossible to be tackled by any conventional discrete methods, may be solved by formulating special differential equations that ensure a specific task is taking place continuously. In my research, several times I have used these tactics to establish feasible numerical algorithms. Continuous methods, in a sense, give a smoother control in realization a problem. This, therefore, opens a new field of applications of numerical ODE techniques.
5. In contrast to the local properties of some discrete methods, the continuous approach usually offers a global method for solving the underlying problem.

In the past few years I have already made some contributions to undertaking the aforementioned tasks. Yet I think there are still many open areas that deserve further investigation. I plan to continue working on the following specific topics:

1. Understand the dynamics of each of the proposed matrix differential equations and its discrete counterpart;
2. Develop new numerical algorithms for inverse problems of data reconstruction arising from physical and engineering applications;
3. Answer the existence question concerning the inverse eigenvalue problems for various specially structured matrices. Consider similar question for the inverse singular value problems;
4. Experiment with new numerical ODE algorithms that preserve invariant manifolds.

2. Past Contributions. One of my early contributions to the continuous realization process has been the homotopy method for algebraic eigenvalue problems and λ-matrix problems [5,15,18]. (Numbers in brackets are referred to articles in my publications.) The idea in [5] recently has been implemented successfully on parallel computers by other researchers and the experimental results are proved to be very significant. Nonetheless, homotopy is just one possible way to establish a realization process. Indeed, many other important mathematical problems may also be formulated in terms of matrix differential equations. Among these, I mention particularly the applications to nonlinear algebraic systems [16], eigenvalue problems [3,4,6,8,10,17], singular value problems [11], least squares approximation problems subject to eigenvalue or singular value constraints [24,29,40,46], inverse eigenvalue problems for Toeplitz matrices [21,42], non-negative matrices [27], or stochastic matrices [46], quadratic programming problems [28], orthogonal Procrustes problems [50,51,56], simultaneous reduction problems [25], and inverse testing problems [55,57]. Along with problems arising out of other fields (See comments on paper [45] in the next section), I believe matrix differential equations should have important impact on many areas of disciplines, including numerical analysis, control theory, signal processing, matrix theory, multivariate statistical analysis, and mathematical programming.

3. Representative Papers. It might help to better perceive the scope of my research if I boldly self-comment on five of my papers. These comments also indicate the chronological changes and developments of my research interest.

[4] *The generalized Toda flow, the QR algorithm and the center manifold theorem, 1984.*

In this paper, I show how the Toda flow can be formulated even for complex-valued, full and non-symmetric matrices.

- I establish the fact that the generalized Toda flow, when sampled at integer times, gives the same sequence of matrices as the QR algorithm applied to the matrix $exp(G(X_0))$ where $G(z)$ is an arbitrary analytic function defined on a domain contain the spectrum of X_0. This paper generalized what was known at that time the connection between Toda-Flaschka flow for symmetric, tridiagonal matrices and the QR algorithm.
- I also establish the fact that the upper triangular matrices with "decreasing main diagonal entries" is the stable center manifold of the Toda flow. Hence, the convergence property of QR algorithm can be understood from the center manifold theory.
- This paper evolves into further discussion in [3] (convergence of QR algorithm for normal matrices), [8] (QZ flow), [10] (Continuous analogue of RQI method) [11] (SVD flow) and [17] (Abstract QR type algorithms).

[24] *The projected gradient method for least squares matrix approximations with spectral constraints, 1990.*

This paper describes a general procedure in using projected gradient method to solve various types of least squares approximation problem subject to spectral constraints. The first and the second order optimality conditions can be explicitly formulated. Important applications include:

- The Wielandt-Hoffman theorem can now be understood geometrically.
- The inverse eigenvalue problem, including the inverse Toeplitz eigenvalue problem, can now be solved numerically despite of the fact that algebraists are still struggling to prove the existence of a solution. My approach is different from that of Overton et al. in that they has to assume the existence of a solution whereas mine is just a generic flow that will find the solution if there is one, and will find the least squares solution if the exact solution does not exist.
- The paper evolves into further discussion in [25] (Simultaneous reduction of real matrices by orthogonal similarity or orthogonal equivalence), [26]

(Nearest normal matrix problem), [27] (Inverse eigenvalue problem for non-negative matrices) and [29] (Inverse singular value problem). So far as I know, problems involved in [25] and [27] have never been touched by numerical analysts before. My approach, on the other hand, is very versatile in the specification of the reduced forms.

[28] *Matrix differential equations: A continuous realization process for linear algebra problems, 1992.*

From mechanics point of view, it would be quite unthinkable that a Hamiltonian system such as the Toda flow would be a gradient flow. The two papers mentioned above should appear to move in two totally different directions. Nevertheless, in this paper I accumulate all the flows that I have considered and recast them in a unified general framework. A portion of this paper was presented at the several meetings, including two 12-lecture series presentations at the Australia National University and the Academia Sinica. Complete lecture notes consist of 256 pages with 169 references. The entire note is available on line at *http://www4.ncsu.edu/~mtchu*. I intend to expend it into a book in the near future. Of particular interesting points in this paper are that

- All the flows are of the type $\frac{dX}{dt} = [X, k(X)]$ where k is an appropriate matrix operator.
- Toda flow is indeed a gradient flow and, hence, the convergence of the QR algorithm and its variations is even easier understandable than before. Further result is given in #35.
- The approach to inverse eigenvalue problems by using differential equations proves to be quite fruitful. Applications include [38] (Schur-Horn theorem), [40] (Least squares problem), [42] (Inverse Toeplitz eigenvalue problem) and [44] (Inverse generalized eigenvalue problem).

[39] *A rank-one reduction formula and its relations to other matrix factorizations, 1995.*

In this paper my co-authors and I prove a powerful result that not only all known matrix factorizations can be derived from a simple rank-one reduction formula but also that new algorithms can be derived from this unified framework. This unification is significant in that it sheds many interesting insights into how a matrix could be simplified. This paper is fairly abstract, so future development and responses from the community are yet to be seen. For the moment, however, I am pleased to mention that the editor of the SIAM Review described this paper as "one of the best papers to come to SIREV in recent years".

[45] *Inverse eigenvalue problems, 1998*

An inverse eigenvalue problem concerns the reconstruction of a structured matrix from prescribed spectral data. The spectral data may involve complete or partial information of eigenvalues or eigenvectors. The structure imposed often is due to a certain physical feasibility requirement. Inverse eigenvalue problems arise in a remarkable variety of applications, including system and control theory, seismic tomography, principal component analysis, antenna array processing, geophysics, molecular spectroscopy, particle physics, data mining, structure analysis, circuit theory, mechanical system simulation and so on. Depending on the application, inverse eigenvalue problems appear in many different forms. Associated with any inverse eigenvalue problem are two fundamental questions — the theoretic issue on solvability and the practical issue on computability. Both questions are difficult and challenging, and the results are scattered around.

- This exposition paper is an accumulation of many years' efforts and experiences from research in [24,27,29,34,38,40,44,46].
- In this paper, I gather 204 references and review a collection of 37 different types of inverse eigenvalue problems and their applications. I classify the general characteristics of each problems. I also discuss up-to-then developments in both the theoretic and the algorithmic aspects of inverse eigenvalue problems.
- More importantly, I identified many open problems in this field that will continue to be my research interest for many years to come. The inverse testing problem [57] is one recent result.

4. Research with Students. Despite the facts that I have been elected twice by students themselves to be an outstanding teacher at North Carolina State University and that I have co-chaired many doctoral committees, I have not produced my own Ph.D. students thus far. I had the chance. Co-authors in [40,44,46] and my M.S. students were all outstanding candidates at N.C. State, but they all decided that the job market was too good to turn away and left for industry. On the other hand, I insist that my M.S. students' work must be publishable in refereed journals. I am pleased to mention four particular joint papers [12], [32], [34] and [37] with my M.S. students. In [12] I believe we were the first to propose and to implement the multi-block method on the Denelcor HEP machine for the parallel computing of ODEs. In [34] we proved a surprising new result that the dimensionality of symmetric Toeplitz matrices satisfying two prescribed eigenpairs was independent of the size of the problem. In [32] we answered two open questions concerning the multivariate eigenvalue problems arising from multivariate statistics. In [37] we solved a long standing non-smooth optimization problem by a much simpler Dikin's method and discovered errors in previously published results. All these papers involve some innovative ideas from which I believe further research can be developed.

5. Current Research. Finally, I want to mention that deviated from my research focus on the continuous realization process, I have broad interest in general mathematics. Most recently, I have been working on two other projects that are more industry oriented. I shall briefly outline them here.

The first one concerns imaging through atmosphere. The wavefront field aberrations induced by atmospheric turbulence can severely degrade the performance of an optical imaging system. The basic idea in adaptive optics is to position the surface of a deformable mirror in such a way as to approximately cancel the atmospheric turbulence. The control of a deformable mirror is based on its linear relationship, called the reconstructor, to the wave front sensor measurement. My work has been on the estimation of this reconstructor in the adaptive optics system. Since the entire process, from the acquisition of wave front measurements to the positioning of the surface of the deformable mirror, must be performed at speeds commensurate with the atmospheric changes, the adaptive optics control imposes several challenging computational problems. The notion of atmosphere turbulence can replaced by turbulence arising in many other circumstances, such as the liquid in the eyeballs, or the noises caused by the motion of engines. The techniques developed for adaptive optics, therefore, can equally be applied to many other important applications in defense, engineering, medicine, and science. A mathematical framework that unifies the description of several different estimators already used in practice is proposed in [58]. Just completed is a recent study on the convergence behavior and its effect of a particular adaptive control algorithm [59].

The second one concerns structured low rank approximation. In a noiseless time-domain signals comprising k components of exponentially decaying sinusoids can be identified by a Hankel matrix H of rank k. The measurement A obtained from, for example, *in vivo* NMR (Nuclear Magnetic Resonance on living objects) often is of full rank because random noises have been added inevitably . The challenge then is to retrieve as much information as possible about H from the observed A. It is known that the truncated singular value decomposition (TSVD) will produce the best low rank approximation. Nevertheless, this TSVD of A generally is no longer Hankel. Thus far, a popular method used by engineers to retrieve a Hankel low rank approximation is to alternate projections between the manifold of rank$-k$ matrices and the space of Hankel matrices. This so called Cadzow's algorithm has been claimed to be "a cleansing process whereby any corrupting noise, measurement distortion, or theoretical mismatch present in the given data set (namely, A) is removed." Recently, my colleagues and I have discovered numerical evidence using new formulations that Cadzow's algorithm usually does *not* give rise to the optimal approximation [54]. The impact of this finding is significant because the structured low rank approximation is also needed in other areas of signal processing application, including model reduction, speech encoding, filter design, and so on. Preliminary results are reported in [54] and we are continuing the study of this subject.