Chapter 9

Toeplitz Inverse Eigenvalue Problem

- Overview
- Spectral properties of $\mathcal{C}(n)$
- An example on the effect of parity assignment
- A Newton method in $\mathcal{C}(n)$
- Numerical experiment

Overview

- Symmetric Toeplitz Matrix $T(r)$:

$$t_{ij} = r_{|i-j|+1}.$$

  e.g.

$$T(r) := \begin{bmatrix} r_1 & r_2 & \dots & r_{n-1} & r_n \\ r_2 & r_1 & & r_{n-2} & r_{n-1} \\ \vdots & \ddots & \ddots & & \vdots \\ r_{n-1} & & & r_1 & r_2 \\ r_n & r_{n-1} & & r_2 & r_1 \end{bmatrix}.$$

- Unit perdiagonal matrix J:

$$j_{ij} = \delta_{i,n-j+1}.$$

  - ◇ Symmetric vector, if $Jv = v$.
  - ◇ Skew-symmetric vector, if $Jv = -v$.

- Centrosymmetric matrix M:

$$m_{ij} = m_{n-i+1,n-j+1}.$$

- $\mathcal{C}(n) := \{M | M = M^T, M = JMJ\}$.

Questions & Known Results

- ToIEP: *Find $r \in R^n$ such that $T(r)$ has a prescribed set of real numbers $\{\lambda_1, \ldots, \lambda_n\}$ as its spectrum.*
- Solvability has been a challenge.
  - ◇ $n \geq 5$ is analytically intractable.
  - ◇ Symmetric Toeplitz matrices can have *arbitrary* real spectra. (Landau '94, nonconstructive proof by topological degree argument).
  - ◇ Eigenvalues *cannot* have arbitrary parity.
- Numerical calculation is possible.
  - ◇ n equations in n unknowns.
    - ▷ Newton's methods.
    - ▷ Continuation methods.
  - ◇ Mostly done in $\mathcal{S}(n)$.
- *Can the calculation be done in $\mathcal{C}(n)$? What are the advantages?*

Spectral Properties of $\mathcal{C}(n)$

- (Cantoni & Butler, '76) Any $M \in \mathcal{C}(n)$:

n	even	odd
M	$\begin{bmatrix} A & C^T \\ C & JAJ \end{bmatrix}$	$\begin{bmatrix} A & x & C^T \\ x^T & q & x^T J \\ C & Jx & JAJ \end{bmatrix}$
$\sqrt{2}K$	$\begin{bmatrix} I & -J \\ I & J \end{bmatrix}$	$\begin{bmatrix} I & 0 & -J \\ 0 & \sqrt{2} & 0 \\ I & 0 & J \end{bmatrix}$
KMK^T	$\begin{bmatrix} A-JC & 0 \\ 0 & A+JC \end{bmatrix}$	$\begin{bmatrix} A-JC & 0 & 0 \\ 0 & q & \sqrt{2}x^T \\ 0 & \sqrt{2}x & A+JC \end{bmatrix}$

  - $\diamond$ $A, C, J \in R^{\lfloor \frac{n}{2} \times \frac{n}{2} \rfloor}$.
  - $\diamond$ $x \in R^{\lfloor \frac{n}{2} \rfloor}$.
  - $\diamond$ $q \in R$.
  - $\diamond$ $A = A^T$.

- Orthonormal basis of eigenvectors:

$$Q = K^T Z$$

  - $Z := Z := \begin{bmatrix} Z_1 & 0 \\ 0 & Z_2 \end{bmatrix}$.
    - $Z_1 :=$ Eigenvectors of $A - JC$.
    - $Z_2 :=$ Eigenvectors of $\begin{bmatrix} q & \sqrt{2}x^T \\ \sqrt{2}x & A + JC \end{bmatrix}$ or $A+JC$.

- Parity of eigenvectors:
  - $K^T \begin{bmatrix} Z_1 \\ 0 \end{bmatrix} = \lfloor \frac{n}{2} \rfloor$ skew-symmetric eigenvectors $\Rightarrow$ "Odd" eigenvalues.
  - $K^T \begin{bmatrix} 0 \\ Z_2 \end{bmatrix} = \lceil \frac{n}{2} \rceil$ symmetric eigenvectors $\Rightarrow$ "Even" eigenvalues.

- *For an ToIEP to be solvable, each given eigenvalue must carry a specific parity.*

Inverse Problem for Centrosymmetric Matrices

- Given *arbitrary*:
  - ◇ Diagonal matrix $\Lambda := \text{diag}\{\lambda_1, \ldots, \lambda_n\}$.
  - ◇ Orthogonal matrix $Z_1 \in R^{\lfloor \frac{n}{2} \rfloor \times \lfloor \frac{n}{2} \rfloor}$.
  - ◇ Orthogonal matrix $Z_2 \in R^{\lceil \frac{n}{2} \rceil \times \lceil \frac{n}{2} \rceil}$.
- Then:
  - ◇ Centrosymmetric matrix:

$$M := K^T \begin{bmatrix} Z_1 & 0 \\ 0 & Z_2 \end{bmatrix} \Lambda \begin{bmatrix} Z_1 & 0 \\ 0 & Z_2 \end{bmatrix}^T K.$$

  - ◇ $\{\lambda_1, \ldots, \lambda_{\lfloor \frac{n}{2} \rfloor}\}$ = Odd eigenvalues of M.
  - ◇ $\{\lambda_{\lfloor \frac{n}{2} \rfloor + 1}, \ldots, \lambda_n\}$ = Even eigenvalues of M.
- Isospectral surface $\mathcal{M}_{\mathcal{C}}(\lambda_1, \ldots, \lambda_n)$:

$$\mathcal{M}_{\mathcal{C}} := \{M \in \mathcal{C}(n) | \text{ eigenvalues} = \lambda_1, \ldots, \lambda_n\}.$$

- M may not be Toeplitz.

An Example of $n = 3$

- $M \in \mathcal{C}(3)$:

$$M = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ \times & m_{22} & \times \\ \times & \times & \times \end{bmatrix}.$$

  - $\diamond$ $\text{trace}(M) = 0 \Longrightarrow$ Three free parameters in $\mathcal{C}(3)$.

- Isospectral subset $\mathcal{M}_{\mathcal{C}}(\lambda_1, \lambda_2, \lambda_3)$:

$$\begin{aligned} (m_{11} - \frac{\lambda_{\sigma_1}}{4})^2 + \frac{1}{2} m_{12}^2 &= \frac{(\lambda_{\sigma_2} - \lambda_{\sigma_3})^2}{16}, \\ m_{13} &= m_{11} - \lambda_{\sigma_1}. \end{aligned}$$

  - $\diamond$ $\sigma =$ A permutation of integers $\{1, 2, 3\}$.

- $\mathcal{M}_{\mathcal{C}} =$ Three ellipses.

  - $\diamond$ One circumscribes the other two.

- Check # of m_{12}-intercepts $\Rightarrow$

$$\#\text{ of solutions} = \begin{cases} 4, & \text{if distinct eigenvalues;} \\ 2, & \text{if multiplicity 2.} \end{cases}$$

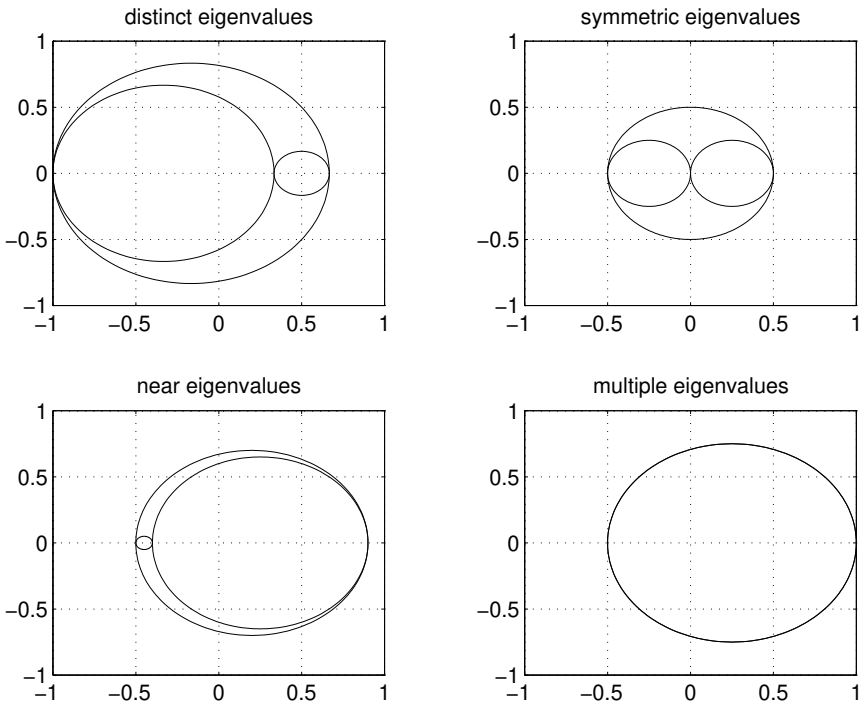

Figure 1: Plots of m_{11} versus m_{12} for $\mathcal{M}_{\mathcal{C}}$ in $\mathcal{C}(3)$.

- Each ellipse = One parity assignment among eigenvalues.
- Wrong assignment $\Rightarrow$ No Toeplitz.
- Magnitude of eigenvalues $\longrightarrow$ Solvability.
- Ordered eigenvalues alternate in parity $\stackrel{?}{\Rightarrow}$ Safeguard.

A Newton Method in $\mathcal{C}(n)$

- A tangent step
- Lift by approximation
- Lift by global ordering
- Lift by local ordering

A Classical Newton Method

- A function:
$$f : R \longrightarrow R.$$
- The scheme:
$$x^{(\nu+1)} = x^{(\nu)} - (f'(x^{(\nu)}))^{-1} f(x^{(\nu)}).$$
- The intercept:
  - $\diamond$ $x^{(\nu+1)}$ = The x-intercept of the tangent line of the graph of f from $(x^{(\nu)}, f(x^{(\nu)}))$.
- The lifting:
  - $\diamond$ $(x^{(\nu+1)}, f(x^{(\nu+1)}))$ = The natural "lift" of the intercept along the y-axis to the the graph of f.

An Analogy of the Newton Method

- Think of
  - $\diamond$ $\mathcal{M}_{\mathcal{C}}(\Lambda)$ = The graph of f.
  - $\diamond$ $\mathcal{T}(n) := \{\text{Toeplitz matrices}\}$ = The x-axis.
  - $\diamond$ Limit the iteration to $\mathcal{C}(n)$.
- Manifold $\mathcal{M}_{\mathcal{C}}(\Lambda)$:
  - $\diamond$ Parametrization:

$$\begin{aligned} M &= Q\Lambda Q^T, \\ Q &= K^T Z, \\ Z &\in \mathcal{O}(\lfloor \frac{n}{2} \rfloor) \times \mathcal{O}(\lceil \frac{n}{2} \rceil). \end{aligned}$$

  - $\diamond$ Tangent vector:

$$\begin{aligned} T_M(\mathcal{M}_{\mathcal{C}}) &= \tilde{S}M - M\tilde{S}, \\ \tilde{S} &:= Q \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix} Q^T. \end{aligned}$$

    - $\triangleright$ S_1 = skew-symmetric in $R^{\lfloor \frac{n}{2} \rfloor \times \lfloor \frac{n}{2} \rfloor}$.
      S_2 = skew-symmetric in $R^{\lceil \frac{n}{2} \rceil \times \lceil \frac{n}{2} \rceil}$

A Tangent Step

- Given $M^{(\nu)} \in \mathcal{M}_{\mathcal{C}}(\Lambda)$ (*Parity fixed*),
  - ◇ Find $\tilde{S}^{(\nu)}$ and $r^{(\nu+1)}$ for

$$M^{(\nu)} + \tilde{S}^{(\nu)} M^{(\nu)} - M^{(\nu)} \tilde{S}^{(\nu)} = T(r^{(\nu+1)}).$$

- Equivalently:

$$\begin{aligned} \Lambda + S^{(\nu)}\Lambda - \Lambda S^{(\nu)} &= {Q^{(\nu)}}^T T(r^{(\nu+1)}) Q^{(\nu)} \\ &= {Z^{(\nu)}}^T K T(r^{(\nu+1)}) K^T Z^{(\nu)}. \end{aligned}$$

  - ◇ Spectral decomposition:

$$\begin{aligned} {Q^{(\nu)}}^T M^{(\nu)} Q^{(\nu)} &= \Lambda, \\ Q^{(\nu)} &= K^T Z^{(\nu)}. \end{aligned}$$

- Key observation:

$$K T(r^{(\nu+1)}) K^T = \begin{bmatrix} T_1^{(\nu+1)} & 0 \\ 0 & T_2^{(\nu+1)} \end{bmatrix}.$$

  $\Rightarrow$ The system is split in half.

Find the Intercept

- The right-hand side of the system is linear in $r^{(\nu+1)}$.
- Diagonal elements in the system $\Rightarrow$ A linear system for $r^{(\nu+1)}$ without reference to $S^{(\nu)}$:

$$J^{(\nu)} r^{(\nu+1)} = \lambda.$$

  - $\diamond$ $\lambda := [\phi_1, \ldots, \phi_{\lfloor \frac{n}{2} \rfloor}, \psi_1, \ldots \psi_{\lceil \frac{n}{2} \rceil}]^T$ (*Fixed parity*).
  - $\diamond$

$$J_{ij}^{(\nu)} := \begin{cases} (Z_1^{(\nu)})_{*i}^T E_1^{[j]} (Z_1^{(\nu)})_{*i}, & \text{if } 1 \le i \le \lfloor \frac{n}{2} \rfloor; \\ \\ (Z_2^{(\nu)})_{*i}^T E_2^{[j]} (Z_2^{(\nu)})_{*i}, & \text{if } \lfloor \frac{n}{2} \rfloor < i \le n. \end{cases}$$

    - $\triangleright$

$$\begin{bmatrix} E_1^{[j]} & 0 \\ 0 & E_2^{[j]} \end{bmatrix} = K T(e_j) K^T.$$

    - $\triangleright$ $(Z_k^{(\nu)})_{*i}$:= the i^{th} column of the matrix $Z_k^{(\nu)}$.

- Only length of $\approx \frac{n}{2}$ in all vector-matrix-vector multiplications.

Compute $S^{(\nu)}$

- Once $T(r^{(\nu+1)})$ is determined, off-diagonal elments in the system $\Rightarrow S^{(\nu)}$:

$$(S_1^{(\nu)})_{ij} = \frac{(Z_1^{(\nu)})_{*i}^T T_1^{(\nu+1)} (Z_1^{(\nu)})_{*j}}{\phi_i - \phi_j},\ 1 \le i < j \le \lfloor \frac{n}{2} \rfloor,$$

$$(S_2^{(\nu)})_{ij} = \frac{(Z_2^{(\nu)})_{*i}^T T_2^{(\nu+1)} (Z_2^{(\nu)})_{*j}}{\psi_i - \psi_j},\ 1 \le i < j \le \lceil \frac{n}{2} \rceil.$$

  - ◇ Eigenvalues within each parity group must be distinct.
  - ◇ $\lambda_1, \ldots, \lambda_n$ need not be totally distinct.

- Multiple eigenvalues $\Longrightarrow$ Basis of eigenspace splits as evenly as possible between symmetric and skew-symmetric eigenvectors. (Delsarte & Genin, '83)
- Multiplicity of each eigenvalue $\le 2 \Longrightarrow S^{(\nu)}$ can be formulated.

Find the Lift

- Coordinate-free lift (Friedland, '87; Chu, '92):

$$M^{(\nu+1)} := Q^{(\nu)} R^{(\nu)^T} Q^{(\nu)^T} M^{(\nu)} Q^{(\nu)} R^{(\nu)} Q^{(\nu)^T}.$$

  - ◇ Lift by approximation:

$$R^{(\nu)} := \left(I + \frac{S^{(\nu)}}{2}\right)\left(I - \frac{S^{(\nu)}}{2}\right)^{-1}.$$

- In calculation, only need

$$Z^{(\nu+1)} := Z^{(\nu)} R^{(\nu)^T}.$$

  - ◇ All matrices involved are 2-block diagonal.

- Quadratic convergence.
- Multiplicity $> 2 \Rightarrow$ No $S^{(\nu)} \Rightarrow$ No lift.
- *Can we by-pass $S^{(\nu)}$ to perform a lift?*

Lift by Global Ordering

- Idea:
  - ◇ Look for matrix $M^{(\nu+1)} \in \mathcal{M}_{\mathcal{C}}$ that is nearest to $T(r^{(\nu+1)})$.

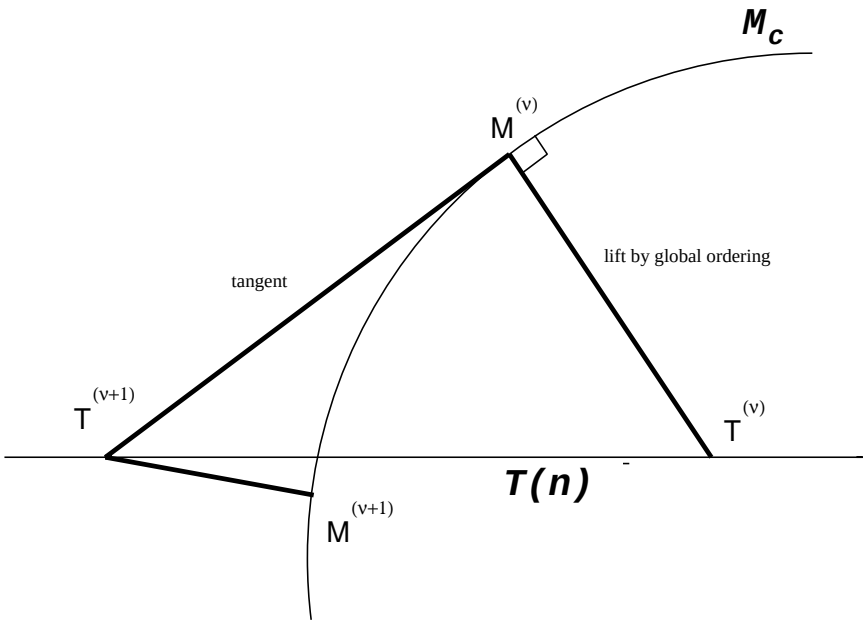

Figure 2: Geometry of Lift by Global Ordering.

- Answer: Wielandt-Hoffman theorem.
  - ◇ Spectral decomposition of $T(r^{(\nu+1)})$ is easy:

$$\overline{Z}^{(\nu+1)^T} K T(r^{(\nu+1)}) K^T \overline{Z}^{(\nu+1)} = \begin{bmatrix} \overline{\Lambda}_1^{(\nu+1)} & 0 \\ 0 & \overline{\Lambda}_2^{(\nu+1)} \end{bmatrix}.$$

  - ◇ Rearrange $\{\lambda_1, \ldots, \lambda_n\}$ in the same ordering as in $\overline{\Lambda}_1^{(\nu+1)}$ and $\overline{\Lambda}_2^{(\nu+1)}$ to obtain $\tilde{\Lambda}_1^{(\nu+1)}$ and $\tilde{\Lambda}_2^{(\nu+1)}$.
  - ◇ Define:

$$M^{(\nu+1)} := K^T \overline{Z}^{(\nu+1)} \begin{bmatrix} \tilde{\Lambda}_1^{(\nu+1)} & 0 \\ 0 & \tilde{\Lambda}_2^{(\nu+1)} \end{bmatrix} \overline{Z}^{(\nu+1)^T} K.$$

- New starting point:

$$\begin{aligned} \Lambda = \Lambda^{(\nu+1)} &:= \begin{bmatrix} \tilde{\Lambda}_1^{(\nu+1)} & 0 \\ 0 & \tilde{\Lambda}_2^{(\nu+1)} \end{bmatrix}, \\ Z^{(\nu+1)} &:= \overline{Z}^{(\nu+1)}. \end{aligned}$$

- Significance:
  - ◇ Parity assignment may be changed.
  - ◇ No $S^{(\nu)}$ is needed.
  - ◇ Multiple eigenvalues with same parity can be handled.

Lift by Local Ordering

- Would like to avoid computing $S^{(\nu)}$ as well as parity switching?
- Idea:
  - $\diamond$ Λ is kept fixed.
  - $\diamond$ Reorganize columns of $\overline{Z}_1^{(\nu+1)}$ and $\overline{Z}_2^{(\nu+1)}$.
- Calculation:
  - $\diamond$ Elements in $\overline{\Lambda}_1^{(\nu+1)}, \overline{\Lambda}_2^{(\nu+1)}$ are in the same ordering as those in Λ_1 and Λ_2.
- New starting point:

$$Z^{(\nu+1)} := \text{The reorganized } \overline{Z}^{(\nu+1)}.$$

- Global ordering = Local ordering, when reaching convergence.
- 1st order projection + 2nd order tangent step $\Longrightarrow$ Quadratic convergence. (Traub)

Numerical Experiment

- Example 1: Wrong parity
- Example 2: Quadratic convergence
- Example 3: Multiplicity = 2
- Example 4: Multiplicity = 3
- Example 5: High order case

Example 1: Wrong Parity

- Test data (*Wrong parity*):

$$\left.\begin{array}{rcl} \lambda_1 &=& -2.4128 \times 10^{+0}(E) \\ \lambda_2 &=& -2.6407 \times 10^{-1}(E) \\ \lambda_3 &=& \quad 2.6769 \times 10^{+0}(O) \end{array}\right\} \textit{Wrong parity}$$

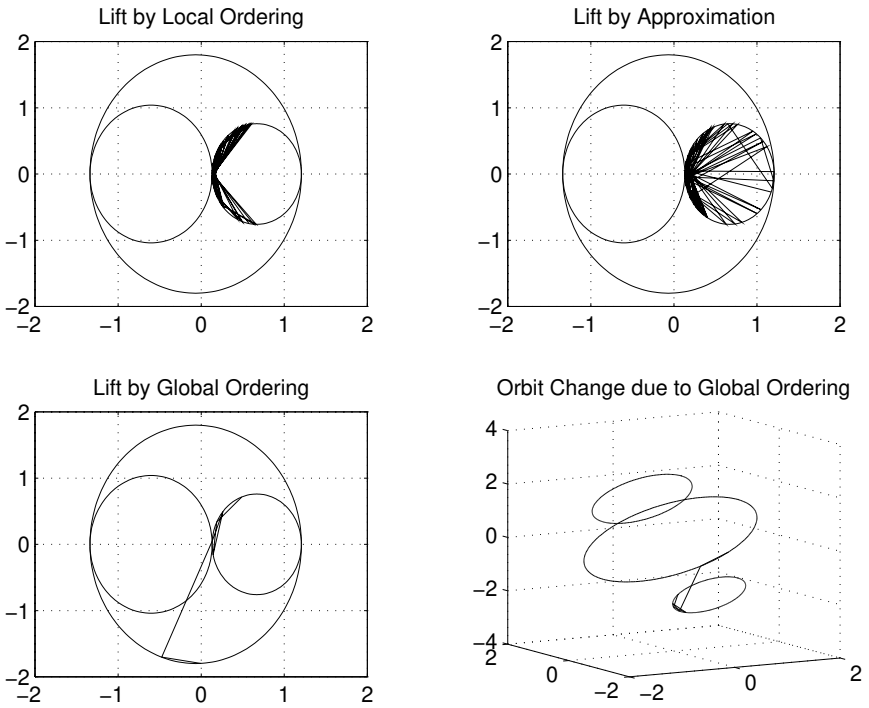

Figure 3: Behaviors of Algorithms When Starting with the Wrong Orbit.

- Lift by approximation $\Rightarrow$ Staying on the *wrong* orbit.
- Local ordering $\Rightarrow$ Wrong orbit, clustering.
- Global ordering $\Rightarrow$ Change orbit, convergence.

Example 2: Quadratic Convergence

- Limit point $r^{(*)}$ may be away from original $r^{(\#)}$, even if $r^{(0)} \approx r^{(\#)}$.
- Limit points may be different among methods, even with the same starting $r^{(0)}$.
- Eigenvalues of $T(r^{(*)}) =$ those of $T(r^{\#})$, but parity may change in the global ordering case.

Example 3: Multiplicity = 2

- Test data (Random number):

$$\begin{cases} -5.8942 \times 10^{-1} & (E) \\ -1.8565 \times 10^{-1} & (O) \\ -1.8565 \times 10^{-1} & (E) \\ 3.7508 \times 10^{-1} & (O) \\ 5.8564 \times 10^{-1} & (E) \end{cases}$$

- Parity unknown.
  - ◇ Assume the possibly safest assignment.
- Multiply eigenvalues split between parities.
- Quadratic convergence.

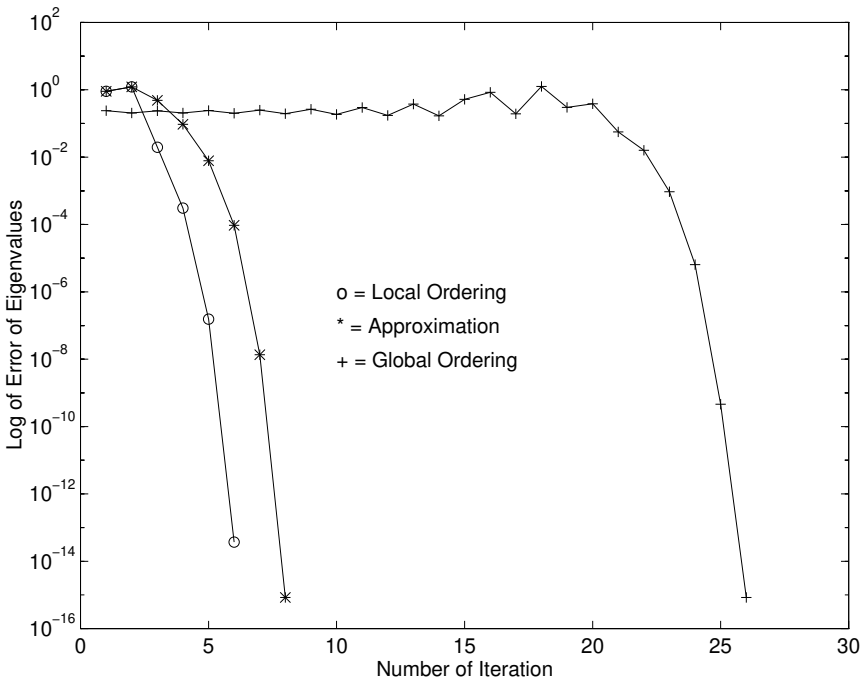

Figure 4: Number of Iteration versus Logarithmic Scale of Errors in Example 3.

Example 4: Multiplicity = 3

- Test data:

$$\begin{matrix} -8.4328 \times 10^{-1} & (E) \\ -1.2863 \times 10^{-1} & (O) \\ -1.2863 \times 10^{-1} & (E) \\ -1.2863 \times 10^{-1} & (O) \\ 1.2292 \times 10^{+0} & (E) \end{matrix}$$

- Lift by approximation fails.
- Methods by local and global ordering converge to

$$[.2204\times10^{-16}, 4.2222\times10^{-1}, 1.2863\times10^{-1}, 4.2222\times10^{-1}, 1.2863\times10^{-1}]$$

with error history

$$\begin{aligned} 2.0327 &\times 10^{+0} \\ 4.0355 &\times 10^{-2} \\ 1.3903 &\times 10^{-4} \\ 3.5477 &\times 10^{-9} \\ 7.8896 &\times 10^{-16}. \end{aligned}$$

Example 5: $n = 20$

- Test data:

$$\begin{array}{llll} -1.0242\times10^{+1} & -9.6736\times10^{+0} & -5.5608\times10^{+0} & -2.2651\times10^{+0} \\ 5.5692\times10^{-1} & 2.1786\times10^{+0} & 3.3867\times10^{+0} & 4.0016\times10^{+0} \\ 6.3594\times10^{+0} & 8.7090\times10^{+0} & & \end{array}$$

$$\begin{array}{llll} -1.0416\times10^{+1} & -9.4352\times10^{+0} & -4.7955\times10^{+0} & -7.7180\times10^{-1} \\ 6.3996\times10^{-1} & 2.6374\times10^{+0} & 4.4879\times10^{+0} & 4.7572\times10^{+0} \\ 6.2222\times10^{+0} & 9.2230\times10^{+0} & & \end{array}$$

  - ◇ Not the safest possible parity assignment, first ten odd, last ten even.

- Method of approximation fails after 100 iterations.
- Method of global ordering performs best.

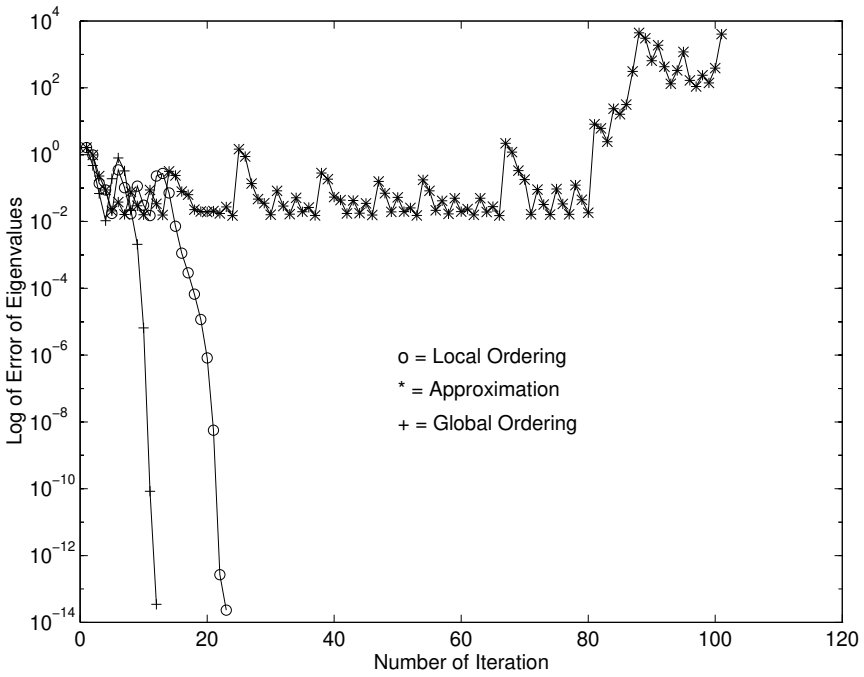

Figure 5: Number of Iteration versus Logarithmic Scale of Errors in Example 5.

Conclusion

- Solving the ToIEP within the subspace $\mathcal{C}(n)$ is possible.
  - ◇ Problem size and cost are halved.
  - ◇ Multiple eigenvalue case can be handled.
- Coordinate-free Newton-like methods are available.
  - ◇ Quadratic convergence is observed.
- Parity assignment of eigenvalues plays an important role in whether an ToIEP is solvable.
- Local ordering and global ordering, based on the Wielandt-Hoffman theorem, permit a new way of lifting.
  - ◇ Higher multiplicity eigenvalue case can now be handled.

	Case (a)	Case (b)	Case (c)
$r^{(\#)}$	0	0	0
	-2.0413×10^{-3}	-9.2349×10^{-1}	-3.3671×10^{-1}
	$1.6065\times10^{+0}$	-7.0499×10^{-2}	4.1523×10^{-1}
Original Value	8.4765×10^{-1}	1.4789×10^{-1}	$1.5578\times10^{+0}$
	2.6810×10^{-1}	-5.5709×10^{-1}	$-2.4443\times10^{+0}$
$r^{(0)}$	0	0	0
	-2.8351×10^{-1}	$-1.8024\times10^{+0}$	6.3658×10^{-1}
	9.3953×10^{-1}	7.3881×10^{-1}	4.0318×10^{-1}
Initial Value	8.2068×10^{-1}	1.5694×10^{-1}	$1.0901\times10^{+0}$
	$1.0634\times10^{+0}$	-5.2451×10^{-1}	$-3.2628\times10^{+0}$
$r^{(*)}$	2.0426×10^{-16}	2.2204×10^{-16}	7.4940×10^{-16}
	-2.0413×10^{-3}	-9.2349×10^{-1}	-3.5391×10^{-1}
	$1.6065\times10^{+0}$	-7.0499×10^{-2}	4.3645×10^{-1}
Local Ordering	8.4765×10^{-1}	1.4789×10^{-1}	$1.5244\times10^{+0}$
	2.6810×10^{-1}	-5.5709×10^{-1}	$-2.4655\times10^{+0}$
$r^{(*)}$	8.6831×10^{-16}	0	4.7184×10^{-16}
	-2.0413×10^{-3}	-9.2349×10^{-1}	-3.3671×10^{-1}
	$1.6065\times10^{+0}$	-7.0499×10^{-2}	4.1523×10^{-1}
Approximation	8.4765×10^{-1}	1.4789×10^{-1}	$1.5578\times10^{+0}$
	2.6810×10^{-1}	-5.5709×10^{-1}	$-2.4443\times10^{+0}$
$r^{(*)}$	2.4113×10^{-16}	-1.1102×10^{-16}	6.1062×10^{-16}
	-9.3778×10^{-2}	-9.2646×10^{-1}	3.5391×10^{-1}
	$1.5174\times10^{+0}$	-6.1419×10^{-2}	4.3645×10^{-1}
Global Ordering	9.9597×10^{-1}	1.3518×10^{-1}	$-1.5244\times10^{+0}$
	5.7042×10^{-1}	-5.4694×10^{-1}	$-2.4655\times10^{+0}$

Table 1: Initial and Final Values of $r^{(\nu)}$ for Example 2.

Iterations	Local Ordering	Approximation	Global Ordering
0	$1.3847\times10^{+0}$	$1.3847\times10^{+0}$	$1.2194\times10^{+0}$
1	7.1545×10^{-1}	7.1545×10^{-1}	4.2739×10^{-1}
2	2.1982×10^{-2}	6.3866×10^{-2}	1.4179×10^{-2}
3	5.1223×10^{-5}	2.0606×10^{-4}	4.3624×10^{-5}
4	4.4931×10^{-10}	7.1037×10^{-9}	4.7985×10^{-10}
5	1.4729×10^{-15}	2.9671×10^{-15}	1.7659×10^{-15}

Table 2: Errors of Eigenvalues for Case (a) in Example 2.