Chapter 2

Applications

- Pole Assignment Problem
- Control of Vibration
- Inverse Strum-Liouville Problem
- Geophysics Application
- Numerical Analysis
- Low Rank Application

Pole Assignment Problem

- Dynamic state equation:

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t).$$

  - ◇ $\mathbf{x}$ = state of the system $\in \mathbb{R}^n$.
  - ◇ $\mathbf{u}$ = input to the system $\in \mathbb{R}^m$.
  - ◇ $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$.

- Want to select $\mathbf{u}(t)$ so as to control the dynamics of $\mathbf{x}(t)$.
  - ◇ Classical problem in control theory.
  - ◇ Extensively studied and very rich in literature.
  - ◇ Several different types.

State Feedback Control

- Choose input $\mathbf{u}$ as a linear function of current state $\mathbf{x}$,

$$\mathbf{u}(t) = F\mathbf{x}(t).$$

- Closed-loop dynamical system:

$$\dot{\mathbf{x}}(t) = (A + BF)\mathbf{x}(t).$$

- Want to choose the *gain matrix* F so as to
  - ◇ Achieve stability.
  - ◇ Speed up response.
- Choose F so as to reassign eigenvalues of $A + BF$.
  - ◇ Usually F carries no structure at all.
  - ◇ It becomes a much harder IEP if F needs to satisfy a certain structural constraint.

Output Feedback Control

- Often $\mathbf{x}(t)$ is not directly observable. Instead, only output $\mathbf{y}(t)$ where

$$\mathbf{y}(t) = C\mathbf{x}(t)$$

  is available.

- Choose input $\mathbf{u}$ as a linear function of current output $\mathbf{y}$,

$$\mathbf{u}(t) = K\mathbf{y}(t).$$

- Closed-loop dynamical system:

$$\dot{\mathbf{x}}(t) = (A + BKC)\mathbf{x}(t).$$

- Want to choose the *output matrix* K so as to reassign the eigenvalues of $A + BKC$.

Control of Vibration

- Area of applications:
  - ◇ Transverse vibrations of masses on a string.
  - ◇ Buckling of structures.
  - ◇ Transient current of electric circuits.
  - ◇ Acoustic vibration in a tube.
- Equation of motion:

$$M\ddot{\mathbf{x}} + C\dot{\mathbf{x}} + K\mathbf{x} = f(\mathbf{x}).$$

  - ◇ $\mathbf{x} \in \mathbb{R}^n$, M, C, $K \in \mathbb{R}^{n\times n}$.
  - ◇ M = diagonal, C, K = symmetric tridiagonal.
- Motion is governed by the homogeneous equation.
  - ◇ Try a solution $\mathbf{x}(t) = \mathbf{v}e^{\lambda t}$.
  - ◇ $\mathbf{v}$ and λ are solutions to the quadratic eigenvalue problem

$$(\lambda^2 M + \lambda C + K)\mathbf{v} = 0.$$

  - ◇ General solution:

$$\mathbf{x}(t) = \sum_{k=1}^{2n} \alpha_k \mathbf{v}_k e^{\lambda_k t}.$$

Undamped System

- $C = 0$, M, K = symmetric and positive definite.
- Try a solution of the form $\mathbf{x}(t) = \mathbf{v}e^{i\omega t}$
- $\mathbf{v}$ and ω solves the generalized eigenvalue problem

$$(K - \omega^2 M)\mathbf{v} = 0$$

  - ◇ ω = the natural frequency.
  - ◇ $\mathbf{v}$ = the natural mode.
- Let $\lambda = \omega^2$, $J := M^{-1/2}KM^{-1/2}$, and $\mathbf{z} = M^{1/2}\mathbf{x}$. Solve the Jacobi eigenvalue problem

$$J\mathbf{z} = \lambda\mathbf{z}.$$

- Two types of inverse eigenvalue problems:
  - ◇ Stiffness matrix K usually is more complicated than the mass matrix M.
    - ▷ Determine K from static constraints, but find M so that some desired natural frequencies are achieved.
    - ▷ This is equivalent to an multiplicative inverse eigenvalue problem.
  - ◇ Construct an unreduced, symmetric, and tridiagonal matrix J from its n eigenvalues and those of its leading principal submatrix of dimension $n-1$.
    - ▷ This IEP can be identified as configuring a spring system from its spectrum and from the spectrum of the same system but the last mass is fixed to have no motion.
    - ▷ This is one kind of Jacobi inverse eigenvalue problems.

Damped System

- Normalize M to identity.
- Define
  - ◇ $Q(\lambda) = \lambda^2 I + \lambda C + K$.
  - ◇ $\tilde{Q}(\lambda)$ = The leading principal submatrix of $Q(\lambda)$ of dimension $n-1$.
- Given scalars
  - ◇ $\{\lambda_1, \ldots, \lambda_{2n}\}$, and
  - ◇ $\{\mu_1, \ldots, \mu_{2n-2}\} \in \mathbb{C}$,
- Find
  - ◇ tridiagonal symmetric matrices C and K, or
  - ◇ real-valued, tridiagonal, symmetric, and weakly diagonally dominant matrices C and K with positive diagonal and negative off-diagonal elements,
- Such that
  - ◇ $\det(Q(\lambda))$ has zeros precisely $\{\lambda_1, \ldots, \lambda_{2n}\}$, and
  - ◇ $\det(\tilde{Q}(\lambda))$ has zeros precisely $\{\mu_1, \ldots, \mu_{2n-2}\}$.

Inverse Sturm-Liouville Problem

- The classical Sturm-Liouville problem:

$$\begin{aligned} \mathcal{L}[u] := -u_n''(x) + p(x)u_n(x) &= \lambda_n u_n(x),\ 0 < x < 1 \\ u_n'(0) - hu_n(0) &= 0 \\ u_n'(1) + Hu_n(1) &= 0. \end{aligned}$$

  - ◇ Eigenvalues of $\mathcal{L}$ are real, simple, countable, and tend to infinity.
  - ◇ Increasing q, h, or H increases all eigenvalues of $\mathcal{L}$.

- Can the function $p(x)$ be determined from eigenvalues? (Two data sequences are required [150].)

Matrix Analogue

- Discretize the BVP by the central difference scheme with mesh $h = \frac{1}{n+1}$,

$$\left(\frac{1}{h^2} \begin{bmatrix} 2 & -1 & 0 & & & \\ -1 & 2 & -1 & & & \\ 0 & -1 & 2 & \dots & & 0 \\ \vdots & & & \ddots & & \\ 0 & & & & 2 & -1 \\ 0 & & & & -1 & 2 \end{bmatrix} + X \right) \mathbf{u} = \lambda \mathbf{u}.$$

  - ◇ X = diagonal matrix representing the discretization of $p(x)$.

- Determine X so that the system processes a prescribed spectrum.

  - ◇ This is an additive inverse eigenvalue problem.

- Caution: There is a significant difference in asymptotic behavior between the discrete and continuous cases.

Applied Physics

- Quantum Mechanics
- Geophysics
- Neutron Transport Theory

Quantum Mechanics

- Computing the electronic structure of an atom requires the spectral and diagonal information of a Hamiltonian matrix H.
- Diagonal elements of H cannot be measured accurately.
- Eigenvalues of H correspond to energy levels of an atom that can be measured to a high degree of accuracy.
- Want to use eigenvalues to correct diagonal elements.
- A least squares IEP [110]:
  - ◇ Given
    - ▷ A real symmetric matrix A,
    - ▷ A set of real eigenvalues $\omega = [\omega_1, \ldots, \omega_n]^T$,
  - ◇ Find a real diagonal matrix D such that

$$\|\sigma(A + D) - \omega\|_2$$

    is minimized.

Geophysics

- Assuming spherical symmetry, want to infer the internal structure of the Earth from the frequencies of spheroidal and torsional modes of oscillations.
- The model involves the generalized Sturm-Liouville problem, i.e.,

$$u^{(2k)} - (p_1 u^{(k-1)})^{(k-1)} + \ldots + (-1)^k p_k u = \lambda u.$$

  - ◇ For well-posedness, $k+1$ spectra associated with $k+1$ distinct sets of boundary conditions are required to construct the unknown coefficients $p_1, \ldots, p_k$ [14, 15].
  - ◇ Theoretical solution can be constructed iteratively for the cases $k = 1, 2$.
  - ◇ Open Question: What is the matrix analogue of this high order problem and how to solve it numerically?

Neutron Transport Theory

- Dynamics in an additive neural network

$$\frac{du_i}{dt} = -a_i u_i + \sum_{j=1}^{n} \omega_{ij} g_j(u_j) + p_i, \quad , i = 1, \ldots n.$$

  - ◇ ω_{ij} = connection coefficient between the ith and jth neurons.
  - ◇ $g_j' > 0$ and g_j is bounded.

- Want to choose W so that a designated point $\mathbf{u}^*$ is a stable equilibrium.
  - ◇ At the critical point, there is a linear constraint

$$-A\mathbf{u}^* + W\mathbf{g}(\mathbf{u}^*) + \mathbf{p} = 0.$$

  - ◇ The eigenvalues of the Jacobian matrix should be in the left half plane.

- An equality constrained IEP [235]:
  - ◇ Given
    - ▷ Two sets of real vectors $\{\mathbf{x}_i\}_{i=1}^p$ and $\{\mathbf{y}_i\}_{i=1}^p$ with $p \leq n$, and
    - ▷ A set of complex numbers $\mathcal{L} = \{\lambda_1, \ldots, \lambda_n\}$, closed in conjugation,
  - ◇ Find a real matrix A such that

$$\begin{aligned} A\mathbf{x}_i &= \mathbf{y}_i, \\ \sigma(A) &= \mathcal{L} \end{aligned}$$

Numerical Analysis

- Preconditioning
- High Order Stable Runge-Kutta Schemes
- Gauss Quadratures

Preconditioning

- Preconditioning the equation $Ax = b$ is a means of transforming the original system into one that has the same solution, but is easier (quicker) to solve with an iterative scheme.
  - ◇ Preconditioning A can be thought of as implicitly multiplying A by M^{-1}.
    - ▷ M is a matrix for which $Mz = y$ can easily be solved, and
    - ▷ $M^{-1}A$ is not too far from normal and its eigenvalues are clustered.
  - ◇ Many types of unstructured preconditioners have been proposed:
    - ▷ Low-order (Coarse-grid) approximation, SOR, incomplete LU factorization, polynomial, and so on.
    - ▷ Open Question: Given a structure of M, what is the best achievable conditioning [166]?

- Precondition by low rank matrices might have applications in practical optimization.
  - ◇ Open Question: Given a matrix matrix $C \in \mathbb{R}^{m\times n}$ and a constant vector $\mathbf{b} \in \mathbb{R}^m$, find a vector $\mathbf{x} \in \mathbb{R}^n$ such that the rank-one updated matrix $\mathbf{b}\mathbf{x}^T + C$ has a prescribed set of singular values.

High Order Stable Runge-Kutta Schemes

- An s-stage Runge-Kutta method is uniquely determined by the Butcher array

$$\begin{array}{c|cccc} c_1 & a_{11} & a_{12} & \dots & a_{1s} \\ c_2 & a_{21} & a_{22} & \dots & a_{2s} \\ \vdots & \vdots & & & \vdots \\ c_s & a_{s1} & a_{s2} & \dots & a_{ss} \\ \hline & b_1 & b_2 & \dots & b_s \end{array}$$

  ◇ The stability function is given by

$$R(z) = 1 + z\mathbf{b}^T(I - zA)^{-1}\mathbf{1}.$$

- To attain stability, implicit methods are preferred.

  ◇ Fully implicit methods are too expensive.

  ◇ Diagonally implicit methods (DIRK), i.e., A is low triangular with *identical* diagonal entries, is computationally more efficient, but is difficult to construct.

  ◇ Singly implicit methods (SIRK) requires that the matrix A, though not lower triangular, should have an s-fold eigenvalue.

- An IEP with prescribed entries [269]:
  - ◇ Given
    - ▷ The number s of stages,
    - ▷ The the desired order p,
    - ▷ Define $k = \lfloor (p-1)/2 \rfloor$,
    - ▷ Constants $\xi_j = 0.5(4J^2 - 1)^{-1/2}$, $j = 1, \ldots, k$,
  - ◇ Find a real number λ and $Q \in \mathbb{R}^{(s-k)\times(s-k)}$ such that
    - ▷ $Q + Q^T$ is positive semi-definite.
    - ▷ $\sigma(X) = \{\lambda\}$ where $X \in \mathbb{R}^{s\times s}$ is of the form

$$X = \left[\begin{array}{cccc|c} 1/2 & -\xi_1 & & & \\ \xi_1 & 0 & & & \\ 0 & & & & \\ \vdots & & \ddots & & \\ & & & 0 & -\xi_k \\ \hline 0 & & & \xi_k & Q \end{array}\right]$$

and $q_{11} = 0$ if p is even.

Gauss Quadratures

- With respect to a given a weight function $\omega(x) \geq 0$ on $[a, b]$, one can define a sequence of orthonormal polynomials $\{p_n(x)\}_{n=0}^{\infty}$ satisfying

$$\int_a^b \omega(x) p_i(x) p_j(x) dx = \delta_{ij}.$$

  ◇ Roots of each $p_n(x)$ are simple, distinct, and lie in the interval $[a, b]$.

  ◇ The roots $\{\lambda_i\}_{i=1}^{n}$ of a fixed $p_n(x)$ define a Gaussian quadrature

$$\int_a^b \omega(x) f(x) dx = \sum_{i=1}^{n} w_i f(\lambda_i),$$

  that has degree of precision up to $2n - 1$.

- With $p_0(x) \equiv 1$ and $p_{-1}(x) \equiv 0$, orthogonal polynomials satisfy a three-term recurrence relationship:

$$p_n(x) = (a_n x + a_n)p_{n-1}(x) - c_n p_{n-2}(x).$$

◇ In matrix form:

$$x \underbrace{\begin{bmatrix} p_0(x) \\ p_1(x) \\ \\ \vdots \\ p_{n-2}(x) \\ p_{n-1}(x) \end{bmatrix}}_{\mathbf{p}(x)} = \underbrace{\begin{bmatrix} \frac{-a_1}{a_1} & \frac{1}{a_1} & 0 & & 0 \\ \frac{c_2}{a_2} & \frac{-a_2}{a_2} & \frac{1}{a_2} & & \\ 0 & & & & \\ \vdots & & & \ddots & \vdots \\ 0 & & & & \frac{1}{a_{n-1}} \\ 0 & & & \dots \frac{c_n}{a_n} & \frac{-a_n}{a_n} \end{bmatrix}}_{\mathbf{T}} \begin{bmatrix} p_0(x) \\ p_1(x) \\ \\ \vdots \\ p_{n-2}(x) \\ p_{n-1}(x) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \\ \vdots \\ 0 \\ p_n(x) \end{bmatrix}$$

◇ $p_n(\lambda_j) = 0$ if and only if

$$\lambda_i \mathbf{p}(\lambda_i) = T\mathbf{p}(\lambda_i).$$

◇ T can be symmetrized by diagonal similarity transformation into a Jacobi matrix J.

◇ It can be shown that the weight w_j in the quadrature is given by

$$w_i = q_{1i}^2, \quad i = 1, \dots n$$

where $\mathbf{q}_i$ is the i-th normalized eigenvector of J.

- The inverse problem:
  - ◇ Given a quadrature, i.e.,
    - ▷ abscissas $\{\lambda_k^*\}_{k=1}^n$, and
    - ▷ weights $\{w_1, \ldots, w_n\}$ with $\sum_{i=1}^n w_i = 1$,
  - ◇ Determine the corresponding orthogonal polynomials.

Low Rank Approximation

- Noise removal in signal/image processing with Toeplitz structure.
  - ◇ rank = noise level where SNR is high.
- Model reduction problem in speech encoding and filter design with Hankel structure.
  - ◇ rank = # of sinusoidal components in the signal.
- GCD approximation for multivariate polynomials with Sylvester structure.
  - ◇ rank = degree of GCD.
- Molecular structure modeling for protein folding with nonnegative matrices.
  - ◇ rank ≤ 5.
- LSI application.
  - ◇ rank = # of factors capturing the random nature of the indexing matrix but structure = ?
- Preconditioning or regularization of ill-posed inverse problems.