Chapter 3

Parameterized Inverse Eigenvalue Problems

- Overview.
- General Results.
- Additive Inverse Eigenvalue Problems.
- Multiplicative Inverse Eigenvalue Problems.

Overview

- The structural constraint is regulated by a set of parameters.
- Most discussion concentrates on linear dependence of the problem on the parameters.

Generic Form

- Given
  - ◇ A *family* of matrices $A(c) \in \mathcal{M}$ with parameters $c \in \mathbf{F}^m$,
  - ◇ A set of scalars $\Omega \subset \mathbf{F}$,
- Find
  - ◇ Values of parameter c such that

$$\sigma(A(c)) \subset \Omega.$$

    - ▷ $\mathcal{M}$ = One particular class of submatrices in $\mathbf{F}^{n\times n}$.
    - ▷ $\mathbf{F}$ = One particular field of scalars.
- Remark:
  - ◇ Degree of free parameters m needs not be the same as the size n of the matrix.
  - ◇ Commonly used Ω:
    - ▷ $\Omega = \{\lambda_k^*\}_{k=1}^n$.
    - ▷ Ω = left-half complex plan.
  - ◇ Depending upon how $A(c)$ is defined, the PIEP can appear in very different form.

Variations

- Linear dependence on parameters (LiPIEP):
$$A(c) = A_0 + \sum_{i=1}^{m} c_i A_i.$$
  - ◇ $A_i \in \mathcal{R}(n)$, $\mathbf{F} = \mathbb{R}$.
  - ◇ $A_i \in \mathcal{S}(n)$, $\mathbf{F} = \mathbb{R}$.
- **(AIEP)** $A(c) = A(X) = A_0 + X$, $X \in \mathcal{N}$.
  - ◇ $\mathcal{N}$ = Some special class of submatrices.
  - ◇ X can be expressed in terms of linear combinations of basis $\{A_i\}$ of $\mathcal{N}$.
- **(MIEP)** $A(c) = A(X) = XA_0$, $X \in \mathcal{N}$.
  - ◇ XA_0 can still be expressed as a linear combination of some A_i, $i = 1, \ldots, m$.
  - ◇ If $X = \mathrm{diag}\{c_1, \ldots, c_n\}$, write $A_0 = [a_1^T, \ldots, a_n^T]^T$ in rows. Then
$$XA_0 = \sum_{i=1}^{n} c_i \underbrace{e_i a_i^T}_{A_i}.$$
- **(Generalized Pole Assignment Problem)** $A(c) = A(K_1, \ldots, K_q) = A_0 + \sum_{i=1}^{q} B_i K_i C_i$.

General Results

- Lot of attention has been paid to the theory and numerical method of the LiPIEP.
  - ⋄ Finding a solution over real field is more complicated and difficult than over complex field.
- Whatever is known about LiPIEP applies to AIEP and MIEP.
- Pole assignment problem itself stands alone as an important application for decades.
  - ⋄ Has been extensively studied already.
  - ⋄ Many theoretical results and numerical techniques are available.
  - ⋄ Approaches include skills from linear system theory, combinatorics, complex analysis to algebraic geometry.
  - ⋄ Will not be discussed in this note.

Existence Theory for Linear PIEP

- Most discussions concentrate on the LiPIEP.

$$A(c) = A_0 + \sum_{i=1}^{m} c_i A_i.$$

- Complex solvability is generally expected by solving polynomial systems.
- Presence of multiple eigenvalues in real case makes a big difference.

Complex Solvability

- Given n complex numbers $\{\lambda_k^*\}_{k=1}^n$,
  - ◇ For almost all $A_i \in \mathbb{C}^{n\times n}$, there exists $c \in \mathbb{C}^n$ such that $A(c) = A_0 + \sum_{k=1}^n c_k A_k$ has eigenvalues $\{\lambda_k^*\}_{k=1}^n$.
  - ◇ There are at most $n!$ distinct solutions.

Real Solvability ($n = m$)

- Notation and Definitions:

$$\begin{aligned} A_k &:= \left[a_{ij}^{(k)}\right], \quad k = 0, 1, \ldots, n, \\ E &:= \left[a_{ii}^{(k)}\right], \quad i, k = 1, \ldots, n, \\ S &:= \sum_{i=1}^{m} |A_k|, \\ \pi(M) &:= \|M - \operatorname{diag}(M)\|_\infty, \\ d(\lambda) &:= \min_{i \neq j} |\lambda_i - \lambda_j| \end{aligned}$$

- Normalize the diagonals of A_j:
  - ◇ Assume $E^{-1} = [\ell_{ij}]$ exists and $\tilde{c} := Ec$.
  - ◇ Rewrite

$$\begin{aligned} A(c) &= A_0 + \sum_{k=1}^{n} c_k A_k = A_0 + \sum_{k=1}^{n} \left(\sum_{j=1}^{n} \ell_{kj} \tilde{c}_j \right) A_k \\ &= A_0 + \sum_{j=1}^{n} \tilde{c}_j \underbrace{\left(\sum_{k=1}^{n} \ell_{kj} A_k \right)}_{\tilde{A}_j}. \end{aligned}$$

$$\operatorname{diag}(\tilde{A}_j) = e_j, \quad j = 1, \ldots n.$$

- [34] Sufficient condition:
  - ◇ Given
    - ▷ n real numbers $\lambda^* = \{\lambda_k^*\}_{k=1}^n$, and
    - ▷ $n+1$ real $n \times n$ matrices A_i, $i = 0, 1, \ldots, n$,
  - ◇ Assume
    - ▷ $\mathrm{diag}(A_k) = e_k$, $k = 1, \ldots, n$,
    - ▷ $\pi(S) < 1$,
    - ▷ The gap $d(\lambda^*)$ is sufficiently large, i.e.,

$$d(\lambda^*) \geq 4\frac{\pi(S)\|\mathrm{diag}(\lambda^*) - diag(A_0)\|_\infty + \pi(A_0)}{1 - \pi(S)}.$$

  - ◇ Then the LiPIEP (with $m = n$) has a real solution $c \in \mathbb{R}^n$.
  - ◇ Idea of proof:
    - ▷ Prove that Gerschgorin circles of $A(c)$ are disjoint.
    - ▷ Use Brouser fixed-point theorem to find a fixed point for the map $T(c) = \lambda^* + c - \lambda(A(c))$.
- Open Question: What can be said if $m > n$?

Multiple Eigenvalue

- Consider the LiPIEP associated with
  - ◇ Matrices $A_i \in \mathbb{R}^{n\times n}$, $i = 0, 1, \ldots, m$, and
  - ◇ k real eigenvalue $\{\lambda_1^*, \ldots, \lambda_k^*\}$,
    - ▷ λ_i^* has multiplicity $r_i \geq 0$.
    - ▷ $r_1 + \ldots + r_k = n$.
- Let $r = \max\{r_1, \ldots, r_k\}$ = maximal multiplicity.
- [310, 332] The LiPIEP is unsolvable almost everywhere if $n - m + r(r-1) > 1$.
  - ◇ If $n = m$, then the LiPIEP is unsolvalbe almost everywhere if and only if $r > 1$.

Sensitivity Analysis

- The solution to an IEP is generally not unique.
- The IEP is generally ill-posed.
    - ◇ Even if a solution depends continuously upon the problem data, the numerical solution could differ by a great deal with small perturbation.

Forward Problem for General $A(c)$

- Assume
  - ◇ $A(c) \in \mathbb{C}^{n\times n}$ is analytic in $c \in \mathbb{C}^m$ over a neighborhood of 0.
  - ◇ λ_0 is a *simple* eigenvalue of $A(0)$.
  - ◇ $\mathbf{x}_0$ and $\mathbf{y}_0$ are the right and left unit eigenvector, respectively, of $A(0)$ corresponding to λ_0.
- Then
  - ◇ There exists an analytic function $\lambda(c)$ in a neighborhood N of $0 \in \mathbb{C}^m$ such that
    - ▷ $\lambda(c)$ is a simple eigenvalue of $A(c)$.
    - ▷ $\lambda(0) = \lambda_0$.
  - ◇ There exist analytic functions $\mathbf{x}(c)$ and $\mathbf{y}(c)$ in N such that
    - ▷ $\mathbf{x}(c)$ is a right eigenvector corresponding to $\lambda(c)$.
    - ▷ $\mathbf{y}(c)$ is a left eigenvector corresponding to $\lambda(c)$.
    - ▷ $\mathbf{x}(0) = \mathbf{x}_0$, $\mathbf{y}(0) = \mathbf{y}_0$.
- Furthermore,

$$\left(\frac{\partial \lambda(c)}{\partial c_i}\right)_{c=0} = \mathbf{y}_0^T \left(\frac{\partial A(c)}{\partial c}\right)_{c=0} \mathbf{x}_0.$$

Inverse Problem for Linear Symmetric $A(c)$

- Assume all matrices are symmetric and the LiPIEP

$$A(c) = A_0 + \sum_{i=1}^{n} c_i A_i$$

is solvable.

- Assume $A(c) = Q(c)\text{diag}\{\lambda_k^*\}_{k=1}^n Q(c)^T$ and define

$$\begin{aligned} J(c) &= \left[\mathbf{q}_i(c)^T A_j \mathbf{q}_i(c)\right], \quad i, j = 1, \ldots, n, \\ b &= \left[\mathbf{q}_1(c)^T A_0 \mathbf{q}_1(c), \ldots, \mathbf{q}_n^T A_0 \mathbf{q}_n(c)\right]^T. \end{aligned}$$

- [360] If

$$\delta = \|\lambda^* - \tilde{\lambda}\|_\infty + \sum_{i=0}^{n} \|A_i - \tilde{A}_i\|_2$$

is sufficiently small, then

  - ◇ The PIEP associated with $\tilde{A}_i$, $i = 0, \ldots, n$ and $\{\tilde{\lambda}_1, \ldots, \tilde{\lambda}_n\}$ is solvable.
  - ◇ There is a solution $\tilde{c}$ near to c,

$$\frac{\|c - \tilde{c}\|_\infty}{\|c\|_\infty} \leq \kappa_\infty(J(c)) \left(\frac{\|\lambda^* - \tilde{\lambda}\|_\infty + \|A_0 - \tilde{A}_0\|_2}{\|\lambda^* - b\|_\infty} + \frac{\sum_{i=1}^n \|A_i - \tilde{A}_i\|_2}{\|J(c)\|_\infty} \right) + O(\delta^2).$$

Numerical Methods

- Direct methods.
  - ◇ Lanczos method.
- Iterative methods.
  - ◇ Newton's method.
  - ◇ Orthogonal reduction method.
- Continuous Methods.
  - ◇ Homotopy method.
  - ◇ Projected gradient method.
  - ◇ ASVD flow method.

Direct Method

- Solution can be found in finite number of steps.
- Formulation exists for IEP with Jacobi structure.
- Will be discussed in Chapter 4.

Iterative Methods

- Netwon's method.
  - ◇ Applicable to real symmetric LiPIEP.
  - ◇ Fast, but only local convergence.
  - ◇ Multiple eigenvalue case needs to be handled more carefully.
- Orthogonal reduction method.
  - ◇ Employs QR-like decomposition.
  - ◇ Can handle multiple eigenvalues easily.

Newton's Method (for real symmetric LiPIEP)

- Assume:
  - ◇ All matrices in
  $$A(c) = A_0 + \sum_{i=1}^{n} c_i A_i$$
  are real and symmetric.
  - ◇ All eigenvalues $\lambda_1^*, \ldots, \lambda_n^*$ are distinct.
- Consider:
  - ◇ The affine subspace
  $$\mathcal{A} := \{A(c) | c \in R^n\}.$$
  - ◇ The isospectral surface
  $$\mathcal{M}_e(\Lambda) := \{Q\Lambda Q^T | Q \in \mathcal{O}(n)\}$$
  where
  $$\Lambda := diag\{\lambda_1^*, \ldots, \lambda_n^*\}.$$
- Any tangent vector $T(X)$ to $\mathcal{M}_e(\Lambda)$ at a point $X \in \mathcal{M}_e(\Lambda)$ must be of the form
$$T(X) = XK - KX$$
for some skew-symmetric matrix $K \in R^{n \times n}$.

A Classical Newton Method

- A function $f : R \longrightarrow R$.
- The scheme:

$$x^{(\nu+1)} = x^{(\nu)} - (f'(x^{(\nu)}))^{-1} f(x^{(\nu)})$$

- The intercept:
  - ◇ The new iterate $x^{(\nu+1)}$ = The x-intercept of the tangent line of the graph of f from $(x^{(\nu)}, f(x^{(\nu)}))$.
- The lifting:
  - ◇ $(x^{(\nu+1)}, f(x^{(\nu+1)}))$ = The natural "lift" of the intercept along the y-axis to the graph of f from which the next tangent line will begin.

An Analogy of the Newton Method

- Think of:
  - ◇ The surface $\mathcal{M}_e(\Lambda)$ as playing the role of the graph of f.
  - ◇ The affine subspace $\mathcal{A}$ as playing the role of the x-axis.
- Given $X^{(\nu)} \in \mathcal{M}_e(\Lambda)$,
  - ◇ There exist a $Q^{(\nu)} \in \mathcal{O}(n)$ such that
$$Q^{(\nu)^T} X^{(\nu)} Q^{(\nu)} = \Lambda.$$
  - ◇ The matrix $X^{(\nu)} + X^{(\nu)}K - KX^{(\nu)}$ with any skew-symmetric matrix K represents a tangent vector to $\mathcal{M}_e(\Lambda)$ emanating from $X^{(\nu)}$.
- Seek an $\mathcal{A}$-intercept $A(c^{(\nu+1)})$ of such a vector with the affine subspace $\mathcal{A}$.
- Lift up the point $A(c^{(\nu+1)}) \in \mathcal{A}$ to a point $X^{(\nu+1)} \in \mathcal{M}_e(\Lambda)$.

Find the Intercept

- Find a skew-symmetric matrix $K^{(\nu)}$ and a vector $c^{(\nu+1)}$ such that

$$X^{(\nu)} + X^{(\nu)}K^{(\nu)} - K^{(\nu)}X^{(\nu)} = A(c^{(\nu+1)}).$$

- Equivalently, find $\tilde{K}^{(\nu)}$ such that

$$\Lambda + \Lambda\tilde{K}^{(\nu)} - \tilde{K}^{(\nu)}\Lambda = {Q^{(\nu)}}^T A(c^{(\nu+1)})Q^{(\nu)}.$$

  ◇ $\tilde{K}^{(\nu)} := {Q^{(\nu)}}^T K^{(\nu)} Q^{(\nu)}$ is skew-symmetric.

- Can find $c^{(\nu)}$ and $K^{(\nu)}$ separately.

- Diagonal elements in the system $\Rightarrow$

$$J^{(\nu)} c^{(\nu+1)} = \lambda^* - b^{(\nu)}.$$

  - $\diamond$ Known quantities:

$$\begin{aligned} J_{ij}^{(\nu)} &:= \mathbf{q}_i^{(\nu)^T} A_j \mathbf{q}_i^{(\nu)}, \text{ for } i, j = 1, \dots, n \\ \lambda^* &:= (\lambda_1^*, \dots, \lambda_n^*)^T \\ b_i^{(\nu)} &:= \mathbf{q}_i^{(\nu)^T} A_0 \mathbf{q}_i^{(\nu)}, \text{ for } i = 1, \dots, n \\ \mathbf{q}_i^{(\nu)} &= \text{ the } i\text{-th column of the matrix } Q^{(\nu)}. \end{aligned}$$

- The vector $c^{(\nu+1)}$ can be solved.
- Off-diagonal elements in the system together with $c^{(\nu+1)}$ $\Rightarrow \tilde{K}^{(\nu)}$ (and, hence, $K^{(\nu)}$):

$$\tilde{K}_{ij}^{(\nu)} = \frac{\mathbf{q}_i^{(\nu)^T} A(c^{(\nu+1)}) \mathbf{q}_j^{(\nu)}}{\lambda_i^* - \lambda_j^*}, \text{ for } 1 \le i < j \le n.$$

Find the Lift-up

- No obvious coordinate axis to follow.
- Solving the IEP $\equiv$ Finding $\mathcal{M}_e(\Lambda) \bigcap \mathcal{A}$.
- Suppose all the iterations are taking place near a point of intersection. Then

$$X^{(\nu+1)} \approx A(c^{(\nu+1)}).$$

- Also should have

$$A(c^{(\nu+1)}) \approx e^{-K^{(\nu)}} X^{(\nu)} e^{K^{(\nu)}}.$$

- Replace $e^{K^{(\nu)}}$ by the Cayley transform:

$$R := (I + \frac{K^{(\nu)}}{2})(I - \frac{K^{(\nu)}}{2})^{-1} \approx e^{K^{(\nu)}}.$$

- Define

$$X^{(\nu+1)} := R^T X^{(\nu)} R \in \mathcal{M}_e(\Lambda).$$

- The next iteration is ready to begin.

Remarks

- Note that

$$X^{(\nu+1)} \approx R^T e^{K^{(\nu)}} A(c^{(\nu+1)}) e^{-K^{(\nu)}} R \approx A(c^{(\nu+1)})$$

  represents a lifting of the matrix $A(c^{(\nu+1)})$ from the affine subspace $\mathcal{A}$ to the surface $\mathcal{M}_e(\Lambda)$.

- The above offers a geometrical interpretation of Method III developed by Friedland el al [145].

- Quadratic convergence even for multiple eigenvalues case.

Continuous Methods

- Homotopy method.
  - ◇ Homotopy theory for some AIEP's can be established.
    - ▷ Open Question: Describe a homotopy for general PIEP.
  - ◇ Provides both an existence proof and a numerical method.
  - ◇ See discussion in AIEP.
- Projection gradient method.
  - ◇ General, least squares setting.
  - ◇ Can be generalized to SIEP with any linear structure.
  - ◇ The method enjoys the globally descent property, but slow.
- ASVD flow method.
  - ◇ Provides stable coordinate transformations for nonsymmetric matrices.
  - ◇ Will be discissed in SIEP for stochastic structure.

Projected Gradient Method (for SIEP)

- The idea works for general *symmetric* $A(c)$ so long as the projection $P(X)$ of a matrix X to $\mathcal{A}$ can be calculated.
- The idea applies to SIEP and is described in that setting.
- Idea:
  - ◇ $X \in \mathcal{M}_e(\Lambda)$ satisfies the spectral constraint.
  - ◇ $P(X) \in \mathcal{V}$ has the desirable structure in $\mathcal{V}$.
  - ◇ Minimize the undesirable part $\|X - P(X)\|$.
- Working with the parameter Q is easier:

$$\begin{aligned}\text{Minimize } F(Q) \ &:= \ \frac{1}{2}\big\langle Q^T\Lambda Q - P(Q^T\Lambda Q), \\ &\qquad Q^T\Lambda Q - P(Q^T\Lambda Q)\big\rangle \\ \text{Subject to } Q^TQ \ &= \ I\end{aligned}$$

  - ◇ $\langle A, B\rangle = \text{trace}(AB^T)$ is the Frobenius inner product.

Feasible Set $O(n)$ & Gradient of F

- The set $O(n)$ is a regular surface.
- The tangent space of $O(n)$ at any orthogonal matrix Q is given by

$$T_Q O(n) = QK(n)$$

  where

$$K(n) = \{\text{All skew-symmetric matrices}\}.$$

- The normal space of $O(n)$ at any orthogonal matrix Q is given by

$$N_Q O(n) = QS(n).$$

- The Fréchet Derivative of F at a general matrix A acting on B:

$$F'(A)B = 2\langle \Lambda A(A^T \Lambda A - P(A^T \Lambda A)), B\rangle.$$

- The gradient of F at a general matrix A:

$$\nabla F(A) = 2\Lambda A(A^T \Lambda A - P(A^T \Lambda A)).$$

The Projected Gradient

- A splitting of $R^{n\times n}$:

$$\begin{aligned} R^{n\times n} &= T_Q O(n) + N_Q O(n) \\ &= QK(n) + QS(n). \end{aligned}$$

- A unique orthogonal splitting of $X \in R^{n\times n}$:

$$X = Q\left\{\frac{1}{2}(Q^T X - X^T Q)\}+Q\{\frac{1}{2}(Q^T X + X^T Q)\right\}.$$

- The projection of $\nabla F(Q)$ into the tangent space:

$$\begin{aligned} g(Q) &= Q\left\{\frac{1}{2}(Q^T \nabla F(Q) - \nabla F(Q)^T Q)\right\} \\ &= Q[P(Q^T \Lambda Q), Q^T \Lambda Q]. \end{aligned}$$

An Isospectral Descent Flow

- A descent flow on the manifold $O(n)$:

$$\frac{dQ}{dt} = Q[Q^T \Lambda Q, P(Q^T \Lambda Q)].$$

- A descent flow on the manifold $M(\Lambda)$:

$$\begin{aligned} \frac{dX}{dt} &= \frac{dQ^T}{dt}\Lambda Q + Q^T \Lambda \frac{dQ}{dt} \\ &= [X, \underbrace{[X, P(X)]}_{k(X)}]. \end{aligned}$$

- The entire concept can be obtained by utilizing the Riemannian geometry on the Lie group $O(n)$.

Additive Inverse Eigenvalue Problems

- Subvariations.
- Solvability Issues.
- Sensitivity Issues.
- Numerical Methods.
- Applications.

Subvariations

- Generic form:
  - ◇ Given
    - ▷ A fixed matrix A and a class of matrices $\mathcal{N}$ in $\mathbf{F}^{n\times n}$,
    - ▷ A subset $\Omega \subset \mathbf{F}$,
  - ◇ Find
    - ▷ $X \in \mathcal{N}$ such that $\sigma(A+X) \subset \Omega$.
- Some special cases:
  - ◇ (AIEP1) A is real, X is real diagonal, and $\mathbf{F}$ is real.
  - ◇ (AIEP2) A is real symmetric, X is real diagonal, and $\mathbf{F}$ is real.
  - ◇ (AIEP3) A is complex general, X is complex diagonal, and $\mathbf{F}$ is complex.
  - ◇ Open Question: $A = 0$, X has prescribed entries at specific location.

Solvability Issues

- If $X = \operatorname{diag}(c_1, \ldots, c_n)$, then consider

$$A + X = A + \sum_{i=1}^{n} c_i \underbrace{e_i e_i^T}_{A_i}.$$

  This is a special PIEP.

- Complex solvability [2, 63, 138]:
  - ◇ For any specified $\{\lambda_k^*\}_{k=1}^n \in \mathbb{C}$, the AIEP3 is solvable.
  - ◇ There are at most $n!$ solutions.
  - ◇ For almost all $\{\lambda_k^*\}_{k=1}^n$, there are exactly $n!$ solutions.

- Real solvability
  - ◇ Some sufficient conditions:
    - ▷ If $d(\lambda^*) > 4\pi(A)$, then AIEP1 is solvable [101].
    - ▷ If $d(\lambda^*) > 2\sqrt{3}\left(\pi(A \circ A)\right)^{1/2}$, then AIEP2 is solvable [170].
  - ◇ Some necessary conditions:
    - ▷ If AIEP1 is solvable, then

$$\sum_{i \neq j} (\lambda_i^* - \lambda_j^*)^2 \geq 2n \sum_{i \neq j} a_{ij} a_{ji}.$$

- Unsolvability:
  - ◇ Both AIEP1 and AIEP2 are unsolvable almost everywhere if an multiple eigenvalue is present [332].

Sensitivity Issues (for AIEP2)

- Suppose that the AIEP2 is solvable.
- Assume
  - ◇ $A(X) := A + X = Q(X)^T \Lambda Q(X)$,
    - ▷ Define

$$J(X) := [q_{ji}^2(X)],$$
$$b(X) := [q_1(X)^T A q_1(X), \ldots, q_n(X)^T A q_n(X)]^T.$$

  - ◇ $J(X)$ is nonsingular,
  - ◇ The perturbation

$$\delta = \|\lambda^* - \tilde{\lambda}\|_\infty + \|A - \tilde{A}\|_2$$

  is small enough.
- Then
  - ◇ The AIEP2 associated with $\tilde{A}$ and $\tilde{\lambda}$ is solvable.
  - ◇ There is a solution $\tilde{X}$ near to X, i.e.,

$$\frac{\|X - \tilde{X}\|_\infty}{\|X\|_\infty} \leq \kappa_\infty(J(X)) \left(\frac{\|\lambda^* - \tilde{\lambda}\|_\infty + \|A - \tilde{A}\|_2}{\|\lambda * - b\|_\infty} \right) + O(\delta^2).$$

Numerical Methods

- Most methods for symmetric or Hermitian problem depend heavily on the fact that the eigenvalues are real and can be totally ordered.
  - ◇ Can consider each eigenvalue λ_i as piecewise differential function $\lambda_i(X)$.
  - ◇ Newton's iteration for AIEP2 is easy to formulate.
- For general matrices where eigenvalues are complex, tracking each eigenvalue requires some kind of matching mechanism.
  - ◇ Homotopy method naturally track each individual eigenvalue curves as are predetermined by initial values.
  - ◇ Homotopy method for AIEP3 gives rise to both an existence proof and a numerical method for finding all solutions.

Newton's Method (for AIEP2)

- At the ν-th iterate, assume $Z^{(\nu)} \in \mathcal{M}_e(\Lambda)$,

$$\begin{aligned} Z^{(\nu)} &= {Q^{(\nu)}}^T \Lambda Q^{(\nu)}, \\ A(X^{(\nu)}) &:= A + X^{(\nu)}, \\ J^{(\nu)} &:= \left[{q_{ji}^{(\nu)}}^2 \right], \\ b^{(\nu)} &:= \left[{q_1^{(\nu)}}^T A q_1^{(\nu)}, \ldots, {q_n^{(\nu)}}^T A q_n^{(\nu)} \right]^T. \end{aligned}$$

- Solve $J^{(\nu)} X^{(\nu+1)} = \lambda^* - b^{(\nu)}$ for $X^{(\nu+1)}$.
- Define skew-symmetric matrix

$$K^{(\nu)} := Q^{(\nu)} \left[\frac{{q_i^{(\nu)}}^T A(X^{(\nu+1)}) q_j^{(\nu)}}{\lambda_i^* - \lambda_j^*} \right] {Q^{(\nu)}}^T.$$

- Update the lift,

$$\begin{aligned} R^{(\nu)} &:= \left(I + \frac{K^{(\nu)}}{2} \right) \left(I - \frac{K^{(\nu)}}{2} \right)^{-1}, \\ Z^{(\nu+1)} &:= {R^{(\nu)}}^T X^{(\nu)} R^{(\nu)}, \\ Q^{(\nu+1)} &:= {R^{(\nu)}}^T Q^{(\nu)}. \end{aligned}$$

Homotopy Method (for AIEP3)

Multiplicative Inverse Eigenvalue Problems

- Subvariations.
- Solvability Issues.
- Sensitivity Issues.
- Numerical Methods.
- Applications.

Subvariations

- Generic form:
  - ◇ Given
    - ▷ A fixed matrix A and a class of matrices $\mathcal{N}$ in $\mathbf{F}^{n\times n}$,
    - ▷ A subset $\Omega \subset \mathbf{F}$,
  - ◇ Find
    - ▷ $X \in \mathcal{N}$ such that $\sigma(XA) \subset \Omega$.
- Some special cases:
  - ◇ (MIEP1) A is real, X is real diagonal, and $\mathbf{F}$ is real.
  - ◇ (MIEP2) A is real, symmetric, and positive definite, X is nonnegative diagonal, and $\mathbf{F}$ is real.
  - ◇ (MIEP3) A is complex general, X is complex diagonal, and $\mathbf{F}$ is complex.
  - ◇ (MIEP4) A is complex Hermitian, $\mathbf{F}$ is real, want $\sigma(X^{-1}AX^{-1}) = \{\lambda_k^*\}_{k=1}^n$ [120].
  - ◇ (MIEP5) Preconditioning applications.

Solvability Issues

- If $X = \mathrm{diag}(c_1, \ldots, c_n)$ and $A = [\mathbf{a}_1^T, \ldots, \mathbf{a}_n^T]^T$, then write

$$XA = \sum_{i=1}^{n} c_i \underbrace{\mathbf{e}_i \mathbf{a}_i^T}_{A_i}.$$

  This is a special PIEP.

- Complex solvability [137]:
  - ◇ Assume that
    - ▷ All principal minors of A are distinct from zero,
  - ◇ Then
    - ▷ For any specified $\{\lambda_k^*\}_{k=1}^n \in \mathbb{C}$, the MIEP3 is solvable.
    - ▷ There are at most $n!$ solutions.

- Real solvability:
  - ◇ Some sufficient conditions:
    - ▷ Suppose
      - · The diagonals of A are normalized to 1, i.e., $a_{ii} = 1$.
      - · $\pi(A) < 1$.
      - · $d(\lambda) \geq \frac{4\pi(A)\|\lambda^*\|_{infty}}{1-\pi(A)}$.
    - ▷ Then the MIEP1 is solvable.
  - ◇ Some necessary conditions:
    - ▷ If MIEP1 is solvable, then

$$
\begin{aligned}
\sum_{i=1}^{n} x_i &= \sum_{i=1}^{n} \lambda_i^*, \\
\det(A) \prod_{i=1}^{n} x_i &= \prod_{i=1}^{n} \lambda_i^*.
\end{aligned}
$$

Optimal Conditioning by Diagonal Matrices

- Friedland's result on MIEP1 suggests that a general complex matrix A can be perfectly conditioned.
  - ◇ Lacks an efficient algorithm to implement the result.
- Open question: Want to know the optimal preconditioner of a given sparsity pattern [165].
  - ◇ Suppose
    - ▷ A is symmetric and positive definite.
    - ▷ A has property A, i.e., A can be symmetrically permuted into
      $$\begin{bmatrix} D_1 & B \\ B^T & D_2 \end{bmatrix}$$
      where D_1 and D_2 are diagonal.
    - ▷ $D = \text{diag}(A)$.
  - ◇ Then [134]
    $$\kappa(D^{-1/2}AD^{-1/2}) = \min_{\widehat{D}>0,\widehat{D}=\text{diagonal}} \kappa(\widehat{D}A\widehat{D}).$$

Sensitivity Issues (for MIEP2)

- Suppose that the MIEP2 is solvable.
- MIEP2 is equivalent to the symmetrized problem:

$$X^{-1/2}A(X)X^{1/2} = X^{1/2}AX^{1/2}.$$

- Define

$$\begin{aligned} X^{1/2}AX^{1/2} &= U(X)^T \Lambda U(X), \\ W(X) &:= [u_{ji}^2(X)]. \end{aligned}$$

- Assume
  - ◇ $W(X)$ is nonsingular,
  - ◇ The perturbation

    $$\delta = \|\lambda^* - \tilde{\lambda}\|_\infty + \|A - \tilde{A}\|_2$$

    is small enough.
- Then
  - ◇ The MIEP2 associated with $\tilde{A}$ and $\tilde{\lambda}$ is solvable.
  - ◇ There is a solution $\tilde{X}$ near to X, i.e.,

$$\frac{\|X - \tilde{X}\|_\infty}{\|X\|_\infty} \leq \frac{\lambda_n^*}{\lambda_1^*}\|W(X)^{-1}\|_\infty \left(\frac{\|\lambda^* - \tilde{\lambda}\|_\infty}{\|\lambda^*|_\infty} + \|A - \tilde{A}\|_2 \right) + O(\delta^2).$$

Numerical Methods

- For preconditioning purpose, no need to solve the MIEP precisely.
  - ◇ There are many techniques for picking up a preconditioner.
  - ◇ Will not be discussed in this note.
- The MIEP is a linear, but not symmetric PIEP even if A is symmetric.
  - ◇ The numerical methods for symmmtric PIEP need to be modified.
  - ◇ If A is a Jacobi matrix, the problem can be solved by direct methods. Will be discussed in Chapter 4.

Reformulate MIEP1 as Nonlinear Equations

- Formulate MIEP1 as solving $f(X) = 0$ for some nonlinear function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$.
- Different ways to formulate $f(X)$:
  - ◇ $f_i(X) := \det(XA - \lambda_i^* I)$.
  - ◇ $f_i(X) := \lambda_i(XA) - \lambda_i^*$.
  - ◇ $f_i(X) := \alpha_n(XA - \lambda_i^* I)$, where $\alpha_n(M) =$ the smallest singular values of M.
- Assume that $\lambda_i \neq 0$ and, therefore, $Y = X^{-1}$ exists.
  - ◇ $g_i(Y) := \det(A - \lambda_i^* Y)$.
  - ◇ $g_i(Y) := \lambda_i(A, Y) - \lambda_i^*$.

Newton's Method (for MIEP2)

- Reformulate the MIEP2 as solving equations

$$\lambda_i(A, Y) - \lambda_i^* = 0, \quad i = 1, \dots n$$

to maintain symmetry.

- At the ν-th stage [213],
  - ◇ Solve the generalized eigenvalue problem

$$\left(A - \lambda(\nu) Y^{(\nu)}\right) \mathbf{x}^{(\nu)} = 0.$$

    - ▷ Normalize $\mathbf{x}_i^{(\nu)}$ so that $\mathbf{x}_i^{(\nu)T} Y^{(\nu)} \mathbf{x}_i^{(\nu)} = 1$.
    - ▷ Denote $Q^{(\nu)} = [\mathbf{x}_1^{(\nu)}, \dots \mathbf{x}_n^{(\nu)}] = [q_{ij}^{(\nu)}]$.
  - ◇ Define (the Jacobian matrix of $\lambda(A, Y)$)

$$J(Y^{(\nu)}) := [-\lambda_i^{(\nu)} q_{ji}^{(\nu)}].$$

  - ◇ Solve $J(Y^{(\nu)}) d^{(\nu)} = \lambda^* - \lambda^{(\nu)}$.
  - ◇ Update $Y^{(\nu+1)} := Y^{(\nu)} + \mathrm{diag}(d^{(\nu)})$.

Numerical Experience

- Open Question: Given the standard Jacobi matrix A with nonzero row entries $[-1, 2, -1]$, what is the set of all reachable spectra of XA via a nonnegative diagonal matrix X?
- Open Question: In structure design, often we are only interested in a few low order natrual frequencies. Indeed, for large structures, it is impractical to calculate all of the frequencies and modes. How should one solve the problem if only a few low order frequencies are given?
- The above Newton method is only a locally convergent method. It appears that in the case of divergence, the Jacobian matrix J becomes highly ill-conditioned and nearly singular.
- To effectively develop an algorithm for controlling the vibration of a string with a specified set of natural frequencies, for example, we need to have another mechanism that can somehow provide a good initial guess before the Newton's method can be employed.