Chapter 6

Partially Described Inverse Eigenvalue Problems

- Overview
- Generic Form
- PDIEP for Toeplitz Matrices

Overview

- The physical system is too complicated or the size is too large,
  - ◇ No reasonable analytical tools are available to evaluate the entire spectral information.
  - ◇ Through vibration tests where the excitation and the response of the structure at selected points are measured experimentally, there are identification techniques to extract a portion of eigenpair information from the measurements.
- It is unwise to use experimental values of high natural frequencies to reconstruct a model because a discrete model with n degrees of freedom can provide accurate values for only about $\frac{n}{3}$ natural frequencies.
- Eigenpair information sometimes provides additional intrinsic relationships. A few eigenpairs can determine much of the desirable reconstruction.

Generic Form

- Given
  - ◇ Vectors $\{\mathbf{v}^{(1)}, \ldots, \mathbf{v}^{(k)}\} \subset \mathbb{F}^n$,
  - ◇ Values $\{\lambda_1, \ldots \lambda_k\} \subset \mathbb{F}$,
  - ◇ A set $\mathcal{N}$ of structured matrices,

  Find $X \in \mathcal{N}$ such that

$$X\mathbf{v}^{(i)} = \lambda_i \mathbf{v}^{(i)}, \quad , i = 1, \ldots, k.$$

- Given
  - ◇ Two sets of real vectors $\{x_1, \ldots, x_p\}$ and $\{y_1, \ldots y_p\}$, $p \leq n$,
  - ◇ An arbitrary set of complex numbers $\mathcal{L} = \{\lambda_1, \ldots, \lambda_n\}$,

  Find a real matrix A such that

$$\begin{aligned} Ax_i &= y_i, \\ \sigma(A) &= \mathcal{L} \end{aligned}$$

- Open Question: How many pairs of information are needed to determine such a structured matrix?

PDIEP for Toeplitz Matrices

- Eigenvectors of a Toeplitz matrix have a special structure [8, 55, 113]:
  - ◇ Eigenvectors of any symmetric and centro-symmetric matrix must be either symmetric or skew-symmetric.
- The dimension of Toeplitz matrices with a single prescribed eigenvector in $\mathbb{R}^n$ should be at least $[\frac{n+1}{2}]$ [85].
- The dimension of Toeplitz matrices with two prescribed eigenvectors is independent of n [75].
- Define

$$\begin{aligned} S(\mathbf{v}) &:= \{\mathbf{c} \in \mathbb{R}^n | T(\mathbf{c})\mathbf{v} = \lambda\mathbf{v} \text{ for some } \lambda \in \mathbb{R}\}, \\ \pi(\mathbf{u}, \mathbf{v}) &:= \dim(S(\mathbf{u}) \bigcap S(\mathbf{v})) \end{aligned}$$

- Let η denote the dimension of the affine subspace of symmetric Toeplitz matrices with two prescribed eigenpairs. Then for almost all eigenvectors $\mathbf{u}$ and $\mathbf{v}$, and for any eigenvalues λ_1 and λ_2, the pair $\pi(\mathbf{u}, \mathbf{v})/\eta$ is summarized in the following table.

$n =$ odd (even)	symmetric	skew-symmetric
symmetric	2/0 (3/1)	2/0 (2/0)
skew-symmetric	2/0 (2/0)	4/0 (3/1)

  - ◇ The symmetric Toeplitz matrix is uniquely determined, if
    - ▷ n is odd and if at least one of the given eigenvectors is symmetric, or
    - ▷ n is even and one eigenvector is symmetric and the other is skew-symmetric.