Chapter 1

Introduction

- Overview
- A Brief History
- Classification
- A Glimpse of Some Known Results
- Conclusion

Overview

- Often a physical process is described by a mathematical model of which parameters represent important physical quantities.
  - ◇ Direct analysis — Analyze or predict the behavior of the underlying physical process from the parameters.
  - ◇ Inverse analysis — Validate, determine, or estimate the parameters adaptively from the behavior of the physical process.

Inverse Eigenvalue Problem (IEP)

- The mathematical model involves matrices whose spectral properties determine the dynamics of the physical system.
- Reconstruct a matrix from prescribed spectral data.
  - ◇ Spectral data may involve a mixture information of eigenvalues or eigenvectors.
  - ◇ Sometimes complete information is difficult to obtain. Only partial information is available.
  - ◇ For feasibility, often necessary to restrict the construction to special classes of matrices.

Fundamental Questions

- Solvability:
  - ◇ Determine a necessary or a sufficient condition under which an IEP has a solution.
- Computability:
  - ◇ Develop a scheme through which, knowing a priori that the given spectral data are feasible, a matrix can be constructed numerically.
- Sensitivity:
  - ◇ Quantify how a solution to an IEP is subject to changes of the spectral data.
- Applicability:
  - ◇ Differentiate whether the given data are exact or approximate, complete or incomplete, and whether only an estimation of the parameters of the system is sufficient.
  - ◇ Decide between physical realizability and physical uncertainty which constraint of the problem should be enforced.

Brief History

- Studies of IEP's have been quite extensive
  - ◇ Engineering application.
  - ◇ Algebraic theorization.
- Mathematical techniques employed in the study are quite sophisticated:
  - ◇ Algebraic curves.
  - ◇ Degree theory.
  - ◇ Differential geometry.
  - ◇ Matrix theory.
  - ◇ Differential equations.
  - ◇ Functional analysis.
  - ◇ ⋮
- Results are quite few and scattered even within the same field of discipline.

Literature Review

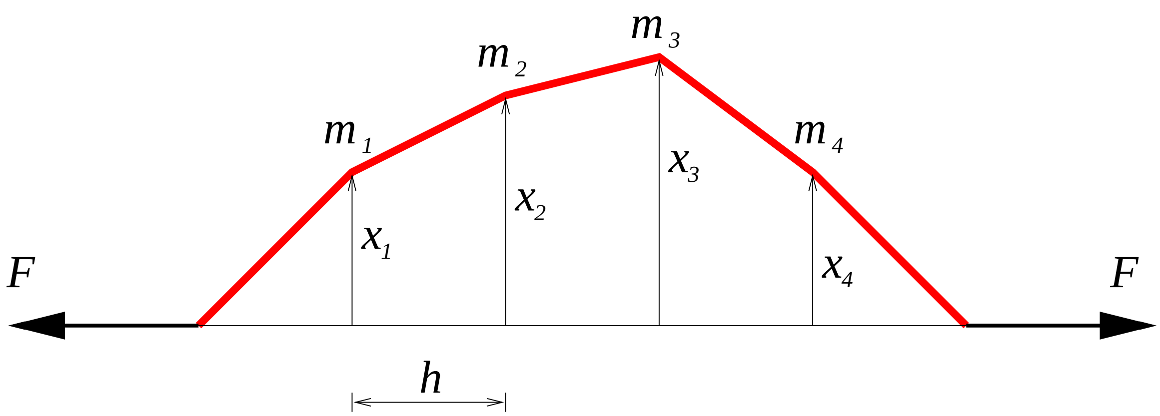
- Inverse Sturm-Liouville problem:
  - ◇ Ambartsumyan'29
  - ◇ Krein'33
  - ◇ Borg'46, Levinson'49
  - ◇ **Gel'fand&Levitan'51**
  - ◇ Kac'66 (Can one hear the shape of a drum?)
  - ◇ Hochstadt'73, Barcilon'74, McLaughlin'76, Hald'78
  - ◇ Zayed'82, Issacson et al'83, **McLaughlin'86**, Andersson'88
  - ◇ Lowe et al'95, **Rundell'97**
- Matrix theory:
  - ◇ Downing&Householder'56, Mirsky'58
  - ◇ Hochstadt'67
  - ◇ de Oliveira'70, Hald'72, Golub'73, Friedland'77, de Boor&Golub'78
  - ◇ Biegler-König'81, Shapiro'83, Barcilon'86, Sun'86, **Boley&Golub'87**
  - ◇ Landau'94, **Chu'98**

- Applied Mechanics:
  - ◇ Barcilon'74
  - ◇ Gottlieb'83, Gladwell'86
  - ◇ Ram'91, **Gladwell'96**, Nylen&Uhlig'97
- Computation:
  - ◇ Morel'76, Boley&Golub'77
  - ◇ Nocedal et al'83, Friedland et al'88, Laurie'88
  - ◇ Chu'90, Zhou&Dai'91, Trench'97, **Xu'98**

Applications

- System identification and control theory.
  - ◇ State/output feedback pole assignment problems.
- Applied mechanics and structure design.
  - ◇ Construct a model of a (damped) mass-spring system with prescribed natural frequencies/modes.
- Applied physics.
  - ◇ Compute the electronic structure of an atom from measured energy levels.
  - ◇ Neutron transport theory.
- Numerical analysis.
  - ◇ Preconditioning.
  - ◇ Computing B-stable RK methods with real poles.
  - ◇ Gaussian quadratures.
- Mathematical analysis.
  - ◇ Inverse Sturm-Liouville problems.

An Example



- Vibration of equally spaced particles (with spacing h and mass m_i) on a string subject to a constant horizontal tension F.
- Equation of motion for 4 particles:

$$
\begin{aligned}
m_1 \frac{d^2 x_1}{dt^2} &= -F\frac{x_1}{h} + F\frac{x_2 - x_1}{h} \\
m_2 \frac{d^2 x_2}{dt^2} &= -F\frac{x_2 - x_1}{h} + F\frac{x_3 - x_2}{h} \\
m_3 \frac{d^2 x_3}{dt^2} &= -F\frac{x_3 - x_2}{h} + F\frac{x_4 - x_3}{h} \\
m_4 \frac{d^2 x_4}{dt^2} &= -F\frac{x_4 - x_3}{h} - F\frac{x_4}{h}
\end{aligned}
$$

- In matrix form:

$$\frac{d^2\mathbf{x}}{dt^2} = -DA\mathbf{x}$$

  ◇ $\mathbf{x} = [x_1, x_2, x_3, x_4]^T$

  ◇ $A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$

  ◇ $D = \text{diag}(d_1, d_2, d_3, d_4)$ with $d_i = \frac{F}{m_i h}$.

- Eigenvalues of DA are the squares of the so called *natural frequencies* of the system.

- Want to place weights m_i so that the system has a prescribed set of natural frequencies.

  ◇ A is symmetric and tridiagonal.

  ◇ D is diagonal.

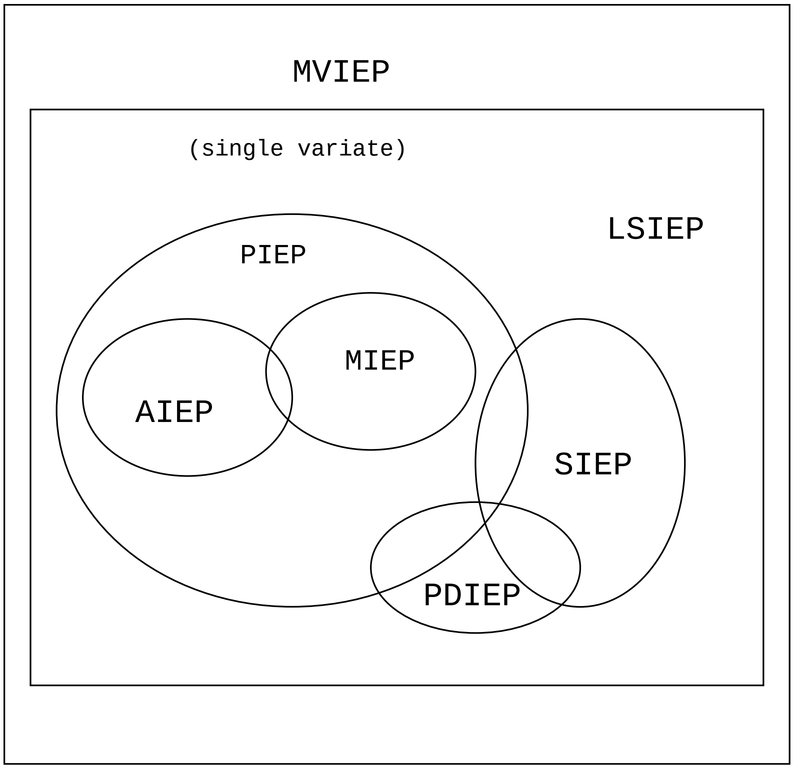
  ◇ This is a multiplicative inverse eigenvalue problem.

- Open Question: Can such a string have arbitrarily prescribed natrual frequencies by adjusting the diagonal matrix D?

Classification

- Based on constraint.
  - ◇ Spectral constraint.
  - ◇ Structure constraint.
- Based on physical suitability.
  - ◇ Physical realizability.
  - ◇ Physical uncertainty.
- Based on discipline.
  - ◇ Essentially mathematical problem.
  - ◇ Essentially engineering problem.
- Based on expectation.
  - ◇ Determination problem.
  - ◇ Estimation problem.

Via Algebraic Characteristics



MVIEP = Multi-Variate IEP
LSIEP = Least Squares IEP
PIEP = Parameterized IEP
SIEP = Structured IEP
PDIEP = Partially Described IEP
AIEP = Additive IEP
MIEP = Multiplicative IEP

PIEP

- Generic form:
  - ◇ Given
    - ▷ A *family* of matrices $A(c) \in \mathcal{M}$ with $c \in \mathbf{F}^m$,
    - ▷ A set of scalars $\Omega \subset \mathbf{F}$,
  - ◇ Find
    - ▷ Values of parameter c such that

$$\sigma(A(c)) \subset \Omega$$

- Remarks:
  - ◇ Not necessarily $m = n$.
  - ◇ Commonly used Ω:
    - ▷ $\Omega = \{\lambda_k^*\}_{k=1}^n$.
    - ▷ $\Omega =$ left-half complex plan.
    - ▷ $\Omega =$ anything but must have a specific number of zeros.

Some Special PIEP's

- $A(c) = A_0 + \sum_{i=1}^{n} c_i A_i$
  - $\diamond$ $A_i \in \mathcal{R}(n)$, $\mathbf{F} = \mathbb{R}$.
  - $\diamond$ $A_i \in \mathcal{S}(n)$, $\mathbf{F} = \mathbb{R}$.
- **(AIEP)** $A(c) = A(X) = A_0 + X$, $X \in \mathcal{N}$.
  - $\diamond$ $A_0 \in \mathcal{C}(n)$, $\mathbf{F} = \mathbb{C}$, $\mathcal{N} = \mathcal{D}_{\mathcal{C}}(n)$.
- **(MIEP)** $A(c) = A(X) = XA_0$, $X \in \mathcal{N}$.
  - $\diamond$ Preconditioning?
- $A(c) = A(K_1, \ldots, K_q) = A_0 + \sum_{i=1}^{q} B_i K_i C_i$.
  - $\diamond$ Pole assignment problem.

SIEP

- Generic form:
  - ◇ Given
    - ▷ A set $\mathcal{N}$ of specially structured matrices,
    - ▷ A set of scalars $\{\lambda_k^*\}_{k=1}^n \in \mathbf{F}$,
  - ◇ Find
    - ▷ $X \in \mathcal{N}$ such that

$$\sigma(X) = \{\lambda_k^*\}_{k=1}^n.$$

- Some special cases:
  - ◇ $\mathcal{N} = \{$Toeplitz matrices in $\mathcal{S}(n)\}$.
  - ◇ $\mathcal{N} = \{$Persymmetric Jacobi matrices in $\mathcal{S}(n)\}$.
  - ◇ $\mathcal{N} = \{$Nonnegative matrices in $\mathcal{S}(n)\}$.
  - ◇ $\mathcal{N} = \{$Row-stochastic matrices in $\mathcal{R}(n)\}$.

A Few More Special SIEP's

- Given scalars $\lambda_i^* \leq \mu_i \leq \lambda_{i+1}^*$, $i = 1, \ldots, n-1$, find a Jacobi matrix J such that

$$\begin{aligned} \sigma(J) &= \{\lambda_k^*\}_{k=1}^n \\ \sigma(J(1:n-1, 1:n-1)) &= \{\mu_1, \ldots, \mu_{n-1}\}. \end{aligned}$$

- Given scalars $\{\lambda_1, \ldots, \lambda_{2n}\}$ and $\{\mu_1, \ldots, \mu_{2n-2}\} \in \mathbb{C}$, find tridiagonal symmetric matrices C and K for the λ-matrix $Q(\lambda) = \lambda^2 I + \lambda C + K$ so that

$$\begin{aligned} \sigma(Q) &= \{\lambda_1, \ldots, \lambda_{2n}\}, \\ \sigma(Q(1:n-1, 1:n-1)) &= \{\mu_1, \ldots, \mu_{2n-2}\}. \end{aligned}$$

- Given distinct scalars $\{\lambda_1, \ldots, \lambda_{2n}\} \subset \mathbb{R}$ and a Jacobi matrix $J_n \in \mathcal{R}(n)$, find a Jacobi matrix a Jacobi matrix $J_{2n} \in \mathcal{R}(2n)$ so that

$$\begin{aligned} \sigma(J_{2n}) &= \{\lambda_1, \ldots, \lambda_{2n}\}, \\ J_{2n}(1:n, 1:n) &= J_n. \end{aligned}$$

- Given a family of matrices $A(c) \in \mathbb{R}^{m \times n}$, with $c \in \mathbb{R}^n$, $m \geq n$, find a parameter c such that the singular values of $A(c)$ are precisely the same as a prescribed set of nonnegative real values $\{\sigma_1, \ldots, \sigma_n\}$.

LSIEP

- Maintain the structure, approximate the eigenvalues:
  - ◇ Given
    - ▷ A set of scalars $\{\lambda_1^*, \ldots, \lambda_m^*\} \subset \mathbf{F}$ $(m \leq n)$,
    - ▷ A set $\mathcal{N}$ of specially structured matrices,
  - ◇ Find
    - ▷ A matrix $X \in \mathcal{N}$
    - ▷ An index subset $\sigma = \{\sigma_1 < \ldots < \sigma_m\}$ such that

$$F(X, \sigma) := \frac{1}{2} \sum_{i=1}^{m} (\lambda_{\sigma_i}(X) - \lambda_i^*)^2,$$

is minimized.

- Maintain the spectrum, approximate the structure:
  - ◇ Given
    - ▷ A set $\mathcal{M}$ of spectrally constrained matrices,
    - ▷ A set $\mathcal{N}$ of specially structured matrices,
    - ▷ A projection P from $\mathcal{M}$ onto $\mathcal{N}$,
  - ◇ Find
    - ▷ $X \in \mathcal{M}$ that minimizes

$$F(X) := \frac{1}{2}\|X - P(X)\|^2.$$

Via Physical Characteristics

- By mechanical types:
  - ◇ Continuous vs. discrete.
  - ◇ Damped vs. undamped.
- By data type:
  - ◇ Spectral, modal, or nodal.
  - ◇ Complete vs incomplete.

A Glimpse of Some Major Issues

- Studies on IEP's have been intensive, ranging from acquiring a pragmatic solution to a real-world application dealing the metaphysical theory of an abstract fromulation.
- Results are scattered even within the same field of discipline.
- Only a handful of the problems have been completely understood.
- Many interesting yet challenging questions remain to be answered.

Complex Solvability

- Solving an IEP over complex field amounts to solving a polynomial system with complex coefficients. Generally speaking, the system is generically solvable.
- Given $A_0 \in \mathcal{C}(n)$ and arbitrary $\{\lambda_k^*\}_{k=1}^n \subset \mathbb{C}$,
  - ◇ There exists $D \in \mathcal{D}_{\mathcal{C}}(n)$ such that
  $$\sigma(A_0 + D) = \{\lambda_k^*\}_{k=1}^n$$
  and there are at most $n!$ solutions.
  - ◇ If $\det(A_0(1:j, 1:j) \neq 0$, $j = 1, \ldots, n$, then there exists $D \in \mathcal{D}_{\mathcal{C}}(n)$ such that
  $$\sigma(DA_0) = \{\lambda_k^*\}_{k=1}^n$$
  and there are at most $n!$ solutions.

Real Solvability

- Solving an IEP over real field is a much harder problem. Sufficient conditions are generally quite restrictive.
- Assume all matrices involved are real,
  - ⋄ If the prescribed *real* eigenvalues are sufficiently different, then there exist $c_1, \ldots, c_n \in \mathbb{R}$ such that

$$\sigma(A_0 + \sum_{i=1}^{n} c_i A_i) = \{\lambda_k^*\}_{k=1}^{n}.$$

  - ⋄ The inverse eigenvalue problem associated with

$$A_0 + \sum_{i=1}^{n} c_i A_i$$

    is unsolvable almost everywhere if and only if any of the prescribed eigenvalues has multiplicity great than 1.
- Symmetric Toeplitz matrices can have arbitrary spectra.

Numerical Methods

- Direct methods
  - ◇ Lanczos method.
  - ◇ Orthogonal reduction methods.
- Iterative methods
  - ◇ Newton-type iteration.
- Continuous methods:
  - ◇ Homotopy approach.
  - ◇ Projected gradient method.
  - ◇ ASVD approach.

Sensitivity Analysis

- Assume all matrices are symmetric and the PIEP for
$$A(c) = A_0 + \sum_{i=1}^{n} c_i A_i$$
is solvable.
- Assume $A(c) = Q(c)\mathrm{diag}\{\lambda_k^*\}_{k=1}^n Q(c)^T$ and define
$$\begin{aligned} J(c) &= \left[q_i(c)^T A_j q_i(c)\right], \quad i, j = 1, \ldots, n, \\ b &= \left[q_1(c)^T A_0 q_1(c), \ldots, q_n^T A_0 q_n(c)\right]^T. \end{aligned}$$
- If
$$\delta = \|\lambda - \tilde{\lambda}\|_\infty + \sum_{i=0}^{n} \|A_i - \tilde{A}_i\|_2$$
is sufficiently small, then
  - ⋄ The PIEP associated with $\tilde{A}_i$, $i = 0, \ldots, n$ and $\{\tilde{\lambda}_1, \ldots, \tilde{\lambda}_n\}$ is solvable.
  - ⋄ There is a solution $\tilde{c}$ near to c,

$$\frac{\|c - \tilde{c}\|_\infty}{\|c\|_\infty} \leq \kappa_\infty(J(c)) \left(\frac{\|\lambda - \tilde{\lambda}\|_\infty + \|A_0 - \tilde{A}_0\|_2}{\|\lambda - b\|_\infty} + \frac{\sum_{i=1}^n \|A_i - \tilde{A}_i\|_2}{\|J(c)\|_\infty} \right) + O(\delta^2).$$

Summary

- An IEP concerns the reconstruction of a matrix satisfying two constraints.
  - *Spectral constraint* – the prescribed spectral data.
  - *Structural constraint* – the desirable structure.
- Different constraints define a variety of IEP's.
- Studies on IEP's have been intensive, ranging from engineering application to algebraic theorization.
  - Many unanswered yet interesting questions.
- A common phenomenon in all applications is that the physical parameters of a certain system are to be reconstructed from knowledge of its dynamical behavior, in particular, of its natural frequencies/modes.
  - Sometimes the constraints can be precisely determined.
  - Sometimes the constraints are only approximate and often incomplete.