On an Adaptive Control Algorithm for Adaptive Optics Applications

by

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Outline

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- Adaptive Optics System
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- Adaptive Optics Control
  - ◇ An Ideal Control
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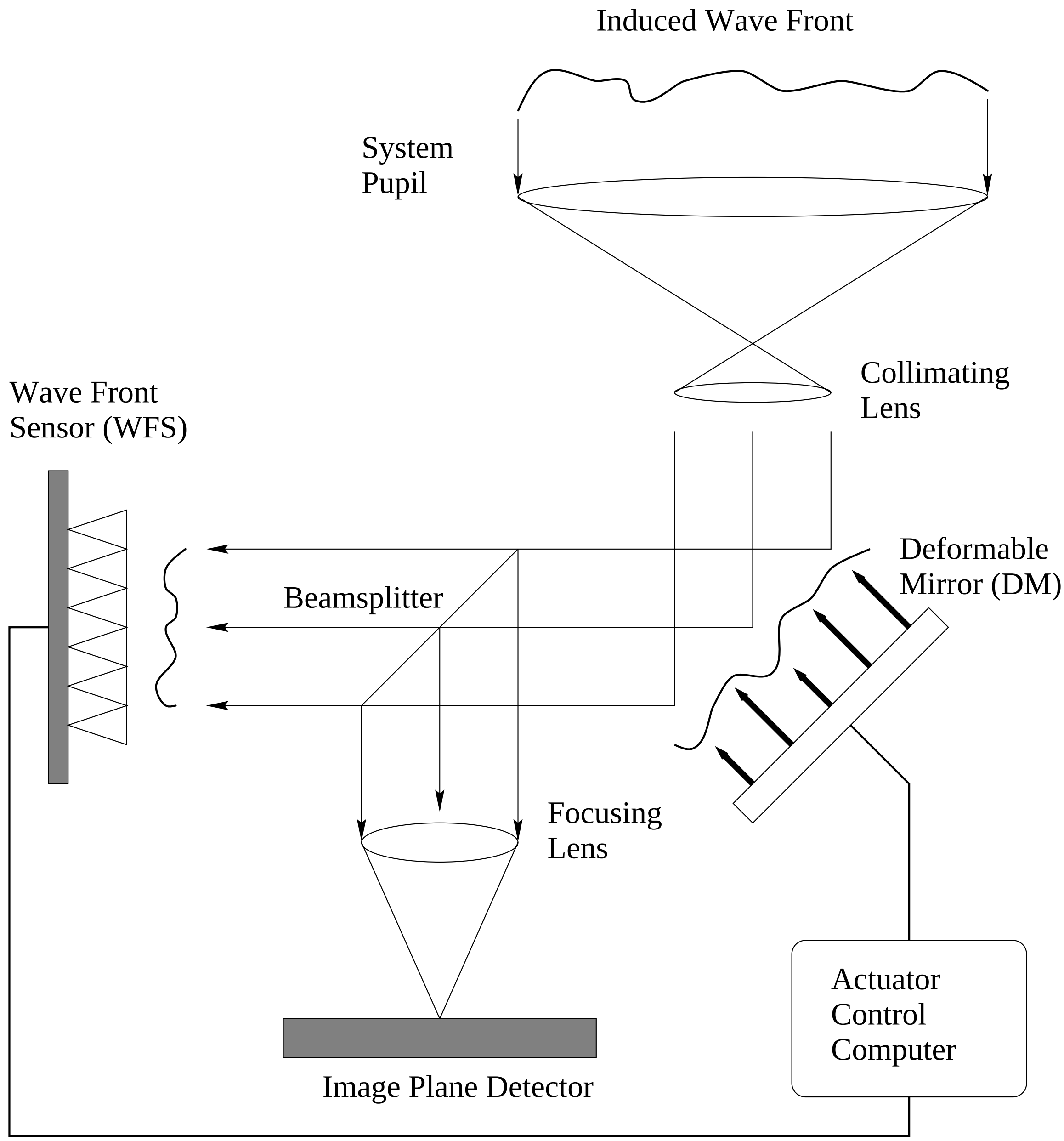
Atmospheric Imaging Computation

- Purpose:
  - ◇ To compensate for the degradation of astronomical image quality caused by the effects of atmospheric turbulence.
- Two stages of approach:
  - ◇ Partially nullify optical distortions by a deformable mirror (DM) operated from a closed-loop adaptive optics (AO) system.
  - ◇ Minimize noise or blur via off-line post-processing deconvolution techniques (not this talk).
- Challenges:
  - ◇ Atmospheric turbulence can only be measured adaptively.
  - ◇ Need theory to pass atmospheric measurements to knowledge of actuating the DM.
  - ◇ Require fast performance of large-scale data processing and computations.

A Simplified AO System



Basic Notation

- Three quantities:
  - ◇ $\phi(t)$ = turbulence-induced phase profile at time t.
  - ◇ $a(t)$ = deformable mirror (DM) actuator command at time t.
  - ◇ $s(t)$ = wavefront slope sensor (WFS) measurement at time t and with no correction.
- Two transformations:
  - ◇ H := transformation from actuator commands to resulting phase profile adjustments.
  - ◇ G := transformation from actuator commands to slope sensor measurement adjustments.

From Actuator to DM Surface

- H is used to describe the DM surface change due to the application of actuators.
- $r_i(\vec{x})$ = influence function on the DM surface at position $\vec{x}$ with an unit adjustment to the ith actuator.
- Assuming m actuators and linear response of actuators to the command, model the DM surface by

$$\hat{\phi}(\vec{x}, t) = \sum_{i=1}^{m} a_i(t) r_i(\vec{x}).$$

  ◇ Sampled at n DM surface positions, can write

$$\boxed{\hat{\phi}(t) = Ha(t)}$$

  ▷ $H = (r_i(\vec{x}_j)) \in R^{n \times m}$.

  ▷ $\hat{\phi}(t) = [\hat{\phi}(\vec{x}_1, t), \ldots, \hat{\phi}(\vec{x}_n, t)]^T \in R^n$ = discrete corrected phase profile at time t.

From Actuator to WFS Measurement

- G is used to describe the WFS slope measurement associated with the actuator command a.
- Consider the H-WFS model where

$$s_j(t) := -\int d\vec{x}(\nabla W_j(\vec{x})\phi(\vec{x},t), \quad j = 1, \ldots, \ell.$$

  ◇ W_j = given specifications of jth subaperture.

- The measurement corresponding to $\hat{\phi}(\vec{x},t)$ would be

$$\hat{s}_j(t) = \sum_{i=1}^{m} \underbrace{\left(-\int d\vec{x}(\nabla W_j(\vec{x})r_i(\vec{x})\right)}_{G_{ji}} a_i(t).$$

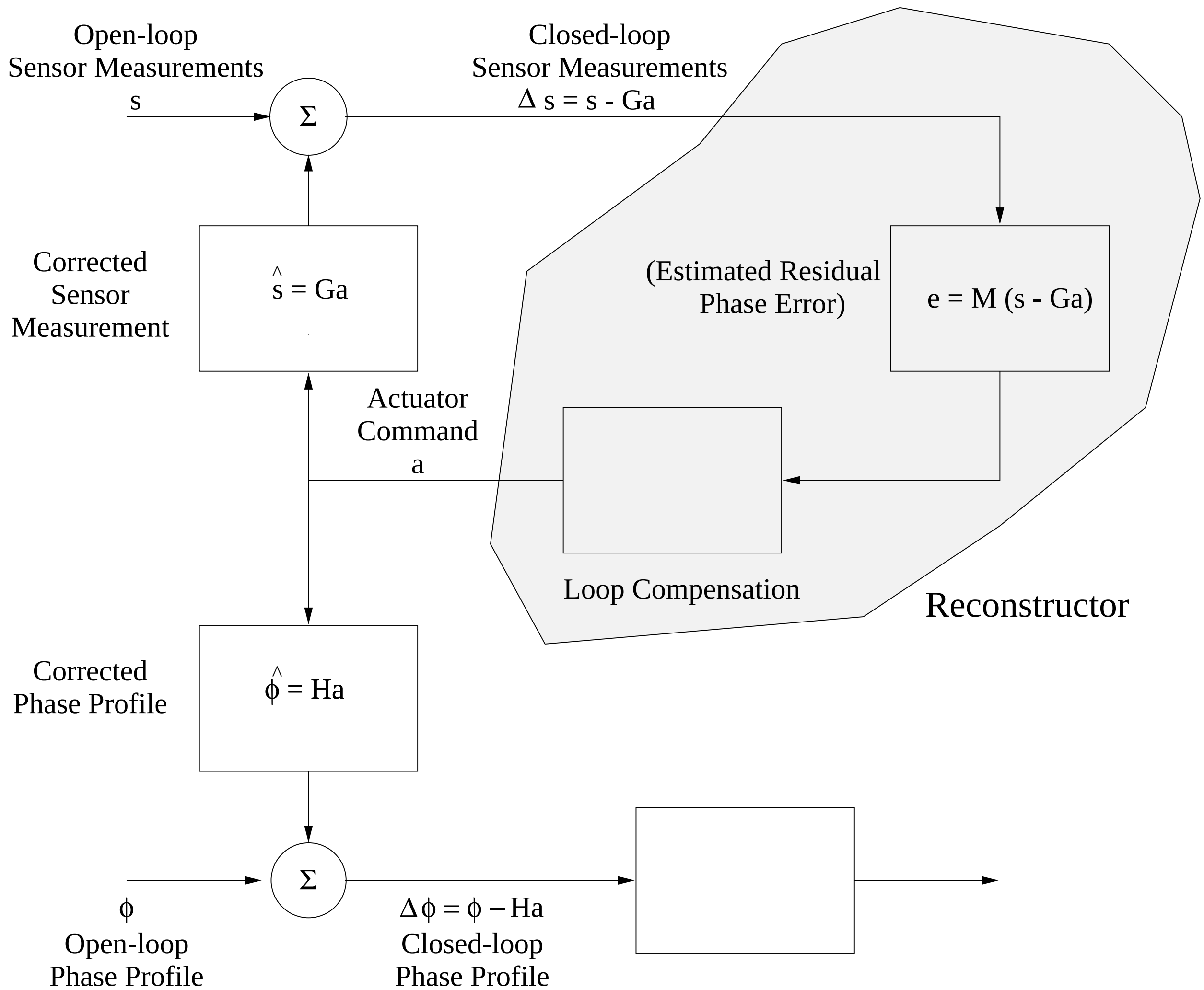
  ◇ Can write

$$\boxed{\hat{s}(t) = Ga(t)}$$

  where $G = [G_{ij}] \in R^{\ell \times m}$.

  ◇ The DM actuators are *not* capable of producing the exact wavefront phase $\phi(\vec{x},t)$ due to its finiteness of degrees of freedom. So $\hat{s} = Ga$ is not an exact measurement.

A Closed-loop AO Control Model



What is Available?

- Two residuals that are available in a *closed-loop* AO system:
  - ◇ $\Delta\phi(t) := \phi(t) - Ha(t)$
    - ▷ Represents the residual phase error remaining after the AO correction.
    - ▷ Also means instantaneous closed-loop wavefront distortion at time t.
  - ◇ $\Delta s(t) := s(t) - Ga(t)$
    - ▷ Represents feedback applied to $s(t)$ by DM actuator adjustment.
    - ▷ Also means *observable* wavefront sensor measurement at time t.
- In practice, there is a servo lag or delay in time Δt, i.e., it is likely
  - ◇ $\Delta\phi(t) := \phi(t) - Ha(t - \Delta t)$.
  - ◇ $\Delta s(t) := s(t) - Ga(t - \Delta t)$.

  Thus the data collected are not perfect.

Open-loop Model

- Assume a linear relationship between open-loop WFS measurement s and turbulence-induced phase profile ϕ:

$$\boxed{s = W\phi + \epsilon}. \tag{1}$$

  - ◇ ϵ = measurement noise with mean zero.
  - ◇ In the H-WFS model, W represents a quadrature of the integral operator evaluated at designated positions $\vec{x}_j$, $j = 1, \dots n$.

- Want to estimate ϕ using $\tilde{\phi}$ from the model

$$\tilde{\phi} = E_{open} s$$

so that the variance

$$\mathcal{E}[\|\phi - \tilde{\phi}\|^2]$$

is minimized.

  - ◇ The wave front reconstruction matrix E_{open} is given by
$$E_{open} = \mathcal{E}[\phi s^T](\mathcal{E}[ss^T])^{-1}.$$
  - ◇ For unbiased estimation, need to enforce the condition that $E_{open} W = I$.

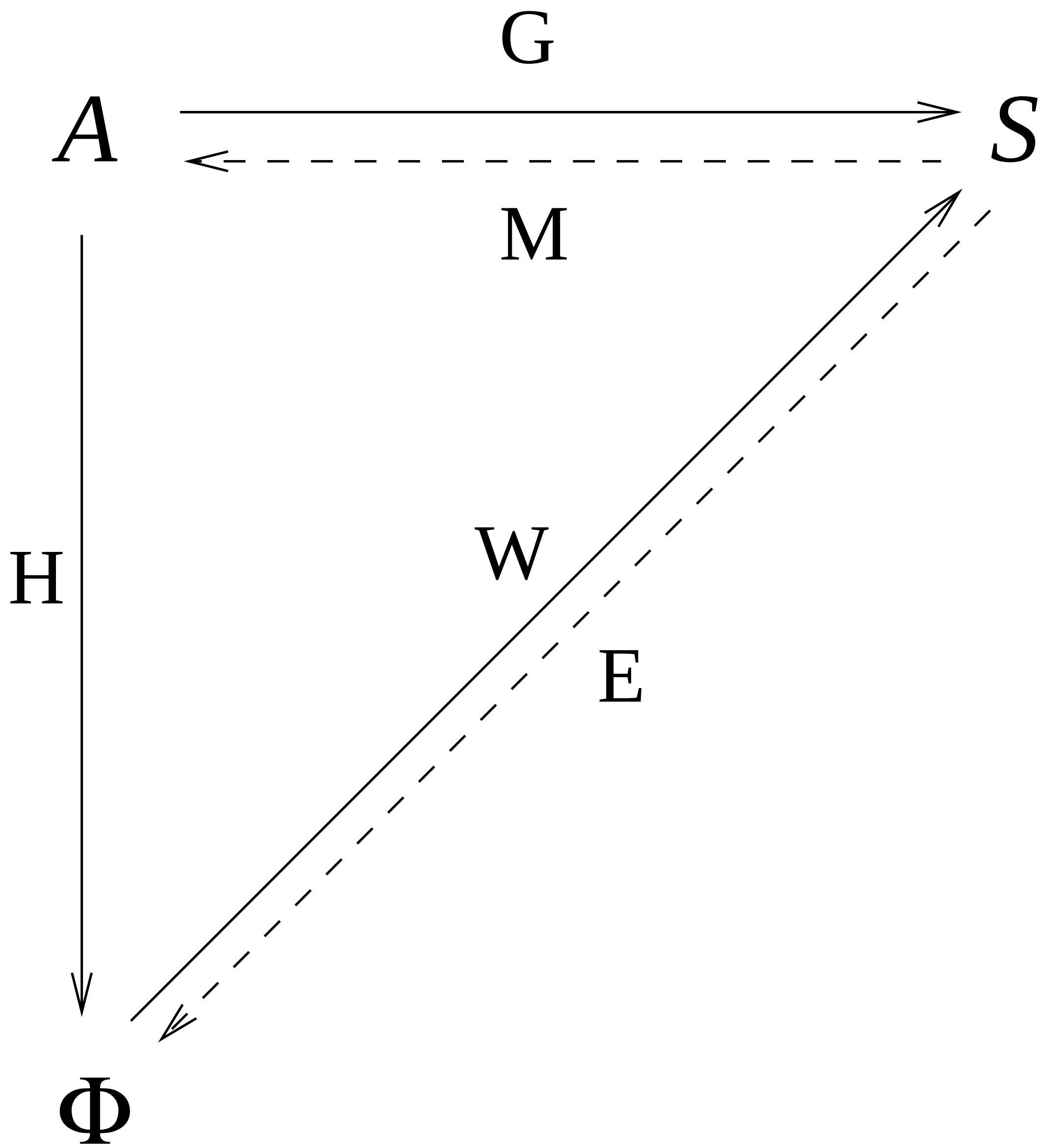
Closed-loop Model

- For the H-WFS model, it is reasonable to assume the relationship
$$WH = G. \tag{2}$$
- Then
$$\begin{aligned} s &= W\phi + \epsilon \\ &= W(Ha + \Delta\phi) + \epsilon \\ &= WHa + (W\Delta\phi + \epsilon). \end{aligned}$$
It follows that
$$\boxed{\Delta s = W\Delta\phi + \epsilon}. \tag{3}$$
  - ◇ The closed-loop relationship (3) is identical to the open-loop relationship (1).
- Can estimate the residual phase error $\Delta\phi(t)$ using $\Delta\tilde{\phi}(t)$ from the model
$$\Delta\tilde{\phi} = E_{closed}\Delta s$$
  - ◇ E_{closed} = wavefront reconstruction matrix.
  - ◇ For unbiased estimation, it requires that $E_{closed}W = I$. Hence
$$E_{closed}G = e_{closed}(WH) = H.$$

Actuator Control

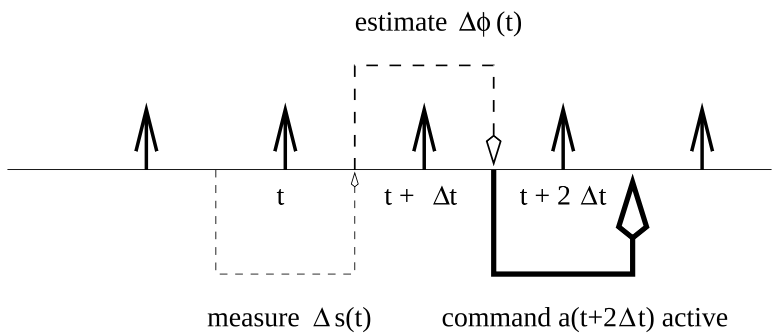
- An Ideal Control:
  - ◇ $\Delta\phi$ = residual error after DM correction by current command a_c.
  - ◇ New command a_+ should reduce the residual error, i.e., want to
  $$\min_a \|Ha - \phi\|.$$
  - ◇ Define $\Delta a := a_+ - a_c$, then want to
  $$\min_{\Delta a} \|H\Delta a - \Delta\phi\|.$$
  - ◇ But $\Delta\phi$ is not observable directly. It has to be estimated from Δs.
- Estimating Δa directly from Δs:
$$\boxed{\Delta a = M\Delta s} \qquad (4)$$

An Inverse Problem



Actuator Control with Temporary Latency

- Due to finite bandwidth of the control loop, Δs is not immediately available.
- Time line for the scenario of a 2-cycle delay,



- ARMA control scheme:

$$a(t+2\Delta t) \;:=\; \sum_{k=0}^{p} c_k a(t+(1-k)\Delta t) + \sum_{j=0}^{q} b_j M_j \Delta s(t - j\Delta t).$$

$$\boxed{a^{(r+2)} = \Sigma_{k=0}^{p} c_k a^{(r+1-k)} + \Sigma_{j=0}^{q} b_j M_j \Delta s^{(r-j)}, r = 0, 1, \ldots}$$

Expected Effect on the AO System

- Suppose
  - ◇ $\mathcal{E}[s(t)]$ is independent of time t throughout the cycle of computation.
  - ◇ Matrix $\Sigma_{j=0}^{q} b_j M_j$ is of full column rank.
- Then
  - ◇ The WFS feedback measurement $\Delta s^{(n)}$ is eventually nullified by the actuators, i.e.,

$$\mathcal{E}[s] = G \lim_{n\to\infty} \mathcal{E}[a^{(n)}].$$

  - ◇ The expected residual phase error is inversely related to the expected WFS measurement noise ϵ via

$$0 = W \lim_{n\to\infty} \mathcal{E}[\Delta\phi^{(n)}] + \mathcal{E}[\epsilon].$$

- Compare with the ideal control:
  - ◇ Even if $\mathcal{E}[\epsilon] = 0$, not necessarily $\mathcal{E}[\| \lim_{n\to\infty} \Delta\phi_n\|^2]$ will be small because W has non-trivial null space.

Almost Sure Convergence

- Each control $a^{(r+j)}$ is a random variable $\Longrightarrow$ The control scheme is a stochastic process.
- Each control $A^{(r+j)}$ is also a realization of the corresponding random variable $\Longrightarrow$ The control scheme is a deterministic iteration.
- Convergence of deterministic iteration on independent random samples $\Longrightarrow$ Almost sure convergence of stochastic process.
- Need fast convergence:
  - ◇ Stationary statistic is not realistic.
  - ◇ Atmospheric turbulence changes rapidly.
  - ◇ Can only assume stationary statistic for a short period of time.

- Define

$$\begin{aligned} \mathbf{a}_{r+2} &:= [a^{(r+2)}, a^{(r+1)}, \ldots a^{(r-q+1)}]^T, \quad r = 0, 1, \ldots \\ \mathbf{b} &:= [\sum_{j=0}^{q} b_j M_j G s', 0, \ldots, 0]^T. \end{aligned}$$

- The ARMA scheme becomes

$$\mathbf{a}_{r+2} = A\mathbf{a}_{r+1} + \mathbf{b}$$

where A is the $m(q+2) \times m(q+2)$ matrix

$$A := \begin{bmatrix} c_0 I_m & c_1 I_m - b_0 M_1 G & \ldots & & c_{q+1} I_m - b_q M_q G \\ I_m & 0 & & & 0 \\ 0 & I_m & & & \\ \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & \ldots & I_m & 0 \end{bmatrix}$$

- Almost convergence $\Longleftrightarrow$ Spectral radius $\rho(A)$ of A is less than one.
- Asymptotic convergence factor is precisely $\rho(A)$.

Numerical Simulation

- Consider the 2-cycle delay scheme

$$a(t + 2\Delta t) = a(t + \Delta t) + 0.6 H^\dagger W^\dagger \Delta s(t).$$

- Test data:

surface positions n	=	5
number of actuators m	=	4
number of subapertures ℓ	=	3
size of random samples z	=	2500
H	=	$rand(n, m)$
W	=	$rand(\ell, n)$
G	=	WH
L_ϕ	=	$rand(n, n)$
L_ϵ	=	$diag(rand(\ell, 1))$
μ_ϕ	=	$zeros(n, 1)$
μ_ϵ	=	$zeros(\ell, 1)$

- Random samples:

$$\begin{aligned} \phi &= \mu_\phi * ones(1, z) + L_\phi * randn(n, z), \\ \epsilon &= \mu_\epsilon * ones(1, z) + L_\epsilon * randn(\ell, z). \end{aligned}$$