Chapter 1

Introduction

- Basic component
- Building the bridge
- Characteristic of a bridge
- Examples

Basic Components

- Two abstract problems:
  - ◇ One is a make-up and is easy.
  - ◇ The other is the real problem and is difficult.
- A bridge:
  - ◇ A continuous path connecting the two problems.
  - ◇ A path that is easy to follow.
- A numerical method:
  - ◇ A method for moving along the bridge.
  - ◇ A method that is readily available.

Building the Bridge

- Specified guidance is available.
  - ◇ The bridge is constructed by monitoring the values of certain specified functions.
  - ◇ The path is guaranteed to work.
  - ◇ e.g. Projected gradient methods.
- Only some general guidance is available.
  - ◇ A bridge is built in a straightforward way.
  - ◇ No guarantee the path will be complete.
  - ◇ e.g. Homotopy methods.
- No guidance at all.
  - ◇ A bridge is built seemingly by accident.
  - ◇ Usually deeper mathematical theory is involved.
  - ◇ e.g. The isospectral flows.

Characteristics of a Bridge

- A bridge, if exists, usually is characterized by an ordinary differential equation.
- The discretization of a bridge, or a numerical method in traveling along a bridge, usually produces an iterative scheme.

Examples

- The eigenvalue problem
- The nonlinear algebraic equation
- The least squares matrix approximation
- List of other applications

The Eigenvalue Problem

- The mathematical problem:
  - ◇ A symmetric matrix A_0 is given.
  - ◇ Solve the equation

$$A_0 x = \lambda x$$

for a nonzero vector x and a scalar λ.

- An iterative method :
  - ◇ The QR decomposition:

$$A = QR$$

where Q is orthogonal and R is upper triangular.

  - ◇ The QR algorithm (Francis '61):

$$\begin{aligned} A_k &= Q_k R_k \\ A_{k+1} &= R_k Q_k. \end{aligned}$$

  - ◇ The sequence $\{A_k\}$ converges to a diagonal matrix.
  - ◇ Every matrix A_k has the same eigenvalues of A_0.

- A continuous method:
  - ◇ Lie algebra decomposition:

    $$X = X^o + X^+ + X^-$$

    where X^o is the diagonal, X^+ the strictly upper triangular, and X^- the strictly lower triangular part of X.
  - ◇ Toda lattice (Symes '82, Deift el al '83):

    $$\begin{aligned} \frac{dX}{dt} &= [X, X^- - X^{-^T}] \\ X(0) &= X_0. \end{aligned}$$

  - ◇ Sampled at integer times, $\{X(k)\}$ gives the same sequence as does the QR algorithm applied to the matrix $A_0 = exp(X_0)$.
- The bridge between X_0 and the limit point of Toda flow is built on the basis of maintaining isospectrum.
  - ◇ What motivates the construction of Toda lattice?
  - ◇ Why is convergence guaranteed?

Nonlinear algebraic equations

- The mathematical problem:
  - ◇ A sufficiently smooth function $f : R^n \to R^n$ is given.
  - ◇ Solve the equation

$$f(x) = 0.$$

- An iterative method:
  - ◇ The Newton method:

$$x_{k+1} = x_k - \alpha_k (f'(x_k))^{-1} f(x_k).$$

  - ◇ The sequence $\{x_k\}$ converges quadratically to a solution, if x_0 is sufficiently close to that solution.

- A continuous method (Smale '76, Keller '78, etc.):
  - ◇ The Newton homotopy:

$$H(x,t) = f(x) - tf(x_0).$$

  - ◇ The zero set $\{(x,t) \in R^{n+1} | H(x,t) = 0\}$ is a smooth curve.
  - ◇ The homotopy curve:

$$\begin{aligned} f'(x)\frac{dx}{ds} - \frac{1}{t}f(x)\frac{dt}{ds} &= 0 \\ x(0) &= x_0 \\ t(0) &= 1 \end{aligned}$$

    where s is the arc length.
  - ◇ Suppose $f'(x)$ is nonsingular. Then written as

$$\frac{dx}{ds} = \frac{dt}{ds}\frac{1}{t}(f'(x))^{-1}f(x).$$

  - ◇ With appropriate step size chosen, an Euler step is equivalent to a regular Newton method.

- The bridge is built upon the hope that the obvious solution will be deformed mathematically into the solution that we are seeking for.
  - ◇ Will this idea always work?
  - ◇ How to mathematically design an appropriate homotopy?

Least Squares Matrix Approximation

- The mathematical problem:
  - ◇ A symmetric matrix N and a set of real values $\{\lambda_1, \ldots, \lambda_n\}$ are given.
  - ◇ Find a least squares approximation of N that has the prescribed eigenvalues.
- A standard formulation:

$$\begin{aligned} \text{Minimize } F(Q) &:= \frac{1}{2}||Q^T \Lambda Q - N||^2 \\ \text{Subject to } Q^T Q &= I \end{aligned}$$

  - ◇ Equality Constrained Optimization:
    - ▷ Augmented Lagrangian methods.
    - ▷ Sequential quadratic programming methods.
  - ◇ None of these techniques is easy.

- A continuous approach (Brockett '88, Chu & Driessel '90):
  - ◇ The projection of the gradient of F can easily be calculated.
  - ◇ Projected gradient flow:

$$\begin{aligned} \frac{dX}{dt} &= [X, [X, N]] \\ X(0) &= \Lambda \end{aligned}$$

    - ▷ $X := Q^T \Lambda Q$.
    - ▷ Flow $X(t)$ moves in a descent direction to reduce $||X - N||^2$.
  - ◇ The optimal solution X can be fully characterized by the spectral decomposition of N and is unique.
- The bridge between a starting point and the optimal point is built on the basis of systematically reducing the difference between the current position and the target position.

Isospectral Flows

- QR flow for normal matrices (Chu '84).
- Generalized Toda flow (Chu '84, Watkins '84).
- QZ flow (Chu '86).
- Continuous Rayleigh quotient flow (Chu '86).
- SVD flow (Chu '86).
- Abstract QR-type flow (Chu '88).
- Scaled Toda-like flow (Chu '95).

Projected Gradient Flows

- Brockett's double bracket flow (Brockett '88).
- Least squares approximation with spectral constraints (Chu & Driessel '90).
- Simultaneous reduction problem (Chu '91).
- Nearest normal matrix problem (Chu '91).
- Inverse eigenvalue problem for nonnegative matrices (Chu & Driessel '91).
- Inverse singular value problem (Chu '92).

Generalized Flows

- Matrix differential equations (Chu ’92).
- Schur-Horn theorem (Chu ’95).
- Least squares inverse eigenvalue problem (Chu & Chen ’96).
- Inverse generalized eigenvalue problem (Chu & Guo ’98).
- Inverse stochastic eigenvalue problem (Chu & Guo ’98).