Chapter 5

Least Squares Inverse Eigenvalue Problems

- Overview
- Least Squares IEP
- Theorem
- Geometric Sketch
- Algorithm

Overview

- The information of spectral constraint and the structrual constraint may not be obtained precisely in practice.
- It may occur in practice that one of the two constraints should be enforced more critically than the other.
  - ◇ Without the physical realizability, the physical system simply cannot be built. So structrual constraint is emphasized.
- It may occur that one constraint could be more relaxed than the other.
  - ◇ For complex system there is no accurate way to measure the spectrum or to obtain the entire information. Physical uncertainty can be tolerated up to a certain degree.
- When the two constraints cannot be settled simultaneously, the IEP could be formulated in a least squares setting.

Formulation

- All IEP's discussed so far have a natural generalization to least squares formulation.
- Depends upon which constraint is to be enforced, there are two ways to formulate the least squares IEP.

Least Squares Approximating the Spectrum

- Given:
  - ◇ Symmetric matrices $A(d) \in \mathbb{R}^{n\times n}$,
  - ◇ Real numbers $\lambda_1^*, \cdots, \lambda_m^*$,
  - ◇ $m \le n$,
- Solve:

$$\min_{d\in\mathbb{R}^l,\sigma} F(d,\sigma) = \sum_{i=1}^{m} (\lambda_{\sigma_i}(d) - \lambda_i^*)^2,$$

$$= \|\Lambda_\sigma(d) - \Lambda_m^*\|_F^2.$$

  - ◇ $\sigma :=$ a permutation;
  - ◇ $\Lambda_m^* := \mathrm{diag}(\lambda_1^*, ..., \lambda_m^*)$.

Least Squares Approximating the Structure

- Let
  - $\Lambda_m^* := \text{diag}(\lambda_1^*, ..., \lambda_m^*)$.
  - $D_{n-m} := \{\text{diag matrices}\} \subseteq \mathbb{R}^{(n-m)\times(n-m)}$.
  - $\mathcal{O}(n) := \{\text{orthogonal matrices}\} \subseteq \mathbb{R}^{n\times n}$
- Define:

$$\Gamma := \left\{ Q \begin{bmatrix} \Lambda_m^* & 0 \\ 0 & \Lambda_{n-m} \end{bmatrix} Q^T \mid Q \in \mathcal{O}(n), \Lambda_{n-m} \in D_{n-m} \right\}.$$

- Given:
  - Symmetric matrices $A(d) \in \mathbb{R}^{n\times n}$,
  - Real numbers $\lambda_1^*, \cdots, \lambda_m^*$,
  - $m \leq n$,
- Solve:

$$\text{dist}(\mathcal{A}(d), \Gamma) = \min_{d\in\mathbb{R}^l, B\in\Gamma} \|A(d) - B\|_F^2.$$

Main Theorem

- At a global minimizer, LSIEP1 and LSIEP2 are equivalent.
  - ◇ Both problems have the same objective values.

$$\min_{d,\sigma} \|\Lambda_\sigma(d) - \Lambda_m^*\|_F^2 = \min_{d,B} \|A(d) - B\|_F^2.$$

  - ◇ If d_1, σ solve **LSIEP1** and d_2, B solve **LSIEP2**, then
    - ▷ $d_1 = d_2$,
    - ▷ $B = Q(d_1) \begin{bmatrix} \Lambda_m^* & 0 \\ 0 & \Lambda_{\overline{\sigma}}(d_1) \end{bmatrix} Q(d_1)^T$.
      - · $Q(d_1)^T A(d_1) Q(d_1) = \begin{bmatrix} \Lambda_\sigma(d_1) & 0 \\ 0 & \Lambda_{\overline{\sigma}}(d_1) \end{bmatrix}$.
      - · $\mathrm{diag}[\Lambda_\sigma(d_1), \Lambda_{\overline{\sigma}}(d_1)]$ has same ordering as $\mathrm{diag}[\Lambda_m^*, \Lambda_{\overline{\sigma}}(d_1)$
      - · $\overline{\sigma} := \{1, ..., n\} - \sigma$.
  - ◇ If so

$$\min_{d,B} \|A(d) - B\|_F^2 = \sum_{i=0}^{m} |\lambda_{\sigma_i} - \lambda_i^*|^2.$$

One Particular Case

- Consider the special case where

$$A(d) := A_0 + \sum_{i=1}^{l} d_i A_i \in \mathcal{A}(d) \text{ (affine subspace)}$$

- Lift & Project:
  1. Find
  $$Z^{(k)} := \arg\min_{B\in\Gamma} \|A(d^{(k)}) - B\|_F \quad \textbf{(Lift)}$$
  by the Wielandt-Hoffman Theorem!
  2. Find
  $$d^{(k+1)} := \arg\min_{d\in\mathbb{R}^l} \|Z^{(k)} - \mathcal{A}(d)\|_F \quad \textbf{(Project)}$$
  by solving a linear system.

- The topology of Γ is not clear. But *Step 1* can be accomplished by considering a "substructure".

$$\mathcal{M}_k := \left\{ Q \begin{bmatrix} \Lambda_m^* & 0 \\ 0 & \Lambda_{\overline{\sigma}}(d^{(k)}) \end{bmatrix} Q^T \mid Q \in \mathcal{O}(n) \right\}$$

which is known.

Geometric Sketch

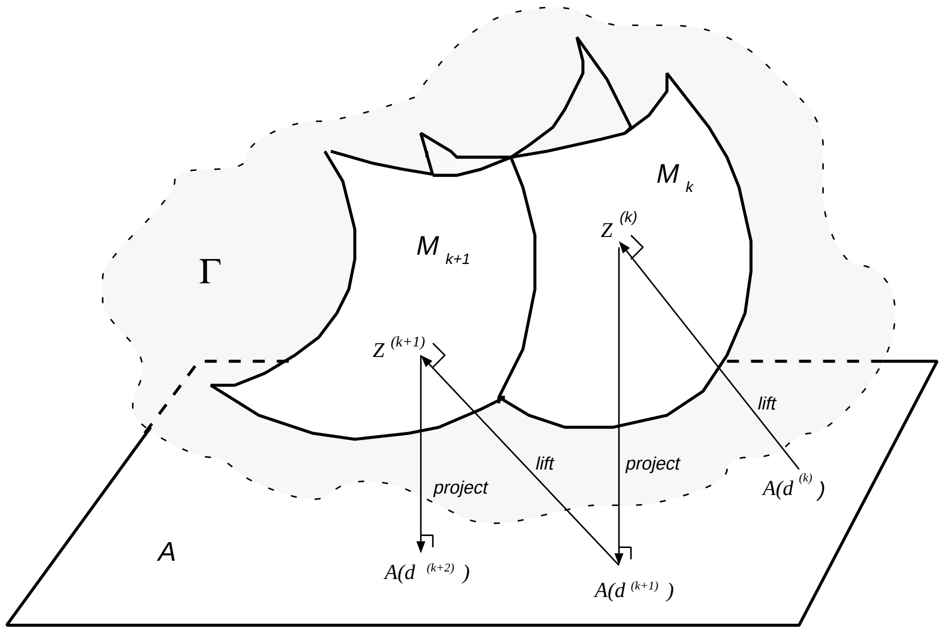

Figure 1: Geometry of life and projection.

- This a descent method!

$$\|Z^{(k+1)} - A(d^{(k+1)})\|_F \leq \|Z^{(k)} - A(d^{(k+1)})\|_F$$

$$\leq \|Z^{(k)} - A(d^{(k)})\|_F$$

Algorithm

- Given $d^{(0)}$.
- For k = 0,1,2,...
  1. Compute $Q(d^{(k)})$ and $\Lambda(d^{(k)})$.
  2. Find $\sigma = \{\sigma_1, ..., \sigma_m\}$. Form $\Lambda_{\overline{\sigma}}$
  3. (**lift**) Form
  $$Z^{(k)} = Q(d^{(k)})diag(\Lambda^*, \Lambda_{\overline{\sigma}}(d^{(k)}))Q(d^{(k)})^T.$$
  4. (**Project**) Compute $d^{(k+1)}$ from
  $$\sum_{i=1}^{l} < A_i, A_j > d_i^{(k+1)} =< Z^{(k)} - A_0, A_j > .$$
  5. stop if $\|d^{(k+1)} - d^{(k)}\| < \varepsilon$