Chapter 10

Miscellaneous Flows

- Flows on S^{n-1}
- Toeplitz annihilator flow
- Linear stationary flow
- SVD flow
- QZ flow
- Scaled Toda flow

Flows on S^{n-1}

- Power method
- Rayleigh quotient method

Power Method

- Iterative method:

$$\begin{aligned} y_{k+1} &= Ax_k \\ x_{k+1} &= \frac{y_{k+1}}{\|y_{k+1}\|}. \end{aligned}$$

  - ◇ Fundamental to the QR algorithm.
  - ◇ $\{x_k\}$ converges to the maximal eigenvector.

- Continuous method:

$$\frac{du}{dt} = Bu - \langle Bu, u\rangle u.$$

  - ◇ Project gradient flow on S^{n-1} for maximizing the Rayleigh quotient

$$\rho(y) := \frac{y^T B y}{y^T y}.$$

  - ◇ Closed form solution:

$$u(t) = \frac{e^{tB} u_0}{\|e^{tB} u_0\|}.$$

- Dynamics of $u(t)$:
  - $\diamond$ Maximal eigenvalue is real $\Rightarrow u(t)$ converges to maximal eigenvector.
  - $\diamond$ Maximal eigenvalue is complex $\Rightarrow u(t)$ is attracted to and oscillating around the 2-dim circle ($= S^{n-1} \cap$ subspace spanned by the real and the imaginary parts of maximal eigenvector).
- Choose $B = \ln A$ if A is nonsingular $\Rightarrow$

$$u(t) = \frac{A^t u_0}{\|A^t u_0\|}.$$

  - $\diamond$ Poincaré map $\equiv$ Classical power method.

Rayleigh Quotient Method

- Iterative method:

$$\begin{aligned} \rho_k &= x_k^T A x_k \\ y_{k+1} &= (A - \rho_k I)^{-1} x_k \\ x_{k+1} &= \frac{y_{k+1}}{\|y_{k+1}\|}. \end{aligned}$$

  ◇ Main objective: Speed up convergence by making $A - \rho_k I$ nearly singular so that $(A - \rho_k I)^{-1}$ has a most dominant eigenvalue.

- Suppose A is normal. Then

  ◇ $\{\rho_k\}$ converges.

  ◇ Either $\{(\rho_k, x_k)\}$ converges to an eigenpair cubically,

  ◇ Or $\{\rho_k\}$ converges linearly to a point equi-distant from $s \geq 2$ eigenvalues of A, but $\{x_k\}$, may or may not have a limit cycle, does not converge.

- Can model RQI by continuous power method in the interval $[k, k+1]$:
  - ◇ Choose $x_k = u(k)$ and $B = \ln(A - \rho_k I)^{-1}$.

$$u(k+1) = \frac{(A - \rho_k I)^{-1} u(k)}{\|(A - \rho_k)^{-1} u(k)\|}.$$

  - ◇ Only piecewise differentiable.
  - ◇ Difficult to analyze asymptotic behavior.

Simulation of RQI

- Behavior of $\ln|z-c|^{-1}$ and $|z-c|^2$ are qualitatively similar near $c \Rightarrow$

$$B = \ln(A - \rho(x_k))^{-1}$$
$$\uparrow$$
$$B = (A - \rho(x_k)I)^{-2}$$
$$\uparrow$$
$$B = B(u) := \left((A - \rho(u)I)^T(A - \rho(u)I\right)^{-1}.$$

- New differential equation:

$$\frac{du}{dt} = B(u)u - \langle B(u)u, u\rangle u.$$

  - ◇ System becomes singular when reaching the set $\Gamma := \{u \in S^{n-1} | \rho(u) \in \sigma(A)\}$.

- Residue function:

$$r(t) := \|\,(A - \rho(u(t))I)\,u(t)\|.$$

Global Dynamics

- Residue is monotone decreasing:

$$
\begin{aligned}
2r\frac{dr}{dt} &= \frac{d}{dt}\langle u, B^{-1}u\rangle \\
&= \langle\frac{du}{dt}, B^{-1}u\rangle + \langle u, B^{-1}\frac{du}{dt}\rangle \\
&\quad +\langle u, -\frac{d\rho}{dt}\left[(A-\rho)+(A-\rho)^T\right]u\rangle \\
&= 2\langle\frac{du}{dt}, B^{-1}u\rangle \\
&= 2\langle Bu - \rangle Bu, u\rangle u, B^{-1}u\rangle \\
&= 2\left(1-\langle u, Bu\rangle\langle u, B^{-1}u\rangle\right) \\
&\le 0.(\text{ Kantorovich Inequality})
\end{aligned}
$$

- $u(t)$ can behave as only one of the following three:
  - ◇ $u(t)$ hits the singular set Γ in finite time.
  - ◇ $u(t) \to$ an eigenvector of A as $t \to \infty$.
  - ◇ $u(t)$ has its ω-limit set contained in the set $E = \{u^*|u^* \text{ is an eigenvector of } B(u^*)\}$.
- Dynamics of the DE are parallel to those of the classical RQI (Chu, '86).

Toeplitz Annihilator Flow

- Isospectral flow on $\mathcal{S}(n)$:

$$\begin{aligned} \frac{dX}{dt} &= [X, k(X)] \\ k : \mathcal{S}(n) &\rightarrow \mathcal{S}(n)^{\perp} \\ &\Updownarrow \\ X(t) &= Q(t)^T X(0) Q(t) \\ &\Updownarrow \\ \frac{dQ}{dt} &= Qk(X),\ Q(0) = I. \end{aligned}$$

- Can take k to so that $k(T) = 0$ iff T is symmetric Toeplitz matrices.

- Ideas:
  - ◇ $X(t)$ stays bounded $\Rightarrow$ Invariant ω-limit set is non-empty.
  - ◇ Simple spectrum of $X(0)$
    $\Rightarrow [X, k(X)] = 0$ iff $k(X) =$ Polynomial of X
    $\Rightarrow k(X) = 0$
    $\Rightarrow X$ is Toeplitz.

Toeplitz Annihilators

- The simplest annihilator:

$$k_{ij} := \begin{cases} x_{i+1,j} - x_{i,j-1}, & \text{if } 1 \leq i < j \leq n \\ 0, & \text{if } 1 \leq i = j \leq n \\ x_{i-1,j} - x_{i,j+1}, & \text{if } 1 \leq j < i \leq n \end{cases}$$

  ◇ Vector field ⇒ Homogeneous polynomials of degree 2 and norm preserving.

- A more complicated way:

$$k(X) := [L(X), C]$$

  ◇ A constant matrix:

$$C := \begin{bmatrix} 0 & 1 & 0 & \dots & & 0 \\ 1 & 0 & 1 & & & \vdots \\ 0 & 1 & \ddots & & & \\ & & & & & \\ \vdots & & & & 0 & 1 \\ 0 & & \dots & & 1 & 0 \end{bmatrix}.$$

  ◇ $L(X)$:= Projection of X onto $\mathcal{T}^C$ along $\mathcal{T}$.

  ◇ $\mathcal{T}^C$:= Any complementary subspace of $\mathcal{T}$ in $\mathcal{S}(n)$.

A Danger

- All annihilators work numerically $\Rightarrow$ Confirming Landau's result (Landau, '94).
- A similar system (Bender et al., '78):

$$\begin{aligned}
\frac{dy_1}{dt} &= y_2y_3 + y_5y_4 - 2\,y_2y_5, \\
\frac{dy_2}{dt} &= y_3y_4 + y_1y_5 - 2y_3y_1, \\
\frac{dy_3}{dt} &= y_5y_4 + y_2y_1 - 2y_2y_4, \\
\frac{dy_4}{dt} &= y_1y_5 + y_2y_3 - 2y_5y_3, \\
\frac{dy_5}{dt} &= y_2y_1 + y_3y_4 - 2y_1y_4,
\end{aligned}$$

  - ◇ Norm preserving.
  - ◇ Random behavior.
- Need to scrutinize the system more!

The System when $n = 3$

- Trace $= 0 \Rightarrow$ Can describe system in terms of $(x_{11}, x_{12}, x_{13}, x_{23}, x_{33})$:

$$\begin{aligned}
\frac{dx_{11}}{dt} &= 4x_{12}x_{11} + 2x_{12}x_{33} - 2x_{13}x_{23} + 2x_{13}x_{12}, \\
\frac{dx_{12}}{dt} &= -4x_{11}^2 - 4x_{11}x_{33} - 2x_{13}x_{33} - x_{13}x_{11} \\
&\quad -x_{33}^2 - x_{23}^2 + x_{23}x_{12}, \\
\frac{dx_{13}}{dt} &= 3x_{11}x_{23} + 3x_{12}x_{33}, \\
\frac{dx_{23}}{dt} &= x_{23}x_{12} - x_{12}^2 - 4x_{11}x_{33} - x_{11}^2 - 4x_{33}^2 \\
&\quad -2x_{13}x_{11} - x_{13}x_{33}, \\
\frac{dx_{33}}{dt} &= 2x_{13}x_{23} - 2x_{13}x_{12} + 4x_{23}x_{33} + 2x_{11}x_{23}.
\end{aligned}$$

Invariant Set

- Isospectral surface is invariant.
- Equilibrium points are invariant:
  - ◇ No isolated equilibrium.
  - ◇ $(c_1, 0, -3c_1, 0, c_1)$
    - ▷ Eigenvalues $= 0, \pm 3\sqrt{6}c_1, \pm 6\sqrt{2}|c_1|i$.
    - ▷ Never stable.
    - ▷ Possible periodic solution.
  - ◇ $(0, c_2, c_3, c_2, 0)$
    - ▷ Eigenvalues $= 0, 0, 6c_2, 2(c_2+c_3), 2(c_2-c_3)$.
    - ▷ Stable when $c_2 < 0$ and $|c_2| \geq c_3$.
- $\mathcal{W} = \{(x_{11}, x_{12}, x_{13}, x_{12}, x_{11}) | x_{11}, x_{12}, x_{13} \in R\}$ is invariant.
  - ◇ $\mathcal{T} \subset \mathcal{W}$ with $x_{11} = 0$.
  - ◇ System in $\mathcal{W} \Rightarrow$ Elliptic orbits within $\mathcal{W}$.

$$\begin{aligned}
\frac{dx_{11}}{dt} &= 6x_{11}x_{12}, \\
\frac{dx_{12}}{dt} &= -9x_{11}^2 - 3x_{11}x_{13}, \\
\frac{dx_{13}}{dt} &= 6x_{11}x_{12}.
\end{aligned}$$

Orbital Stability

- Numerical integration always solves ITEP.
- Ellipses are orbitally unstable:

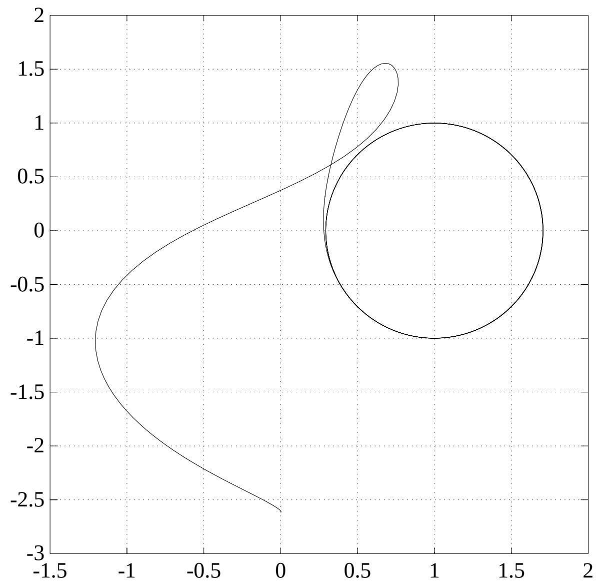

Figure 1: Plot of $x_{11}(t)$ versus $x_{12}(t)$.

- Orbital unstability can be confirmed by the *characteristic exponents* at a periodic solution (Coddington et al., '55).

- Orbits stay in isospectral surface ⇒ Eigenvalues are not lost.

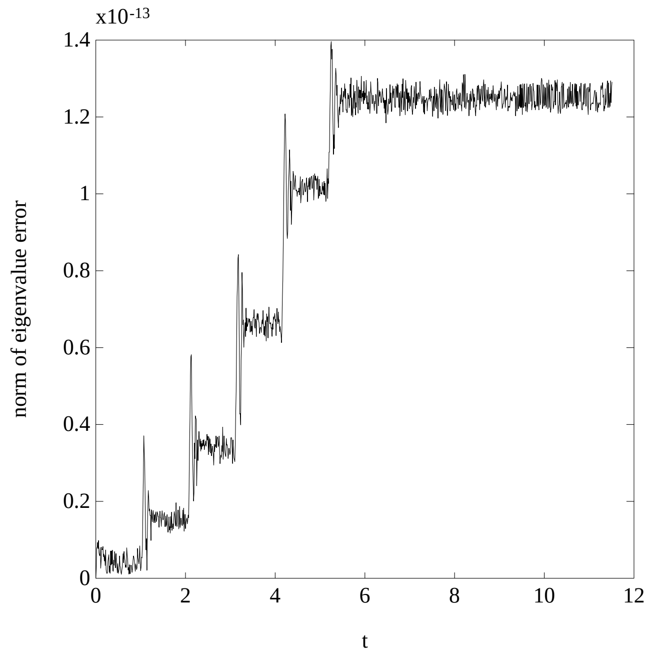

Figure 2: Error of eigenvalues between $X(0)$ and $X(t)$.

Linear Stationary Flow

- Stationary iteration for solving $Ax^* = b$:

$$x_{k+1} = Gx_k + c.$$

  - ◇ $G = I - Q^{-1}A$, $c = Q^{-1}b$.
  - ◇ Fixed point of iteration $= x^*$.
  - ◇ Iteration converges $\Leftrightarrow \rho(G) < 1$.

- A differential system:

$$\frac{dx}{dt} = (G - I)x + c = -Q^{-1}Ax + c.$$

  - ◇ Euler step with unit stepsize $\Rightarrow$ Stationary iteration.
  - ◇ $\Re\lambda(Q^{-1}A) > 0 \Rightarrow x^* =$ Global attractor.
  - ◇ The choice of Q exists (Pole assignment).

- Discrete methods for DE $\Rightarrow$ Plenty of new iterative schemes.
- Three-term polynomial acceleration methods $\equiv$ Two-step, explicit, variable coefficient ODE methods.

$$x_1 = \beta_1(Gx_0 + c) + (1 - \beta_1)x_0$$
$$x_{k+1} = \alpha_{k+1}\{\beta_{k+1}(Gx_k + c) + (1 - \beta_{k+1})x_k\} + (1 - \alpha_{k+1})x_{k-1}$$

iff

$$\begin{aligned} x_1 &= x_0 + \beta_1 f_0 \\ x_{k+1} &= \alpha_{k+1}x_k + (1 - \alpha_{k+1})x_{k-1} + \alpha_{k+1}\beta_{k+1}f_k. \end{aligned}$$

◇ Function evaluation $f_k := (G - I)x_k + c$.

SVD Flow

- Golub and Kahan algorithm:
  - ◇ Use Householder transformations to reduce $A \in R^{m\times n}$ to upper bidiagonal form.
  - ◇ Apply implicit-shift QR steps to the tridiagonal matrix $B^T B$ to compute the SVD of B.
- A one-parameter family of matrices:

$$
\begin{aligned}
Y(t) &= U(t)BV(t), \\
&\Updownarrow \\
\frac{dY}{dt} &= YN - MY \\
M, N &= \text{skew-symmetric}, \\
&\Updownarrow \\
\frac{dU}{dt} &= -MU, \\
\frac{dV}{dt} &= VN.
\end{aligned}
$$

- One special choice:

$$\begin{aligned} M(t) &:= \Pi_0\left(Y(t)Y(t)^T\right) \\ N(t) &:= \Pi_0\left(Y(t)^T Y(t)\right). \end{aligned}$$

  ◇ $Y(t)$ maintains the bidiagonal form.

$$\begin{aligned} \frac{dy_{i,i}}{dt} &= y_{i,i}\left(y_{i,i+1}^2 - y_{i-1,i}^2\right) \\ \frac{dy_{i,i+1}}{dt} &= y_{i,i+1}\left(y_{i+1,i+1}^2 - y_{i,i}^2\right). \end{aligned}$$

  ◇ Poincaré map $\equiv$ SVD algorithm.

QZ Flow

- Moler and Stewart algorithm:
  - ◇ Simultaneously educe A to upper Hessenberg form and B to upper triangular form.
  - ◇ Apply QR steps to reduce A to upper quasi-triangular form while preserving triangularity of B.
- A one-parameter family of matrices:

$$
\begin{aligned}
X(t) &= Q(t)AZ(t), \\
Y(t) &= Q(t)BZ(t), \\
&\Updownarrow \\
\frac{dX}{dt} &= XM - NX \\
\frac{dY}{dt} &= YM - NY \\
M, N &= \text{skew-symmetric}, \\
&\Updownarrow \\
\frac{dQ}{dt} &= -MM, \\
\frac{dZ}{dt} &= ZN.
\end{aligned}
$$

- One special choice:

$$\begin{aligned} M(t) &:= \Pi_0\left(X(t)Y(t)^{-1}\right) \\ N(t) &:= \Pi_0\left(Y(t)^{-1}X(t)\right). \end{aligned}$$

  - ◇ $X(t)$ maintains upper Hessenberg form.
  - ◇ $Y(t)$ maintains upper triangular form.
  - ◇ Poincaré map $\equiv$ QZ algorithm.

Scaled Toda Flow

- Componentwise scaling:

$$\frac{dX}{dt} = [X, A \circ X].$$

  - $\circ$ = Hadamard product.

- Isospectral flow:

$$\begin{aligned} X(t) &= L(t)^{-1} X_0 L(t) = R(t) X_0 R(t)^{-1}, \\ &\Updownarrow \\ \frac{dL}{dt} &= L(A \circ X), \\ \frac{dR}{dt} &= ((\mathbf{1} - A) \circ X) R. \end{aligned}$$

- QR-like iteration:

$$\begin{aligned} e^{X_0 t} &= L(t) R(t), \\ e^{X(t) t} &= R(t) L(t). \end{aligned}$$

Choices of A

- Continuous power method:

$$a_{ij} := \begin{cases} 1, & \text{if } j = 1 \text{ and } i > j; \\ -1, & \text{if } i = 1 \text{ and } j > i; \\ 0, & \text{otherwise.} \end{cases}$$

- Toda lattice:

$$a_{ij} := \begin{cases} 1, & \text{if } i > j; \\ -1, & \text{if } j > i; \\ 0, & \text{otherwise.} \end{cases}$$

- Arbitrary patten:

$$a_{ij} := \begin{cases} 1, & \text{if } (i, j) \in \Delta; \\ -1, & \text{if } (j, i) \in \Delta; \\ 0, & \text{otherwise.} \end{cases}.$$

  - ◇ Δ = Arbitrary index subset.

- Brockett's double bracket flow:

$$a_{ij} := d_i - d_j$$

  - ◇ $d_i \geq d_j$ if $i \leq j$.

Convergence

- $X(0)$symmetric, A skew-symmetric and tril(A) non-negative $\Rightarrow$

$$\lim_{t\to\infty} A \circ X(t) = 0.$$

  ◇ $x_{ij}(t) \longrightarrow 0$ whenever $a_{ij} \neq 0$.