On the Differential Equation $\frac{dX}{dt} = [X, k(X)]$ where k Is a Toeplitz Annihilator

Moody T. Chu

Department of Mathematics
North Carolina State University
Raleigh, North Carolina 27695-8205

Inverse Eigenvalue Problem (IEP)

- Given
  - Real symmetric matrices $A_0, A_1, \ldots, A_n \in R^{n\times n}$;
  - Real numbers $\lambda_1^* \geq \ldots \geq \lambda_n^*$,
- Find
  - Values of $c := (c_1, \ldots, c_n)^T \in R^n$,
- Such that
  - Eigenvalues of the matrix

    $$A(c) := A_0 + c_1 A_1 + \ldots + c_n A_n$$

    are precisely $\lambda_1^*, \ldots, \lambda_n^*$.

Existence Question

- Not always will the (IEP) have a solution.
- Inverse Toeplitz Eigenvalue Problem (ITEP):
  - A special case of the (IEP) where $A_0 = 0$ and $A_k := (A_{ij}^{(k)})$ with

$$A_{ij}^{(k)} := \begin{cases} 1, & \text{if } |i-j| = k-1; \\ 0, & \text{otherwise.} \end{cases}$$

  - Existence question for (ITEP) remains open for $n \geq 5$.

Notation

- $\mathcal{S}(n)$:= The subspace of all symmetric matrices in $R^{n\times n}$.
- $\mathcal{O}(n)$:= The manifold of all orthogonal matrices in $R^{n\times n}$.
- $\mathcal{T}$:= The subspace of all Toeplitz matrices in $\mathcal{S}(n)$.
- $\Lambda := diag\{\lambda_1, \ldots, \lambda_n\}$.
- $\mathcal{M}(\Lambda) := \{Q\Lambda Q^T | Q \in \mathcal{O}(n)\}$
  - Contains all matrices in $\mathcal{S}(n)$ whose eigenvalues are precisely $\lambda_1^*, \ldots, \lambda_n^*$.
- $\mathcal{A} := \{A(c) | c \in R^n\}$.
  - Solving the (IEP) is equivalent to finding an intersection of the two sets $\mathcal{M}(\Lambda)$ and $\mathcal{A}$.

A Descent Method for ITP

- Assume
  - Matrices $A_1, \ldots, A_n$ are mutually orthonormal.
  - Matrix A_0 is perpendicular to all A_k, for $k = 1, \ldots, n$.
- The distance from X to the affine subspace $\mathcal{A}$ is

$$dist(X, \mathcal{A}) = \|X - (A_0 + P(X))\|$$

where

$$P(X) = \sum_{k=1}^{n} < X, A_k > A_k.$$

- Approach the (IEP) by solving the optimization problem:

$$\begin{array}{rl} \text{Minimize} & F(Q) := \frac{1}{2}\|Q^T \Lambda Q - A_0 - P(Q^T \Lambda Q)\|^2 \\ \text{Subject to} & Q \in \mathcal{O}(n). \end{array}$$

Compute the Projected Gradient

- The gradient ∇F can be calculated:

$$\nabla F(Q) = 2\Lambda Q\{Q^T\Lambda Q - A_0 - P(Q^T\Lambda Q)\}.$$

- Projection is easy because:

$$\begin{aligned} R^{n\times n} &= T_Q\mathcal{O}(n) \oplus T_Q\mathcal{O}(n)^{\perp} \\ &= Q\mathcal{S}(n)^{\perp} \oplus Q\mathcal{S}(n) \end{aligned}$$

- The vector field

$$\frac{dQ}{dt} = Q[Q^T\Lambda Q, A_0 + P(Q^T\Lambda Q)]$$

where

$$[A, B] := AB - BA$$

defines a steepest descent flow on the manifold $\mathcal{O}(n)$ for the objective function $F(Q)$.

A Descent Flow for the IEP

- Define
$$X(t) := Q(t)^T \Lambda Q(t).$$
- $X(t)$ is governed by:
$$\frac{dX}{dt} = [X, [X, A_0 + P(X)]].$$
- Starting with any given $X(0) \in \mathcal{M}(\Lambda)$, the solution $X(t)$ of the initial value problem will
  - Stay on the surface $\mathcal{M}(\Lambda)$.
  - Move in the steepest descent direction to minimize $dist(X(t), \mathcal{A})$.

- Suppose
  - $X(t) \longrightarrow \hat{X}$ as $t \longrightarrow \infty$.
  - $\hat{X}$ is also in $\mathcal{A}$.

  Then $c_i =< \hat{X}, A_i >$ for $i = 1, \ldots n$ is a putative solution to the (IEP).

- UNFORTUNATELY,œ the flow $X(t)$ for the (ITEP) sometimes converges to a stable equilibrium point that is not Toeplitz.

A New Approach

- To stay on the surface $\mathcal{M}(\Lambda)$, a differential equation must take the form
$$\frac{dX}{dt} = [X, k(X)]$$
where $k : \mathcal{S}(n) \longrightarrow \mathcal{S}(n)^{\perp}$.
- Require k to be a linear Toeplitz annihilator:
  - $k(X) = 0$ if and only if $X \in \mathcal{T}$.

- What is the idea?
  - Suppose all elements in Λ are distinct.
  - $[X, k(X)] = 0$ if and only if $k(X)$ is a polynomial of X.
  - $k(X) \in \mathcal{S}(n) \cap \mathcal{S}(n)^{\perp} = \{0\}$.
  - $||X(t)|| = ||\Lambda||$ for all $t \in R$.
  - A bounded flow on a compact set must have a non-empty ω-limit set.
- Can such a k be defined?
  - The simpliest choice:

$$k_{ij} := \begin{cases} x_{i+1,j} - x_{i,j-1}, & \text{if } 1 \leq i < j \leq n \\ 0, & \text{if } 1 \leq i = j \leq n \\ x_{i,j-1} - x_{i+1,j}, & \text{if } 1 \leq j < i \leq n \end{cases}$$

The Differential Equation

- Only consider
  - The upper triangular part of a matrix.
  - $n = 3$.
- The differential equation is invariant under the translation $X + \sigma I$. Thus
  - Assume

$$\lambda_1 + \lambda_2 + \lambda_3 = 0.$$

  - Eliminate one variable

$$x_{22} = -x_{11} - x_{33}.$$

- The differential system is equivalent to

$$
\begin{aligned}
\frac{dx_{11}}{dt} &= 4x_{12}x_{11} + 2x_{12}x_{33} - 2x_{13}x_{23} + 2x_{13}x_{12}, \\
\frac{dx_{12}}{dt} &= -4x_{11}^2 - 4x_{11}x_{33} - 2x_{13}x_{33} - x_{13}x_{11} \\
&\quad -x_{33}^2 - x_{23}^2 + x_{23}x_{12}, \\
\frac{dx_{13}}{dt} &= 3x_{11}x_{23} + 3x_{12}x_{33}, \\
\frac{dx_{23}}{dt} &= x_{23}x_{12} - x_{12}^2 - 4x_{11}x_{33} - x_{11}^2 - 4x_{33}^2 \\
&\quad -2x_{13}x_{11} - x_{13}x_{33}, \\
\frac{dx_{33}}{dt} &= 2x_{13}x_{23} - 2x_{13}x_{12} + 4x_{23}x_{33} + 2x_{11}x_{23}.
\end{aligned}
$$

Critical Points

- The vector field is a system of homogeneous polynomials of degree 2.
  - No isolated equilibrium for the differential system.
- Only two kinds of real equilibria (by using the theory of Gröbner bases):
  - $(c_1, 0, -3c_1, 0, c_1)$, $c_1 \in R$.
  - $(0, c_2, c_3, c_2, 0)$, $c_2, c_3 \in R$.
- Local stability:
  - Eigenvalues at $(c_1, 0, -3c_1, 0, c_1)$ are

$$0,\ \pm 3\sqrt{6}c_1,\ \pm 6\sqrt{2}|c_1|i.$$

    * This kind of equilibrium can never be stable.
    * Near such an equilibrium, there should be periodic solutions.
    * The number of zero eigenvalue indicates the dimension of the manifold of equilibria.

– Eigenvalues at $(0, c_2, c_3, c_2, 0)$ are

$$0,\ 0,\ 6c_2,\ 2(c_2 + c_3),\ 2(c_2 - c_3).$$

∗ The equilibrium can be stable only if

$$c_2 < 0 \ \text{ and } \ |c_2| \geq c_3.$$

∗ The equilibrium corresponds to a Toeplitz matrix

$$\begin{bmatrix} 0 & c_2 & c_3 \\ c_2 & 0 & c_2 \\ c_3 & c_2 & 0 \end{bmatrix}$$

whose eigenvalues are:

$$\frac{c_3 - \sqrt{c_3^2 + 8c_2^2}}{2},\ -c_3,\ \frac{c_3 + \sqrt{c_3^2 + 8c_2^2}}{2}$$

in ascending order.

∗ The case $|c_2| = c_3$ corresponds to multiple eigenvalues.

Invariant Sets

- Due to the homogeneity,
  - If $X(t)$ is a solution, so is $Y(t) := \frac{1}{\alpha}X(\frac{t}{\alpha})$ for any real constant α.
  - If $\mathcal{I}$ is any set invariant under the differential equation, so is the set $\alpha\mathcal{I}$.
- A matrix is said to be
  - (Skew-) Persymmetric if it is (skew-) symmetric about the NE-SW diagonal.
- The subspace $\mathcal{W}$ of all persymmetric matrices in $\mathcal{S}(n)$ is invariant.
  - $\mathcal{W}$ is a 3-dimensional subspace:

$$\mathcal{W} = \{(x_{11}, x_{12}, x_{13}, x_{12}, x_{11}) | x_{11}, x_{12}, x_{13} \in R\}.$$

- The intersection of$\mathcal{W}$ and $\mathcal{M}(\Lambda)$ consists of three "ellipses":

$$\begin{aligned}(x_{11} - \frac{\lambda_3}{4})^2 + \frac{1}{2}x_{12}^2 &= \frac{(2\lambda_1 + \lambda_3)^2}{16}, \\ x_{13} &= x_{11} - \lambda_3;\end{aligned}$$

$$\begin{aligned}(x_{11} - \frac{\lambda_1}{4})^2 + \frac{1}{2}x_{12}^2 &= \frac{(\lambda_1 + 2\lambda_3)^2}{16}, \\ x_{13} &= x_{11} - \lambda_1;\end{aligned}$$

$$\begin{aligned}(x_{11} + \frac{\lambda_1 + \lambda_3}{4})^2 + \frac{1}{2}x_{12}^2 &= \frac{(\lambda_1 - \lambda_3)^2}{16}, \\ x_{13} &= x_{11} + \lambda_1 + \lambda_3.\end{aligned}$$

  - The projections of these ellipses onto the (x_{11}, x_{12})-plane must be such that one circumscribes the other two.
  - For $n = 3$ the (ITEP) has exactly
    - Four real solutions if all given eigenvalues are distinct.
    - Two real solutions if one eigenvalue has multiplicity 2.

Flows on $\mathcal{W}$

- The differential equation restricted on $\mathcal{W}$ is given by:

$$\begin{aligned}\frac{dx_{11}}{dt} &= 6x_{11}x_{12},\\ \frac{dx_{12}}{dt} &= -9x_{11}^2 - 3x_{11}x_{13},\\ \frac{dx_{13}}{dt} &= 6x_{11}x_{12}.\end{aligned}$$

- $\mathcal{W}$ itself consists of layers of 2-dimensional invariant affine subspaces:
  - Each affine subspace is determined by

$$x_{13} = x_{11} + c_4$$

    for a certain real constant c_4.
  - For any given c_4, the integral curves on the invariant affine subspace are determined by

$$(x_{11} + \frac{c_4}{4})^2 + \frac{1}{2}x_{12}^2 = c_5^2$$

    for real constants c_5.

- These elliptic orbits are concentric with the center $(-\frac{c_4}{4}, 0, \frac{3c_4}{4})$ which is an equilibrium of the first kind.
  - There are periodic solutions near that equilibrium.
  - For large enough c_5^2, a non-periodic solution of will converge as is shown.
    - The limit point corresponds to an equilibrium of the second kind

$$(0, -\sqrt{2c_5^2 - \frac{c_4^2}{8}}, c_4, -\sqrt{2c_5^2 - \frac{c_4^2}{8}}, 0).$$

    - The limit point can be stable for the entire system only if

$$c_5^2 \geq \frac{9c_4^2}{16}.$$

- For any given $\{\lambda_1, \lambda_2, \lambda_3\}$, the surface $\mathcal{M}(\Lambda)$ can have one and only one equilibrium which is stable for the differential system.

Orbital Stability

- The existence of periodic solutions is disappointing.
- A computer plot of $x_{12}(t)$ versus $x_{11}(t)$ for a single trajectory where
  - Initial values $x_{11}(0) = 1.0, x_{12}(0) = 1.0, x_{13} = -3.0, x_{14} = 1.0, x_{15} = 1.0$ indicate the true trajectory should be an ellipse.
  - Interval of integration is $0 \leq t \leq 11.05$.
  - A variable-step variable-order numerical method with high accuracy of error control ($\leq 10^{-14}$) fails to stay close to the ellipse.
- Calculate the characteristic exponents of the linearized system:
  - The period is estimated to be $T = 1.04719755120$ (a ccurate up to the 11-th digit).
  - The fundamental matrix $\Phi(t) = P(t)e^{tR}$ for the linear system is calculated with $\Phi(0) = I$.

- The corresponding characteristic exponents (eigenvalues of R) are estimated to be:

$$\pm 7.8998,\ \pm 2.0222 \times 10^{-5},\ 4.5297 \times 10^{-14}.$$

- The first positive characteristic exponent clearly indicates the orbital unstability.

• Although the numerical solution fails to track down the elliptic orbit, it stays close to the surface $\mathcal{M}(\Lambda)$.

  - See the plot of the difference of eigenvalues (measured in the 2-norm) between $X(0)$ and $X(t)$.The error is certainly acceptable within machine roundoff.
  - Although the numerical solution is meaningless to the original initial value problem, the final false limit point does solve the (ITEP).

More Questions

- Are there any invariant sets other than the ones we have found?
- With $X(0) = \Lambda$, the solution $X(t)$ displays a special feature — diagonals of $X(t)$ alternate symmetry of evenness with oddness. Does this mean anything?
- Starting with $X(0) = \Lambda$, the solution flow has been observed numerically to always converge a symmetric Toeplitz matrix as $t \longrightarrow \infty$. How to argue analytically that this is the case?
- How much can the understanding for $n = 3$ be generalized to higher dimensional case?