Structured Low Rank Approximation

Lecture IV: Singular Value Assignment with Low Rank Matrices

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Syllabus

- Objectives:
  - ◇ To provide some preliminaries.
  - ◇ To treat some mathematics.
  - ◇ To point out some applications.
  - ◇ To describe some algorithms.
- Topics:
  - ◇ Lecture I: Introduction
  - ◇ Lecture II: General Approach
  - ◇ Lecture III: Distance Geometry and Protein Structure
  - ⧫ Lecture IV: Singular Value Assignment with Low Rank Matrices
  - ◇ Lecture V: Nonnegative Matrix Factorization
- Assignments:
  - ◇ Theory and computation of the approximation problem are yet to be studied.

Lecture IV

Singular Value Assignment with Low Rank Matrices

Joint Work with Delin Chu

Outline

- Introduction
  - ◇ Pole Assignment Problem
  - ◇ Singular Value Assignment Problem
- Rank One Update — the Building Block
  - ◇ Necessary and Sufficient Condition
  - ◇ Complete Characterization
  - ◇ Unsolvability
- Main Result
- Recursive Algorithm
- Minimum Low Rank Approximation

State Feedback Control Pole Assignment Problem

- Given matrices $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$, consider the state $\mathbf{x}(t) \in \mathbb{R}^n$ under the dynamic state displaymath:

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t). \tag{1}$$

  - ◇ Select the input $\mathbf{u}(t) \in \mathbb{R}^m$ so that the dynamics of the resulting $\mathbf{x}(t)$ is driven into to a certain desired state.
  - ◇ In state feedback control,

$$\mathbf{u}(t) = F\mathbf{x}(t).$$

  - ◇ System (1) is changed to a closed-loop dynamical system:

$$\dot{\mathbf{x}}(t) = (A + BF)\mathbf{x}(t).$$

- Choose the *gain matrix* $F \in \mathbb{R}^{m\times n}$ so as to achieve stability and to speed up response.
  - ◇ The problem can be translated into choosing F so as to reassign eigenvalues of the matrix $A + BF$.

Known Results

- Well studied subject. (Byrnes'89, Kautsky, Nichols, and Van Dooren'85, Sun'96, Wonham'85)
- Given any set of n complex numbers $\{\lambda_1, \ldots, \lambda_n\}$, closed under complex conjugation, a matrix $F \in \mathbb{R}^{m\times n}$ exists such that

$$\lambda(A + BF) = \{\lambda_1, \ldots, \lambda_n\}$$

  if and only if

$$\operatorname{rank}[A - \mu I, B] = n, \quad \text{for all } \mu \in \mathbb{C}.$$

  - ⋄ Also known as the pair (A, B) being *controllable*.
  - ⋄ If $m = 1$, the pole assignment problem, if solvable, has a unique solution.
- It can be proved that

$$\bigcap_{F\in\mathbb{R}^{m\times n}} \lambda(A + BF) = \{\mu \in \lambda(A) \mid \operatorname{rank}[A - \mu I, B] < n\},$$

  - ⋄ For a certain peculiar pair (A, B) of matrices the eigenvalues of A cannot be reassigned by any F.
  - ⋄ Unassignable matrix pairs form a zero measure set.

Singular Value Assignment Problem (ISVPrk)

- Given
  - ◇ A matrix $A \in \mathbb{R}^{m\times n}$ $(m \geq n)$,
  - ◇ An integer $n \geq \ell > 0$, and
  - ◇ Real numbers $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_n \geq 0$,
- Find a matrix $F \in \mathbb{R}^{m\times n}$ such that

$$\begin{aligned} \operatorname{rank}(F) &\leq \ell, \\ \sigma(A+F) &= \{\beta_1, \beta_2, \cdots, \beta_n\}. \end{aligned}$$

Literature Search

- The state feedback pole assignment problem is a special case of the inverse eigenvalue problems.
  - See the book by Chu and Golub'04.
  - http://www4.ncsu.edu/ mtchu/Research/Lectures/lecture.html
- The inverse singular value problems have not received as many studies.
- Some related inverse problems:
  - The de Oliveira theorem (de Oliveira'71) on the principal elements and singular values.
  - The Weyl-Horn theorem (Horn'54, Weyl'49) on the relationship between singular values and eigenvalues.
  - The Sing-Thompson theorem (Sing'76, Thompson'77) on the majorization between the diagonal elements and singular values.
- An inverse singular value problem can be recast as a specially structured inverse eigenvalue problem.
  - The existing theory does not provide us a clue on when the ISVPrk is solvable.

Our Contributions

- We completely characterize the necessary and sufficient condition under which the above ISVPrk is solvable.
- We offer a constructive proof which can be implemented as a numerical means to find the solution.

Rank One Update — the Building Block

- A rank one update is of the form $F = \mathbf{b}\mathbf{f}^{\top}$.
  - ◇ Two controls, $\mathbf{b} \in \mathbb{R}^{m\times 1}$ and $\mathbf{f} \in \mathbb{R}^{n\times 1}$.
- Assume the scenario where
  - ◇ The column vector $\mathbf{b} \in \mathbb{R}^{m}$ is temporarily given and fixed.
  - ◇ The column vector $\mathbf{f} \in \mathbb{R}^{n}$ is to be determined.
  - ◇ What conditions must be imposed on $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_n \geq 0$ for existence?
- How the vector $\mathbf{b}$ could be adjusted to maximally relax the condition?

First Scenario Where b Is Fixed

- Let $Q_{\mathbf{b}} \in \mathbb{R}^{m\times m}$ be the Householder transformation such that that

$$Q_{\mathbf{b}}^\top \mathbf{b} = \begin{bmatrix} b_0 \\ \mathbf{0} \end{bmatrix},$$

  where $b_0 = \|\mathbf{b}\|_2 \in \mathbb{R}$.

- Write

$$Q_{\mathbf{b}}^\top A = \begin{bmatrix} \mathbf{a}_{\mathbf{b}}^\top \\ A_{\mathbf{b}} \end{bmatrix},$$

  with $\mathbf{a}_{\mathbf{b}} \in \mathbb{R}^n$ and $A_{\mathbf{b}} \in \mathbb{R}^{(m-1)\times n}$.

- Denote the SVD of $A_{\mathbf{b}}$:

$$\underbrace{A_{\mathbf{b}} = U_{\mathbf{b}} \begin{bmatrix} \gamma_1 & & & & 0 \\ & \gamma_2 & & & 0 \\ & & \ddots & & \vdots \\ & & & \gamma_{n-1} & 0 \end{bmatrix} V_{\mathbf{b}}^\top}_{m=n} \quad \text{or} \quad \underbrace{A_{\mathbf{b}} = U_{\mathbf{b}} \begin{bmatrix} \gamma_1 & & & \\ & \gamma_2 & & \\ & & \ddots & \\ & & & \gamma_n \\ \mathbf{0} & & & \mathbf{0} \end{bmatrix} V_{\mathbf{b}}^\top}_{m>n}.$$

- For any $\mathbf{f} \in \mathbb{R}^n$, denote

$$(\mathbf{a}_{\mathbf{b}}^\top + b_0 \mathbf{f}^\top) V_{\mathbf{b}} = [f_1, f_2, \cdots, f_n].$$

Equivalence

- Define $A(\mathbf{f}) \in \mathbb{R}^{(n+1)\times n}$ by

$$A(\mathbf{f}) = \begin{bmatrix} f_1 & f_2 & \cdots & f_{n-1} & f_n \\ \gamma_1 & & & & \\ & \gamma_2 & & & \\ & & \ddots & & \\ & & & \gamma_{n-1} & \\ & & & & \gamma_n \end{bmatrix},$$

  - ◇ $\gamma_n = 0$ if $m = n$.

- Obtain the equivalence

$$\sigma(A + \mathbf{b}\mathbf{f}^\top) = \{\beta_1, \beta_2, \cdots, \beta_n\} \iff \sigma(A(\mathbf{f})) = \{\beta_1, \beta_2, \cdots, \beta_n\}.$$

- For each given $\mathbf{b}$ the matrix $A_{\mathbf{b}}$ is known and hence values of γ_i's are also known.
- To solve the ISVPrk for the case of $F = \mathbf{b}\mathbf{f}^\top$, it suffices to determine the values of $f_1, \ldots, f_n$.

First Necessary and Sufficient Condition

- Given any fixed $\mathbf{b} \in \mathbb{R}^m$, there exists a vector $\mathbf{f} \in \mathbb{R}^n$ such that

$$\sigma(A(\mathbf{f})) = \{\beta_1, \beta_2, \cdots, \beta_n\}$$

if and only if

$$\beta_i \geq \gamma_i \geq \beta_{i+1}, \quad i = 1, 2, \cdots, n, \tag{2}$$

where $\beta_{n+1} := 0$.

Ideas of Proof

- The necessity of the interlacing inequality (2) is a well known property of singular value decompositions. (Golub and Van Loan'96)
- Observe that

$$\mathcal{A}_{\mathbf{f}} := A(\mathbf{f})A(\mathbf{f})^\top = \begin{bmatrix} \sum_{i=1}^n f_i^2 & f_1\gamma_1 & f_2\gamma_2 & \cdots & f_{n-1}\gamma_{n-1} & f_n\gamma_n \\ f_1\gamma_1 & \gamma_1^2 & & & & \\ f_2\gamma_2 & & \gamma_2^2 & & & \\ \vdots & & & \ddots & & \\ f_{n-1}\gamma_{n-1} & & & & \gamma_{n-1}^2 & \\ f_n\gamma_n & & & & & \gamma_n^2 \end{bmatrix}$$

  is a bordered matrix in $\mathbb{R}^{(n+1)\times(n+1)}$.
- Consider the fact

$$\begin{aligned} \sigma(A(\mathbf{f})) = \{\beta_1, \cdots, \beta_n\} \quad &\Longleftrightarrow \quad \lambda(\mathcal{A}_{\mathbf{f}}) = \{\beta_1^2, \cdots, \beta_n^2, 0\} \\ &\Longleftrightarrow \quad p(\mu) := \mu \prod_{i=1}^n (\mu - \beta_i^2) - \det(\mu I - \mathcal{A}_{\mathbf{f}}) \equiv 0. \end{aligned}$$

- Note that

$$p(\mu) = \left(\mathrm{trace}(\mathcal{A}_{\mathbf{f}}) - \sum_{i=1}^{n} \beta_i^2 \right) \mu^n + \text{low degree terms in } \mu$$

  is a polynomial of degree at most n in μ.

- Note also that we can expand the determinant of $\mathcal{A}_{\mathbf{f}}$ and solve the equation,

$$\mu \prod_{j=1}^{n} (\mu - \beta_j^2) = (\mu - \sum_{i=1}^{n} f_i^2) \prod_{j=1}^{n} (\mu - \gamma_j^2) - \sum_{i=1}^{n} \left((f_i \gamma_i)^2 \prod_{\substack{j=1 \\ j \neq i}}^{n} (\mu - \gamma_j^2) \right). \tag{3}$$

- Need to consider four cases:
  - ◇ That all γ_k, $k = 1, \dots n$, are distinct and nonzero.
  - ◇ That $\gamma_1 > \cdots > \gamma_t > \gamma_{t+1} = \cdots = \gamma_n = 0$ for some integer t.
  - ◇ That the set $\{\gamma_1, \gamma_2, \cdots, \gamma_n\}$ consists of t many distinct non-zero elements.
  - ◇ That the set $\{\gamma_1, \gamma_2, \cdots, \gamma_n\}$ consists of $t + 1$ distinct elements including one zero.

- Will illustrate only the first case. (Boley and Golub'87)
  - ◇ For each k, set $\mu = \gamma_k^2$ in (3) to obtain

$$\gamma_k^2 \prod_{j=1}^{n} (\gamma_k^2 - \beta_j^2) = -(f_k \gamma_k)^2 \prod_{\substack{j=1 \\ j \neq k}}^{n} (\gamma_k^2 - \gamma_j^2).$$

  - ◇ f_k^2 is uniquely determined by

$$f_k^2 = -\frac{\prod_{j=1}^{n} (\gamma_k^2 - \beta_j^2)}{\prod_{\substack{j=1 \\ j \neq k}}^{n} (\gamma_k^2 - \gamma_j^2)}, \quad k = 1, \cdots, n. \tag{4}$$

  - ◇ The interlacing property (2) guarantees that the right hand side of (4) is nonnegative and hence real-valued f_k can be defined.
  - ◇ With this choice of $f_1, \cdots, f_n$, we see that $p(\mu)$ has $n+1$ zeros at $\mu = 0, \gamma_1^2, \cdots, \gamma_n^2$ and hence $p(\mu) \equiv 0$.
- For other cases, f_k can be defined slightly differently. (Chu and Chu'04)

Second Scenario Where b Is Relaxed

- Recall $\sigma(A_{\mathbf{b}}) = \{\gamma_1, \ldots, \gamma_n\}$ are determined by $Q_{\mathbf{b}}$ associated with the vector $\mathbf{b}$.
- If $\mathbf{b}$ is changed, then is the interlacing inequality (2).
  - ⋄ How much room can the inequality be adjusted by changing $\mathbf{b}$?

The Effect of $\mathbf{b}$

- Let $A \in \mathbb{R}^{m\times n}$ $(m \geq n)$ be given and fixed.
  - ◇ Denote

$$\sigma(A) = \{\alpha_1, \cdots, \alpha_n\}, \quad \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n.$$

- There exist a unit vector $\mathbf{b} \in \mathbb{R}^m$ and an orthogonal matrix $Q_{\mathbf{b}} \in \mathbb{R}^{m\times m}$ such that

$$Q_{\mathbf{b}}^\top \mathbf{b} = \begin{bmatrix} 1 \\ \mathbf{0} \end{bmatrix}, \quad Q_{\mathbf{b}}^\top A = \begin{bmatrix} \mathbf{a}_{\mathbf{b}}^\top \\ A_{\mathbf{b}} \end{bmatrix},$$

  with

$$\sigma(A_{\mathbf{b}}) = \begin{cases} \{\gamma_1, \cdots, \gamma_{n-1}\}, & \text{if } m = n, \\ \{\gamma_1, \cdots, \gamma_n\}, & \text{if } m > n. \end{cases}$$

  if and only if values γ_i satisfy the interlacing inequality

$$\alpha_i \geq \gamma_i \geq \alpha_{i+1}, \quad i = 1, \cdots, n, \tag{5}$$

  where $\alpha_{n+1} = 0$ and $\gamma_n = 0$ if $m = n$.

Ideas of Proof (Constructive)

- The necessity of (5) is due to the fact that $A_{\mathbf{b}}$ is a submatrix of A.
- Define

$$\underbrace{\tilde{A} = \begin{bmatrix} 0 & 0 & \cdots & & 0 \\ \gamma_1 & & & & \\ & \gamma_2 & & & \\ & & \ddots & & \vdots \\ & & & \gamma_{n-1} & 0 \end{bmatrix}}_{m=n} \quad \text{or} \quad \underbrace{\tilde{A} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \gamma_1 & & & \\ & \gamma_2 & & \\ & & \ddots & \\ & & & \gamma_n \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix}}_{m>n}.$$

  - ⋄ Pad the last row with $\mathbf{0}$'s, if $m > n + 1$.

- Consider the first scenario with $\tilde{A}$ and $\tilde{\mathbf{b}} = \begin{bmatrix} 1 \\ \mathbf{0} \end{bmatrix} \in \mathbb{R}^m$,
  - ⋄ There exists $\mathbf{c} \in \mathbb{R}^n$ such that

$$\sigma(\tilde{A} + \tilde{\mathbf{b}}\mathbf{c}^\top) = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}.$$

- Denote

$$\begin{aligned} A &= U_1 \Sigma V_1^\top, \\ \tilde{A} + \tilde{\mathbf{b}}\mathbf{c}^\top &= U_2 \Sigma V_2^\top. \end{aligned}$$

- Define

$$Q_{\mathbf{b}} := U_1 U_2^\top \quad \text{and} \quad \mathbf{b} := Q_{\mathbf{b}} \tilde{\mathbf{b}}.$$

- Observe the partition

$$Q_{\mathbf{b}}^\top A = (\tilde{A} + \tilde{\mathbf{b}}\mathbf{c}^\top)(V_2 V_1^\top) = \begin{bmatrix} \mathbf{a}_{\mathbf{b}}^\top \\ A_{\mathbf{b}} \end{bmatrix}.$$

- The desired properties are built in.

$$\mathbf{a}_{\mathbf{b}} = A^\top \mathbf{b} \quad (= V_1 V_2^\top \mathbf{c})$$

and

$$A_{\mathbf{b}} = \begin{bmatrix} \gamma_1 & & & & 0 \\ & \gamma_2 & & & \\ & & \ddots & & \vdots \\ & & & \gamma_{n-1} & 0 \end{bmatrix} V_2 V_1^\top \quad \text{or} \quad \begin{bmatrix} \gamma_1 & & & \\ & \gamma_2 & & \\ & & \ddots & \\ & & & \gamma_n \\ 0 & 0 & \cdots & 0 \end{bmatrix} V_2 V_1^\top .$$

Complete Characterization

- The following three statements are equivalent:
  1. The ISVPrk with $\ell = 1$ is solvable.
  2. For each $i = 1, \cdots, n$, there exists a value γ_i satisfying both inequalities

$$\alpha_i \geq \gamma_i \geq \alpha_{i+1}, \tag{6}$$
$$\beta_i \geq \gamma_i \geq \beta_{i+1}, \tag{7}$$

     where $\alpha_{n+1} := 0$ and $\beta_{n+1} := 0$.
  3. For each $i = 1, \cdots, n-1$,

$$\beta_{i+1} \leq \alpha_i \quad \text{and} \quad \alpha_{i+1} \leq \beta_i.$$

- Note that α_i's and β_i's do not necessarily satisfy any interlacing property.

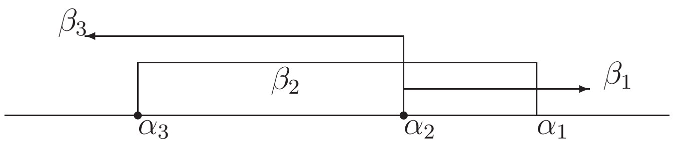

Figure 1: Feasible range of α_i's and β_i's for the case $n = 3$.

Proof

- By keeping the ordering $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$ and $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_n$, the equivalence of Statements 2 and 3 is obvious.
- Assume the ISVPrk has a rank one solution $F \in \mathbb{R}^{m\times n}$.
  - ⋄ There exists orthogonal matrix Q_F such that
$$Q_F^\top F = \begin{bmatrix} \mathbf{f}^\top \\ \mathbf{O} \end{bmatrix},$$
    with $\mathbf{f} \in \mathbb{R}^n$.
  - ⋄ Write
$$Q_F^\top A = \begin{bmatrix} \mathbf{a}_\mathbf{f}^\top \\ A_F \end{bmatrix},$$
    with $\mathbf{a}_\mathbf{f} \in \mathbb{R}^n$ and $A_F \in \mathbb{R}^{(m-1)\times n}$.
  - ⋄ Let
$$\sigma(A_F) = \begin{cases} \{\gamma_1, \gamma_2, \cdots, \gamma_{n-1}\} & \text{if } m = n, \\ \{\gamma_1, \gamma_2, \cdots, \gamma_n\}, & \text{if } m > n, \end{cases}$$
    with the descending order $\gamma_1 \geq \gamma_2 \geq \cdots$.
  - ⋄ A_F is a submatrix of both $Q_F^\top A$ and $Q_F^\top(A + F)$, the singular values of A_F interlace with those of both $Q_F^\top A$ and $Q_F^\top(A + F)$.
  - ⋄ The interlacing properties follow.

- Assume the interlacing inequality (6) holds.
  - $\diamond$ By relaxation, there exist $\mathbf{b} \in \mathbb{R}^m$ and $Q_{\mathbf{b}} \in \mathbb{R}^{m\times m}$ such that $Q_{\mathbf{b}}^\top \mathbf{b} = \begin{bmatrix} 1 \\ \mathbf{0} \end{bmatrix}$, $Q_{\mathbf{b}}^\top A = \begin{bmatrix} \mathbf{a}_{\mathbf{b}}^\top \\ A_{\mathbf{b}} \end{bmatrix}$, and $\sigma(A_{\mathbf{b}}) = \{\mu_1, \ldots, \mu_n\}$.
- With this $\mathbf{b}$, by the first scenario, a vector $\mathbf{f} \in \mathbb{R}^n$ can be constructed such that

$$\sigma(A + \mathbf{b}\mathbf{f}^\top) = \{\beta_1, \beta_2, \cdots, \beta_n\}.$$

Unsolvability

- When is the case of A where its singular values absolutely cannot be reassigned by any rank one matrices?
- Denote the multiplicity of distinct singular values $\alpha_1(A)$, $\cdots$, $\alpha_t(A)$ of A as $s_1, \cdots, s_t$. Then

$$\bigcap_{\text{rank}(F)\leq 1} \sigma(A+F) = \{\alpha_k(A) \text{ with algebraic multiplicity } (s_k - 2) \,|s_k > 2,\ 1 \leq k \leq t\}.$$

- Values in $\bigcap_{\text{rank}(F)\leq 1} \sigma(A+F)$ are those which are invariant under rank one update.

$$\bigcap_{\text{rank}(F)\leq 1} \sigma(A+F) = \emptyset \quad \Longleftrightarrow \quad s_k \leq 2,\ k = 1, \cdots, t.$$

Main Result

- Recall the problem.
  - Given
    - A matrix $A \in \mathbb{R}^{m\times n}$ $(m \geq n)$,
    - An integer $n \geq \ell > 0$, and
    - Real numbers $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_n \geq 0$,
  - Find a matrix $F \in \mathbb{R}^{m\times n}$ such that

$$\begin{aligned} \operatorname{rank}(F) &\leq \ell, \\ \sigma(A+F) &= \{\beta_1, \beta_2, \cdots, \beta_n\}. \end{aligned}$$

- The ISVPrk is solvable if and only if for each $i = 1, \cdots n - \ell$,

$$\beta_{i+\ell} \leq \alpha_i, \quad \text{and} \quad \alpha_{i+\ell} \leq \beta_i.$$

- The necessary condition is related to the classical Weyl inequality for singular values of sums of matrices.
- The simplicity of the condition is surprisingly pleasant.

Proof

- The case for $\ell = 1$ has already been established.
- Assume that the assertion is true for $\ell = k$.
  - ◇ Want to establish the case $\ell = k + 1 \leq n$.
- The necessity.
  - ◇ Any $F \in \mathbb{R}^{m \times n}$ with $\operatorname{rank}(F) \leq k + 1$ can be factorized as
$$F = F_1 + F_2,$$
    with $\operatorname{rank}(F_1) \leq k$ and $\operatorname{rank}(F_2) \leq 1$.
  - ◇ Denote
$$\sigma(A + F_1) = \{\gamma_1, \gamma_2, \cdots, \gamma_n\}.$$
    - ▷ By assumption,
$$\gamma_{i+k} \leq \alpha_i \quad \text{and} \quad \alpha_{i+k} \leq \gamma_i, \quad i = 1, \cdots, n - k.$$
    - ▷ As a rank one update of $A + F_1$,
$$\beta_{i+1} \leq \gamma_i \quad \text{and} \quad \gamma_{i+1} \leq \beta_i, \quad i = 1, \cdots, n - 1.$$
  - ◇ Together,
$$\beta_{i+k+1} \leq \alpha_i \quad \text{and} \quad \alpha_{i+k+1} \leq \beta_i, \quad i = 1, \cdots, n - k - 1.$$

- The sufficiency.
  - ◇ Note that

$$\left\{\begin{array}{ccc} \beta_{k+2} & \leq & \alpha_1 \\ \beta_{k+3} & \leq & \alpha_2 \\ & \vdots & \\ \beta_n & \leq & \alpha_{n-k-1} \end{array}\right. \quad \text{and} \quad \left\{\begin{array}{ccc} \alpha_{k+2} & \leq & \beta_1 \\ \alpha_{k+3} & \leq & \beta_2 \\ & \vdots & \\ \alpha_n & \leq & \beta_{n-k-1} \end{array}\right. .$$

  - ◇ By inspection that there exist γ_i, $i = 1, \cdots, n$, with

$$\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_n,$$

such that

$$\left\{\begin{array}{ccccc} \max\{\beta_1, \alpha_1\} & < & \gamma_1 & & \\ \max\{\alpha_{k+2}, \beta_3\} & \leq & \gamma_2 & \leq & \beta_1 \\ \max\{\alpha_{k+3}, \beta_4\} & \leq & \gamma_3 & \leq & \beta_2 \\ & & \vdots & & \\ \max\{\alpha_n, \beta_{n-k+1}\} & \leq & \gamma_{n-k} & \leq & \beta_{n-k-1} \\ \beta_{n-k} & \leq & \gamma_{n+1-k} & \leq & \beta_{n-k} \\ & & \vdots & & \\ \beta_{k+1} & \leq & \gamma_k & \leq & \beta_{k-1} \\ \beta_{k+2} & \leq & \gamma_{k+1} & \leq & \min\{\beta_k, \alpha_1\} \\ \beta_{k+3} & \leq & \gamma_{k+2} & \leq & \min\{\beta_{k+1}, \alpha_2\} \\ & & \vdots & & \\ \beta_n & \leq & \gamma_{n-1} & \leq & \min\{\beta_{n-2}, \alpha_{n-1-k}\} \\ & & \gamma_n & = & 0 \end{array}\right. , \quad \text{if } n-k < k+1,$$

or

$$\left\{\begin{array}{ccccc} \max\{\alpha_1, \beta_1\} & < & \gamma_1 & & \\ \max\{\alpha_{k+2}, \beta_3\} & \leq & \gamma_2 & \leq & \beta_1 \\ \max\{\alpha_{k+3}, \beta_4\} & \leq & \gamma_3 & \leq & \beta_2 \\ & & \vdots & & \\ \max\{\alpha_{2k}, \beta_{k+1}\} & \leq & \gamma_k & \leq & \beta_{k-1} \\ \max\{\alpha_{2k+1}, \beta_{k+2}\} & \leq & \gamma_{k+1} & \leq & \min\{\alpha_1, \beta_k\} \\ \max\{\alpha_{2k+2}, \beta_{k+3}\} & \leq & \gamma_{k+2} & \leq & \min\{\alpha_2, \beta_{k+1}\} \\ & & \vdots & & \\ \max\{\alpha_n, \beta_{k+J+2}\} & \leq & \gamma_{k+J+1} & \leq & \min\{\alpha_{J+1}, \beta_{k+J}\} \\ \beta_{k+J+3} & \leq & \gamma_{k+J+2} & \leq & \min\{\alpha_{J+2}, \beta_{K+J+1}\} \\ \beta_{k+J+4} & \leq & \gamma_{k+J+3} & \leq & \min\{\alpha_{J+3}, \beta_{k+J+2}\} \end{array}\right. \quad \text{if } n-k \geq k+1 \text{ with } J = (n-k)-(k+1).$$

- ◇ These values of γ_i satisfy

$$\gamma_{i+k} \leq \alpha_i, \quad \alpha_{i+k} \leq \gamma_i, \quad i = 1, \cdots, n-k,$$

and

$$\beta_{i+1} \leq \gamma_i, \quad \gamma_{i+1} \leq \beta_i, \quad i = 1, \cdots, n-1.$$

- ◇ By the inductive assumption, there exists a matrix $F_1 \in \mathbb{R}^{m \times n}$ such that

$$\mathrm{rank}(F_1) \leq k, \quad \sigma(A + F_1) = \{\gamma_1, \gamma_2, \cdots, \gamma_n\}.$$

- ◇ By rank one update, there exists a matrix $F_2 \in \mathbb{R}^{m \times n}$ such that

$$\mathrm{rank}(F_2) \leq 1, \quad \sigma((A + \underbrace{F_1) + F_2}_{F}) = \{\beta_1, \beta_2, \cdots, \beta_n\}.$$

Numerical Algorithm

- The proofs given above can be implemented as numerical means to compute a solution for the ISVPrk.
- Once a rank one update algorithm is available, the entire induction process can easily be implemented in any programming language that supports a routine to call itself recursively.
  - ◇ The main feature in the routine should be a single divide and conquer mechanism.
  - ◇ See the pseudo-code.

```
function [F]=svd_update(A,alpha,beta,ell);

if ell == 1                                % The rank one case
    [b,f] = svd_update_rank_one(A,alpha,beta); % Algorithm 4.1
    F = b*f';

else
    k = ell-1;                             % The general case
    choose gamma(1) >= gamma(2) >= ... >= gamma(n) such that

        gamma(i+k) <= alpha(i); alpha(i+k) <= gamma(i); i = 1, ..., n-k
        beta(i+1) <= gamma(i); gamma(i+1) <= beta(i); i = 1, ..., n-1

    [F1] = svd_update(A,alpha,gamma,k);

    [b,f] = svd_update_rank_one(A+F1,gamma,beta);
    F2 = b*f';

    F = F1+F2;

end
```

Table 1: A pseudo-MATLAB program for the recursive algorithm.

Singular Value Reassignment with Rank One Update

1. Compute the singular value decomposition
$$A = U_1 \Sigma V_1^\top$$
and denote $\sigma(A) = \{\alpha_1, \cdots, \alpha_n\}$ with $\alpha_1 \geq \cdots \geq \alpha_n$.
2. For $i = 1, \cdots, n-1$, check to see if
$$\beta_{i+1} \leq \alpha_i \quad \text{and} \quad \alpha_{i+1} \leq \beta_i.$$
If not, stop.
3. For $i = 1, \cdots, n-1$, define
$$\gamma_i := \frac{\min\{\alpha_i, \beta_i\} + \max\{\alpha_{i+1}, \beta_{i+1}\}}{2}.$$
and
$$\gamma_n := \begin{cases} 0, & \text{if } m = n, \\ \frac{\min\{\alpha_n, \beta_n\}}{2}, & \text{otherwise.} \end{cases}$$
4. If $\gamma_1 > \cdots > \gamma_n > 0$, define for each $k = 1, \cdots, n$
$$c_k := \sqrt{-\frac{\prod_{j=1}^{n}(\gamma_k^2 - \alpha_j^2)}{\prod_{\substack{j=1 \\ j \neq k}}^{n}(\gamma_k^2 - \gamma_j^2)}};$$
else modify c_k's according to the remaining three cases.

5. Define

$$\hat{A} := \begin{cases} [\mathbf{c}^\top; \mathrm{diag}(\gamma_1, \cdots, \gamma_{n-1}), zeros(m-1,1)], & \text{if } m = n, \\ [\mathbf{c}^\top; diag(\gamma_1, \cdots, \gamma_n)], & \text{if } m = n+1, \\ [\mathbf{c}^\top; diag(\gamma_1, \cdots, \gamma_n); zeros(m-n-1, n)], & \text{otherwise.} \end{cases}$$

6. Compute the singular value decomposition

$$\hat{A} := U_2 \Sigma V_2^\top.$$

7. Define

$$\begin{aligned} \mathbf{b} &:= U_1 U_2(1,:)^\top, \\ V_{\mathbf{b}} &:= V_1 V_2^\top, \\ \mathbf{a}_{\mathbf{b}} &:= A^\top \mathbf{b} \text{ (or } V_{\mathbf{b}}\mathbf{c}). \end{aligned}$$

8. If $\gamma_1 > \cdots > \gamma_n > 0$, define for each $k = 1, \cdots, n$

$$\hat{f}_k := \sqrt{-\frac{\prod_{j=1}^{n}(\gamma_k^2 - \beta_j^2)}{\prod^{n}_{\substack{j=1 \\ j \neq k}} (\gamma_k^2 - \gamma_j^2)}};$$

else modify $\hat{f}_k$'s according to the remaining three cases.

9. Define

$$\mathbf{f} := V_{\mathbf{b}} \hat{\mathbf{f}} - \mathbf{a}_{\mathbf{b}}.$$

Numerical Stability

- Many choices in Step 3.
- The computation of $\mathbf{c}$ and $\hat{\mathbf{f}}$ is numerically unstable.
- Similar remedy for Jacobi inverse eigenvalue problems are available. (de Boor and Golud'78, Gragg and Harrod'86), if so desired.

Minimum Low Rank Approximation

- Given
  - ◇ A matrix $A \in \mathbb{R}^{m\times n}$ $(m \geq n)$,
  - ◇ An integer $n \geq \ell > 0$, and
  - ◇ Real numbers $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_n \geq 0$,
- Find a matrix $F \in \mathbb{R}^{m\times n}$ such that

$$\begin{array}{rcl} \operatorname{rank}(F) & \leq & \ell, \\ \sigma(A+F) & = & \{\beta_1, \beta_2, \cdots, \beta_n\}, \\ \text{and} & & \|F\|_F \text{ is minimized.} \end{array}$$

Conclusion

- We have provided a rigorous theoretic basis for the singular value reassignment problem.
- A simple yet both necessary and sufficient condition () completely settles the issue of solvability for the ISVPrk.
- Our proof is constructive so it can be exploited to provide a possible means for computing the solution numerically.
- Using the rank one case as the building block, the algorithm features a divide-and-conquer scheme.
- The numerical procedure as it stands now might not be stable when there are close-by singular values. Remedies are available in the literature. We mainly concentrates on the general ideas.