ON THE GLOBAL CONVERGENCE OF THE HIGH-ORDER POWER METHOD FOR RANK-ONE TENSOR APPROXIMATION

DRAFT AS OF January 8, 2013

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Abstract. Tensor decomposition has consequential applications in various fields of disciplines, but it remains to be an extremely challenging task even to this date. A slightly more manageable endeavor has been to find a low rank approximation in place of the decomposition. Even for this less stringent task, it is an established fact that tensors beyond matrices can fail to have best low rank approximations, with the notable exception that the best rank-one approximation always exists for tensors of any order. Toward the latter, the most popular approach is via the notion of alternating projection. The specific numerical scheme appears as a variant of the power method. The so called proofs of convergence in the literature, however, have been only on the limiting behavior of the objective values. This is not enough, as the dynamics of the iterates itself should also be analyzed. The purpose of this paper is to fill the gap by presenting a rigorous convergence analysis of the power method for the rank-one approximation. Also briefly discussed is a comparison of the power method with the global optimization.

Key words. tensor decomposition, low rank approximation, alternating least squares, power method, convergence analysis, singular value decomposition

AMS subject classifications. 15A69, 41A50, 68W25, 65J10

1. Introduction.

A real-valued tensor of order k can be represented by a k-way array

$$T = [\tau_{i_1,\ldots,i_k}] \in \mathbb{R}^{I_1 \times I_2 \times \ldots \times I_k}$$

with elements $\tau_{i_1,\ldots,i_k}$ accessed via k indices. A tensor of the form

$$\mathbf{u}^{(1)} \otimes \ldots \otimes \mathbf{u}^{(k)} := \left[u_{i_1}^{(1)} \ldots u_{i_k}^{(k)} \right],$$

where elements are the products of entries from vectors $\mathbf{u}^{(j)} \in \mathbb{R}^{I_j}$, $j = 1, \ldots, k$, is said to be of rank one. By a tensor decomposition we refer to a generalization of the singular value decomposition of matrices to multi-dimensional arrays, that is, we want to rewrite the given tensor T as the sum of some rank-one tensors. Depending on the desirable structure, the summation itself may assume different formats which thus lead to different types of decompositions. The two most prevalent types are the Tucker decomposition [6, 7, 24]

$$T = \sum_{j_1,j_2,\ldots,j_k} g_{j_1,j_2,\ldots,j_k} \mathbf{u}_{j_1}^{(1)} \otimes \ldots \otimes \mathbf{u}_{j_k}^{(k)} \tag{1.1}$$

and the CP decomposition [9, 10, 12, 13]

$$T = \sum_{j} \mathbf{u}_{j}^{(1)} \otimes \ldots \otimes \mathbf{u}_{j}^{(k)}. \tag{1.2}$$

Any of these decomposition finds a wide range of applications. So that we may stay focused on our main point, we only mention without giving details that decomposing a tensor, especially when subject to the constraint of maintaining minimum number of terms in the summation, amounts to finding real-valued solutions for a high-degree polynomial system of equations, which certainly is not a trivial task. Tensor decomposition has been a major ongoing research topic with numerous results available in the literature.

Instead of requiring a complete decomposition, an alternative approach by the low rank approximation has also attracted considerable interest in the field. The basic setup consists of two parts. Firstly, the number of

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terms in the summation is to be fixed a priori, whose choice is a difficult problem in itself [4, 16, 18]. Secondly, the difference between the given T and the summation is to be minimized in the sense of Frobenius norm. In contrast to the algebraic problem of finding an exact decomposition which is NP-hard, the analytic problem of finding an approximation is more computationally feasible. Nonetheless, it must be mentioned that, except for the case of order-2 tensors which are simply matrices or the case of rank-1 approximation for general tensors, the best low rank approximation for high-order tensors may not exist at all [8, 16, 19]. This paper concerns the rank-1 approximation only. Our main contribution is on the convergence analysis of the high-order power method which has been regarded as the most prevalent technique but whose proof has been elusive.

2. Basics. To facilitate the discussion, we first briefly review the idea of regarding tensors as linear operators so as to take advantage of what is already established in linear theory, such as the associated adjoint operators. We then review the basic setup of the high-order power method.

2.1. Linear operator. We begin the discussion with a linear operator $\mathscr{T}$ from the matrix space $\mathbb{R}^{m\times n}$ to the vector space $\mathbb{R}^p$, which allows us to visualize how an order-3 tensor representation acts on matrices. Let $\{E_J\}$, where J is a double index, denote the standard basis $E_J = [e^J_{st}] \in \mathbb{R}^{m\times n}$ with

$$e^J_{st} = \begin{cases} 1, & \text{if } J = (s,t), \\ 0, & \text{otherwise,} \end{cases}$$

and likewise $\{\mathbf{e}_i\}$ the standard basis of $\mathbb{R}^p$. Suppose that the action of $\mathscr{T}$ at a basis matrix $E_J \in \mathbb{R}^{m\times n}$ is expressed in terms of basis $\{\mathbf{e}_i\}$ as

$$\mathscr{T}(E_J) = \sum_i \tau_{i,J}\mathbf{e}_i$$

with $\tau_{i,J} \in \mathbb{R}$. Then, similar to the usual linear transformations, we have for any $X = [x_{ij}] \in \mathbb{R}^{m\times n}$ the relationship

$$\mathscr{T}(X) = \mathscr{T}\left(\sum_J x_J E_J\right) = \sum_J x_J \left(\sum_i \tau_{i,J} e_i\right) = \sum_i \left(\sum_J \tau_{i,J} x_J\right)\mathbf{e}_i.$$

Note that $\tau_{i,:} \in \mathbb{R}^{m\times n}$ and that the inner summation $\sum_J \tau_{i,J}x_J$ is precisely the Frobenius inner product $\langle \tau_{i,:}, X\rangle$ between the $m \times n$ matrices $\tau_{i,:}$ and X. In this way, the operator $\mathscr{T}$ is represented by an order-3 tensor T with the action $\circledast$ defined by

$$\mathscr{T}(X) \equiv \underbrace{\begin{bmatrix} \boxed{\tau_{1,:}} \\ \boxed{\tau_{2,:}} \\ \vdots \\ \boxed{\tau_{p,:}} \end{bmatrix}}_{T} \circledast X := [\langle \tau_{i,:}, X\rangle] \in \mathbb{R}^p.$$

The order-3 tensor T may be visualized as a $p \times 1$ block matrix of which each block is an $m \times n$ matrices, whereas the action $\circledast$ is a 2-dimensional contraction. It follows naturally that the adjoint operator $\mathscr{T}^* : \mathbb{R}^p \to \mathbb{R}^{m\times n}$ should be defined as

$$\mathscr{T}^*(\mathbf{v}) := \sum_i v_i \tau_{i,:} \tag{2.1}$$

which satisfies the requirement that

$$\langle \mathscr{T}(X), \mathbf{v}\rangle = \langle X, \mathscr{T}^*(\mathbf{v})\rangle \tag{2.2}$$

for every $X \in \mathbb{R}^{m\times n}$ and $\mathbf{v} \in \mathbb{R}^p$.

In a similar manner, an order-k tensor $T \in \mathbb{R}^{I_1\times\ldots\times I_k}$ may be thought of as a linear operator mapping order-$(k-1)$ tensors to vectors. Of course, because now we have multiple "facets", it should be specified that with respect to which facet of the tensor T that the $(k-1)$-dimensional contraction is taking place. For this matter, we identify T as the operator $\mathscr{T} \equiv \mathscr{T}_\ell$, $\ell = 1,\ldots,k$, with

$$\mathscr{T}_\ell : \mathbb{R}^{I_1\times\ldots\times\widehat{I}_\ell\times\ldots\times I_k} \to \mathbb{R}^{I_\ell}, \tag{2.3}$$

where $\widehat{I}_\ell$ means that quantities associated with this particular index are taken out from the remaining list, and define its action at any $S \in \mathbb{R}^{I_1\times\ldots\times\widehat{I}_\ell\times\ldots\times I_k}$ by the operation

$$\mathscr{T}_\ell(S) := T \circledast_\ell S = [\langle \tau_{:,i_\ell,:}, S\rangle] \in \mathbb{R}^{I_\ell}, \tag{2.4}$$

where $\tau_{:,i_\ell,:}$ denotes the i_ℓ-th "slice" of the tensor T in the ℓ-th direction and $\langle\cdot,\cdot\rangle$ is the Frobenius inner product generalized to multi-dimensional arrays. The following result generalizes the spirit of (2.2) whereas the notion of "transpose" is embedded in the tensor product $\circledast_\ell$. Its proof is only a rearrangement of terms in the summation by the associative law.

LEMMA 2.1. *For any $\ell = 1,\ldots,k$, let $\mathbf{h}^{(\ell)} \in \mathbb{R}^{I_\ell}$ be an arbitrary vector. Then it is true that*

$$\left\langle T, \mathbf{u}^{(1)}\otimes\ldots\otimes\mathbf{h}^{(\ell)}\otimes\ldots\otimes\mathbf{u}^{(k)}\right\rangle = \left\langle T\circledast_\ell\left(\mathbf{u}^{(1)}\otimes\ldots\otimes\widehat{\mathbf{u}}^{(\ell)}\otimes\ldots\otimes\mathbf{u}^{(k)}\right), \mathbf{h}^{(\ell)}\right\rangle. \tag{2.5}$$

Because (2.5) holds for any index ℓ, we have the reciprocating relationship

$$\left\langle T\circledast_i\left(\mathbf{u}^{(1)}\otimes\ldots\otimes\widehat{\mathbf{u}}^{(i)}\otimes\ldots\otimes\mathbf{u}^{(k)}\right), \mathbf{u}^{(i)}\right\rangle = \left\langle T\circledast_j\left(\mathbf{u}^{(1)}\otimes\ldots\otimes\widehat{\mathbf{u}}^{(j)}\otimes\ldots\otimes\mathbf{u}^{(k)}\right), \mathbf{u}^{(j)}\right\rangle. \tag{2.6}$$

for any two distinct indices i and j. Both identities will be extremely useful in the subsequent discussion.

2.2. High-order power method. The problem of finding a best rank-1 approximation to T is to determine unit vectors $\mathbf{u}^{(j)} \in \mathbb{R}^{I_j}$, $j = 1,\ldots k$ and a scalar λ such that the function

$$f\left(\lambda, \mathbf{u}^{(1)},\ldots,\mathbf{u}^{(k)}\right) := \left\|T - \lambda\mathbf{u}^{(1)}\otimes\ldots\otimes\mathbf{u}^{(k)}\right\|_F^2 = \sum_{i_1,i_2,\ldots,i_k}\left(\tau_{i_1,\ldots,i_k} - \lambda u_{i_1}^{(1)}\ldots u_{i_k}^{(k)}\right)^2 \tag{2.7}$$

is minimized. It is not difficult to see that for any fixed unit vectors $\mathbf{u}^{(1)},\ldots,\mathbf{u}^{(k)}$, the optimal value of λ for (2.7) is necessarily given by

$$\lambda = \lambda\left(\mathbf{u}^{(1)},\ldots,\mathbf{u}^{(k)}\right) = \left\langle T, \mathbf{u}^{(1)}\otimes\ldots\otimes\mathbf{u}^{(k)}\right\rangle. \tag{2.8}$$

To put it differently, by regarding the order-k tensor space as a vector space equipped with the Frobenius inner product, the expression for λ in (2.8) is precisely the length of the projection of the "vector" T onto the direction of the "unit vector" $\mathbf{u}^{(1)}\otimes\ldots\otimes\mathbf{u}^{(k)}$. Thus, minimizing the orthogonal component of T, as is so desired in (2.7), is equivalent to maximizing the parallel component λ. In [28], the expression (2.8) is called the generalized Rayleigh quotient.

Using the Lagrange multiplier theory, we can easily argue with the aid of (2.5) that the first order optimality condition for a stationary point is to satisfy the system of equations [20]

$$T\circledast_\ell\left(\mathbf{u}^{(1)}\otimes\ldots\otimes\widehat{\mathbf{u}}^{(\ell)}\otimes\ldots\otimes\mathbf{u}^{(k)}\right) = \lambda\mathbf{u}^{(\ell)}, \quad \ell = 1,\ldots,k. \tag{2.9}$$

To solve (2.9), a commonly used tactic is to alternate directions among the variables in a way similar to the classical Gauss-Seidel method. See Algorithm 1 for the description, where the subscript $\cdot_{[p]}$ indicates the quantities resulting from the p-th iteration.

Algorithm 1 (High-order power method.)

Given $\mathbf{u}_{[0]}^{(1)} \in \mathbb{R}^{I_1}, \ldots, \mathbf{u}_{[0]}^{(k)} \in \mathbb{R}^{I_k}$,
for $p = 0, 1, \cdots,$ **do**
  for $\ell = 1, 2, \cdots, k$ **do**
    $\mathbf{u}_{[p+1]}^{(\ell)} = T \circledast_\ell \left(\mathbf{u}_{[p+1]}^{(1)} \otimes \ldots \otimes \mathbf{u}_{[p+1]}^{(\ell-1)} \otimes \widehat{\mathbf{u}}^{(\ell)} \otimes \mathbf{u}_{[p]}^{(\ell+1)} \ldots \otimes \mathbf{u}_{[p]}^{(k)} \right)$
    $\lambda_{[p+1]}^{(\ell)} := \|\mathbf{u}_{[p+1]}^{(\ell)}\|_2$
    $\mathbf{u}_{[p+1]}^{(\ell)} := \frac{\mathbf{u}_{[p+1]}^{(\ell)}}{\lambda_{[p+1]}^{(k)}}$
  end for
end for

It is interesting to note that in the case $k = 2$, T is a matrix and the alternating direction iteration dictated in Algorithm 1 can be expressed as

$$\left\{ \begin{array}{rcl} \mathbf{u}_{[p+1]}^{(1)} & = & \frac{1}{\lambda_{[p+1]}^{(1)}} T \mathbf{u}_{[p]}^{(2)}, \\ \mathbf{u}_{[p+1]}^{(2)} & = & \frac{1}{\lambda_{[p+1]}^{(2)}} T^\top \mathbf{u}_{[p+1]}^{(1)}, \end{array} \right. \tag{2.10}$$

which implies

$$\left\{ \begin{array}{rcl} \mathbf{u}_{[p+1]}^{(1)} & = & \frac{1}{\lambda_{[p+1]}^{(1)} \lambda_{[p]}^{(2)}} T T^\top \mathbf{u}_{[p]}^{(1)}, \\ \mathbf{u}_{[p+1]}^{(2)} & = & \frac{1}{\lambda_{[p+1]}^{(1)} \lambda_{[p+1]}^{(2)}} T^\top T \mathbf{u}_{[p]}^{(2)}. \end{array} \right. \tag{2.11}$$

The scalings in (2.11) are insignificant because their sole purpose is for normalization, but the scheme itself is precisely the familiar power method applied to the matrices $TT^\top$ and $T^\top T$, respectively. For this reason, Algorithm 1 is known as the high-order power method. The convergence of both sequences $\left\{ \mathbf{u}_{[p]}^{(1)} \right\}$ and $\left\{ \mathbf{u}_{[p]}^{(2)} \right\}$ in (2.11) is well understood in the literature and the limit point of either $\left\{ \lambda_{[p]}^{(1)} \right\}$ or $\left\{ \lambda_{[p]}^{(2)} \right\}$ is precisely the largest singular value of T. Such a convergence behavior when Algorithm 1 is applied to high-order tensors, however, has not been sufficiently analyzed in the literature. The primary purpose of this paper is to fill that gap.

2.3. Alternating least squares. Actually, Algorithm 1 has an inherent optimization property — it can be regarded as an alternative least squares (ALS) method [2, 10] applied to (2.7). To see this descending property, consider the maximization of the function $\theta_\ell : \mathbb{R}^{I_\ell} \to \mathbb{R}$ defined by

$$\theta_\ell \left(\mathbf{y}^{(\ell)} \right) := \left\langle T, \mathbf{u}_{[p+1]}^{(1)} \otimes \ldots \otimes \mathbf{u}_{[p+1]}^{(\ell-1)} \otimes \mathbf{y}^{(\ell)} \otimes \mathbf{u}_{[p]}^{(\ell+1)} \otimes \ldots \otimes \mathbf{u}_{[p]}^{(k)} \right\rangle, \tag{2.12}$$

subject to the constraint $\|\mathbf{y}^{(\ell)}\|_2 = 1$ for each fixed p. Applying the Lagrange multiplier theory and using (2.5), it is easy to see that the first order optimality condition for (2.12) is the equation

$$T \circledast_\ell \left(\mathbf{u}_{[p+1]}^{(1)} \otimes \ldots \otimes \mathbf{u}_{[p+1]}^{(\ell-1)} \otimes \widehat{\mathbf{u}}^{(\ell)} \otimes \mathbf{u}_{[p]}^{(\ell+1)} \ldots \otimes \mathbf{u}_{[p]}^{(k)} \right) = \mu_\ell \mathbf{y}^{(\ell)}$$

for $\mathbf{y}^{(\ell)}$ and some Lagrangian multiplier μ_ℓ. Since μ_ℓ is necessarily positive, we see that after normalization the choice $\mathbf{y}^{(\ell)} = \mathbf{u}_{[p+1]}^{(\ell)}$ is the unique maximizer for the function θ_ℓ. In this case, by (2.5) again, we can verify that

$$\mu_\ell = \theta_\ell \left(\mathbf{u}_{[p+1]}^{(\ell)} \right) = \lambda_{[p+1]}^{(\ell)}.$$

By repeating this argument to successive ℓ, we conclude that

$$\lambda_{[p]} := \lambda\left(\mathbf{u}_{[p]}^{(1)}, \ldots, \mathbf{u}_{[p]}^{(k)}\right) \leq \lambda_{[p+1]}^{(1)} \leq \lambda_{[p+1]}^{(2)} \leq \ldots \leq \lambda_{[p+1]}^{(k)} = \lambda\left(\mathbf{u}_{[p+1]}^{(1)}, \ldots, \mathbf{u}_{[p+1]}^{(k)}\right). \quad (2.13)$$

In other words, each sweep in the inner loop of Algorithm 1 has the effect of increasing the functional value of λ. Because the range of λ is bounded by $\|T\|_F$, the sequence $\left\{\lambda_{[p]}\right\}$ must converge.

The ALS method as is formulated in Algorithm 1 is cast upon rank-1 approximation only. Its appearance in the form of a (high-order) power method requires no inversion and makes the computation straightforward. The idea can be generalized in various ways and has been used as the "workhorse" algorithm for computing rank decomposition of general tensors and low rank approximation of arbitrary ranks. More detailed discussions can be found in [4, 16]. The above argument about the nondecreasing property of $\lambda_{[p]}$ is essentially the so called proof of convergence in the literature [14, 28]. This certainly is not enough, because the convergence of objective values does not guarantee the convergence of iterates to a global minimizer nor even to a stationary point [16]. Because of this incompleteness in general, it was claimed in [5] that "Their [the ALS algorithms] extensive use is thus unexplainable." More need be done. We offer the following convergence analysis as only a small step toward that goal.

3. Convergence analysis. We have read about a discussion on the convergence of the ALS iterates for rank-1 approximation of order-3 tensors in [28, Theorem 4.3]. The proof, however, is indirect as it calls for a comparison with the Newton method which is known to converge only locally. In fact, we think the argument there is not rigorous enough in itself at multiple places. We hereby present our proof for tensors of arbitrary order.

At each given $\mathbf{u}^{(2)} \in \mathbb{R}^{I_2}, \ldots, \mathbf{u}^{(k)} \in \mathbb{R}^{I_k}$, define

$$\mathbf{w} = \mathbf{w}\left(\mathbf{u}^{(2)}, \ldots, \mathbf{u}^{(k)}\right) := T \circledast_1 \left(\mathbf{u}^{(2)} \otimes \ldots \otimes \mathbf{u}^{(k)}\right) \in \mathbb{R}^{I_1}, \quad (3.1)$$

and

$$g\left(\mathbf{u}^{(2)}, \ldots, \mathbf{u}^{(k)}\right) := \left\langle T, \mathbf{w} \otimes \mathbf{u}^{(2)} \otimes \ldots \otimes \mathbf{u}^{(k)} \right\rangle. \quad (3.2)$$

It is easy to check that

$$g\left(\mathbf{u}^{(2)}, \ldots, \mathbf{u}^{(k)}\right) = \sum_{i_1=1}^{I_1} \left\langle \tau_{i_1,:,:}, \mathbf{u}^{(2)} \otimes \ldots \otimes \mathbf{u}^{(k)} \right\rangle^2 = \left\| \mathbf{w}\left(\mathbf{u}^{(2)}, \ldots, \mathbf{u}^{(k)}\right) \right\|_2^2 \leq \|T\|_F^2. \quad (3.3)$$

One of our key steps is to show that the evaluation of the function g at the iterates generated from Algorithm 1 forms a nondecreasing sequence.

For any fixed p, define the incremental quantities

$$\delta\mathbf{u}_{[p]}^{(\ell)} := \mathbf{u}_{[p+1]}^{(\ell)} - \mathbf{u}_{[p]}^{(\ell)}, \quad \ell = 1, \ldots, k. \quad (3.4)$$

The following simple fact will prove handy in subsequent development.

LEMMA 3.1. *Suppose that* $\mathbf{u}_{[p+1]}^{(\ell)}$ *and* $\mathbf{u}_{[p]}^{(\ell)}$ *are unit vectors. Then*

$$\left\langle \delta\mathbf{u}_{[p]}^{(\ell)}, \delta\mathbf{u}_{[p]}^{(\ell)} \right\rangle = 2\left\langle \mathbf{u}_{[p+1]}^{(\ell)}, \delta\mathbf{u}_{[p]}^{(\ell)} \right\rangle. \quad (3.5)$$

Proof. By the definition (3.4), it is trivial to see that $\left\langle \delta\mathbf{u}_{[p]}^{(\ell)}, \delta\mathbf{u}_{[p]}^{(\ell)} \right\rangle = 2 - 2\left\langle \mathbf{u}_{[p+1]}^{(\ell)}, \mathbf{u}_{[p]}^{(\ell)} \right\rangle$. The assertion follows from $\left\langle \mathbf{u}_{[p+1]}^{(\ell)}, \delta\mathbf{u}_{[p]}^{(\ell)} \right\rangle = \left\langle \mathbf{u}_{[p+1]}^{(\ell)}, \mathbf{u}_{[p+1]}^{(\ell)} - \mathbf{u}_{[p]}^{(\ell)} \right\rangle = 1 - \left\langle \mathbf{u}_{[p+1]}^{(\ell)}, \mathbf{u}_{[p]}^{(\ell)} \right\rangle$. ∎

It is instructive to first observe the following relationship whose derivation illustrates a critical mechanism that will be used repeatedly to gradually interchange and eventually simplify terms.

$$\begin{aligned}
g\left(\mathbf{u}_{[p+1]}^{(2)},\mathbf{u}_{[p]}^{(3)},\ldots,\mathbf{u}_{[p]}^{(k)}\right) &= \left\|T\circledast_1\left(\left(\mathbf{u}_{[p]}^{(2)}+\delta\mathbf{u}_{[p]}^{(2)}\right)\otimes\mathbf{u}_{[p]}^{(3)}\otimes\ldots\otimes\mathbf{u}_{[p]}^{(k)}\right)\right\|_2^2 \\
&= g\left(\mathbf{u}_{[p]}^{(2)},\mathbf{u}_{[p]}^{(3)},\ldots,\mathbf{u}_{[p]}^{(k)}\right)+\left\|T\circledast_1\left(\delta\mathbf{u}_{[p]}^{(2)}\otimes\mathbf{u}_{[p]}^{(3)}\otimes\ldots\otimes\mathbf{u}_{[p]}^{(k)}\right)\right\|_2^2 \\
&\quad +2\left\langle\underbrace{T\circledast_1\left(\mathbf{u}_{[p]}^{(2)}\otimes\mathbf{u}_{[p]}^{(3)}\otimes\ldots\otimes\mathbf{u}_{[p]}^{(k)}\right)}_{\lambda_{[p+1]}^{(1)}\mathbf{u}_{[p+1]}^{(1)}},T\circledast_1\left(\delta\mathbf{u}_{[p]}^{(2)}\otimes\mathbf{u}_{[p]}^{(3)}\otimes\ldots\otimes\mathbf{u}_{[p]}^{(k)}\right)\right\rangle \\
&= g\left(\mathbf{u}_{[p]}^{(2)},\mathbf{u}_{[p]}^{(3)},\ldots,\mathbf{u}_{[p]}^{(k)}\right)+\left\|T\circledast_1\left(\delta\mathbf{u}_{[p]}^{(2)}\otimes\mathbf{u}_{[p]}^{(3)}\otimes\ldots\otimes\mathbf{u}_{[p]}^{(k)}\right)\right\|_2^2 \\
&\quad +2\lambda_{[p+1]}^{(1)}\left\langle\delta\mathbf{u}_{[p]}^{(2)},\underbrace{T\circledast_2\left(\mathbf{u}_{[p+1]}^{(1)}\otimes\mathbf{u}_{[p]}^{(3)}\otimes\ldots\otimes\mathbf{u}_{[p]}^{(k)}\right)}_{\lambda_{[p+1]}^{(2)}\mathbf{u}_{[p+1]}^{(2)}}\right\rangle \\
&= g\left(\mathbf{u}_{[p]}^{(2)},\mathbf{u}_{[p]}^{(3)},\ldots,\mathbf{u}_{[p]}^{(k)}\right)+\left\|T\circledast_1\left(\delta\mathbf{u}_{[p]}^{(2)}\otimes\mathbf{u}_{[p]}^{(3)}\otimes\ldots\otimes\mathbf{u}_{[p]}^{(k)}\right)\right\|_2^2 \\
&\quad +\lambda_{[p+1]}^{(1)}\lambda_{[p+1]}^{(2)}\left\|\delta\mathbf{u}_{[p]}^{(2)}\right\|_2^2. \qquad (3.6)
\end{aligned}$$

In the above, the third equality follows from (2.5), whereas the fourth equality follows from (3.5). What we have achieved in (3.6) is to calculate the exact effect on g when only the vector $\mathbf{u}_{[p]}^{(2)}$ is updated.

In a similar vein, we now build a telescoping relationship for the remaining cases of $\ell=3,\ldots,k$. For convenience, we shall adopt the following abbreviations to simplify the expressions which become repetitive and tedious. For integers $s\le t$, let

$$U_{[p]}^{\{s..t\}}:=\left(\mathbf{u}_{[p]}^{(s)},\mathbf{u}_{[p]}^{(s+1)},\ldots,\mathbf{u}_{[p]}^{(t)}\right)\in\mathbb{R}^{I_s}\times\ldots\times\mathbb{R}^{I_t} \qquad (3.7)$$

denote the list of vectors $\mathbf{u}_{[p]}^{(\ell)}$, $\ell=s,\ldots,t$, and

$$\bigotimes U_{[p]}^{\{s..t\}}:=\mathbf{u}_{[p]}^{(s)}\otimes\mathbf{u}_{[p]}^{(s+1)}\otimes\ldots\otimes\mathbf{u}_{[p]}^{(t)} \qquad (3.8)$$

the outer product of these vectors. We then can write

$$\begin{aligned}
&g\left(\mathbf{u}_{[p+1]}^{(2)},\ldots,\mathbf{u}_{[p+1]}^{(\ell-1)},\mathbf{u}_{[p+1]}^{(\ell)},\mathbf{u}_{[p]}^{(\ell+1)},\mathbf{u}_{[p]}^{(k)}\right)=g\left(U_{[p+1]}^{\{2..\ell\}},U_{[p]}^{\{(\ell+1)..k\}}\right) \\
&=\left\|T\circledast_1\left(\bigotimes U_{[p+1]}^{\{2..(\ell-1)\}}\otimes\left(\mathbf{u}_{[p]}^{(\ell)}+\delta\mathbf{u}_{[p]}^{(\ell)}\right)\otimes\bigotimes U_{[p]}^{\{(\ell+1)..k\}}\right)\right\|_2^2 \\
&=g\left(U_{[p+1]}^{\{2..(\ell-1)\}},U_{[p]}^{\{\ell..k\}}\right)+\left\|T\circledast_1\left(\bigotimes U_{[p+1]}^{\{2..(\ell-1)\}}\otimes\delta\mathbf{u}_{[p]}^{(\ell)}\otimes\bigotimes U_{[p]}^{\{(\ell+1)..k\}}\right)\right\|_2^2 \\
&\quad +2\left\langle T\circledast_1\left(\bigotimes U_{[p+1]}^{\{2..(\ell-1)\}}\otimes\bigotimes U_{[p]}^{\{\ell..k\}}\right),T\circledast_1\left(\bigotimes U_{[p+1]}^{\{2..(\ell-1)\}}\otimes\delta\mathbf{u}_{[p]}^{(\ell)}\otimes\bigotimes U_{[p]}^{\{(\ell+1)..k\}}\right)\right\rangle.
\end{aligned}$$

The last term in the third equation above needs some additional manipulation. For clarity, we single this term out for special treatment. We analyze the expression by employing the mechanism illustrated in (3.6)

recursively as follows.

$$
\begin{aligned}
&\left\langle T\circledast_1\left(\bigotimes U_{[p+1]}^{\{2..(\ell-1)\}}\otimes\bigotimes U_{[p]}^{\{\ell..k\}}\right), T\circledast_1\left(\bigotimes U_{[p+1]}^{\{2..(\ell-1)\}}\otimes\delta\mathbf{u}_{[p]}^{(\ell)}\otimes\bigotimes U_{[p]}^{\{(\ell+1)..k\}}\right)\right\rangle\\
&=\left\langle T\circledast_1\left(\bigotimes U_{[p+1]}^{\{2..(\ell-2)\}}\otimes\bigotimes U_{[p]}^{\{(\ell-1)..k\}}\right), T\circledast_1\left(\bigotimes U_{[p+1]}^{\{2..(\ell-1)\}}\otimes\delta\mathbf{u}_{[p]}^{(\ell)}\otimes\bigotimes U_{[p]}^{\{(\ell+1)..k\}}\right)\right\rangle\\
&\quad+\left\langle T\circledast_1\left(\bigotimes U_{[p+1]}^{\{2..(\ell-2)\}}\otimes\delta\mathbf{u}_{[p]}^{(\ell-1)}\bigotimes U_{[p]}^{\{\ell..k\}}\right), T\circledast_1\left(\bigotimes U_{[p+1]}^{\{2..(\ell-1)\}}\otimes\delta\mathbf{u}_{[p]}^{(\ell)}\otimes\bigotimes U_{[p]}^{\{(\ell+1)..k\}}\right)\right\rangle\\
&=\ldots\\
&=\left\langle \underbrace{T\circledast_1\left(\bigotimes U_{[p]}^{\{2..k\}}\right)}_{\lambda_{[p+1]}^{(1)}\mathbf{u}_{[p+1]}^{(1)}}, T\circledast_1\left(\bigotimes U_{[p+1]}^{\{2..(\ell-1)\}}\otimes\delta\mathbf{u}_{[p]}^{(\ell)}\otimes\bigotimes U_{[p]}^{\{(\ell+1)..k\}}\right)\right\rangle\\
&\quad+\sum_{j=2}^{\ell-1}\left\langle T\circledast_1\left(\bigotimes U_{[p+1]}^{\{2..(j-1)\}}\otimes\delta\mathbf{u}_{[p]}^{(j)}\bigotimes U_{[p]}^{\{(j+1)..k\}}\right), T\circledast_1\left(\bigotimes U_{[p+1]}^{\{2..(\ell-1)\}}\otimes\delta\mathbf{u}_{[p]}^{(\ell)}\otimes\bigotimes U_{[p]}^{\{(\ell+1)..k\}}\right)\right\rangle\\
&=\lambda_{[p+1]}^{(1)}\left\langle \delta\mathbf{u}_{[p]}^{(\ell)}, \underbrace{T\circledast_\ell\left(\bigotimes U_{[p+1]}^{\{1..(\ell-1)\}}\otimes\widehat{\delta\mathbf{u}}_{[p]}^{(\ell)}\otimes\bigotimes U_{[p]}^{\{(\ell+1)..k\}}\right)}_{\lambda_{[p+1]}^{(\ell)}\mathbf{u}_{[p+1]}^{(\ell)}}\right\rangle\\
&\quad+\sum_{j=2}^{\ell-1}\left\langle T\circledast_1\left(\bigotimes U_{[p+1]}^{\{2..(j-1)\}}\otimes\delta\mathbf{u}_{[p]}^{(j)}\bigotimes U_{[p]}^{\{(j+1)..k\}}\right), T\circledast_1\left(\bigotimes U_{[p+1]}^{\{2..(\ell-1)\}}\otimes\delta\mathbf{u}_{[p]}^{(\ell)}\otimes\bigotimes U_{[p]}^{\{(\ell+1)..k\}}\right)\right\rangle\\
&=\frac{\lambda_{[p+1]}^{(1)}\lambda_{[p+1]}^{(\ell)}}{2}\left\|\delta\mathbf{u}_{[p]}^{(\ell)}\right\|_2^2\\
&\quad+\sum_{j=2}^{\ell-1}\left\langle T\circledast_1\left(\bigotimes U_{[p+1]}^{\{2..(j-1)\}}\otimes\delta\mathbf{u}_{[p]}^{(j)}\bigotimes U_{[p]}^{\{(j+1)..k\}}\right), T\circledast_1\left(\bigotimes U_{[p+1]}^{\{2..(\ell-1)\}}\otimes\delta\mathbf{u}_{[p]}^{(\ell)}\otimes\bigotimes U_{[p]}^{\{(\ell+1)..k\}}\right)\right\rangle
\end{aligned}
$$

Introducing an additional abbreviation

$$\Delta_{[p]}^{(\ell)}:=T\circledast_1\left(\bigotimes U_{[p+1]}^{\{2..(\ell-1)\}}\otimes\delta\mathbf{u}_{[p]}^{(\ell)}\otimes\bigotimes U_{[p]}^{\{(\ell+1)..k\}}\right),\tag{3.9}$$

the above relationships can be summarized as

$$
\begin{aligned}
g\left(U_{[p+1]}^{\{2..\ell\}},U_{[p]}^{\{(\ell+1)..k\}}\right)&=g\left(U_{[p+1]}^{\{2..(\ell-1)\}},U_{[p]}^{\{\ell..k\}}\right)\\
&\quad+\left\|\Delta_{[p]}^{(\ell)}\right\|_2^2+\lambda_{[p+1]}^{(1)}\lambda_{[p+1]}^{(\ell)}\left\|\delta\mathbf{u}_{[p]}^{(\ell)}\right\|_2^2+2\sum_{j=2}^{\ell-1}\left\langle\Delta_{[p]}^{(j)},\Delta_{[p]}^{(\ell)}\right\rangle.
\end{aligned}\tag{3.10}
$$

Applying (3.10) recursively, we have established our first major result.

THEOREM 3.2. *Let* $U_{[p]}^{\{2..k\}}$ *denote the current iterate* $\left(\mathbf{u}_{[p]}^{(2)},\mathbf{u}_{[p]}^{(3)},\ldots,\mathbf{u}_{[p]}^{(k)}\right)$. *Then*

$$
\begin{aligned}
g\left(U_{[p+1]}^{\{2..k\}}\right)&=g\left(U_{[p]}^{\{2..k\}}\right)+\sum_{\ell=2}^{k}\left(\left\|\Delta_{[p]}^{(\ell)}\right\|_2^2+\lambda_{[p+1]}^{(1)}\lambda_{[p+1]}^{(\ell)}\left\|\delta\mathbf{u}_{[p]}^{(\ell)}\right\|_2^2+2\sum_{j=2}^{\ell-1}\left\langle\Delta_{[p]}^{(j)},\Delta_{[p]}^{(\ell)}\right\rangle\right)\\
&=g\left(U_{[p]}^{\{2..k\}}\right)+\lambda_{[p+1]}^{(1)}\sum_{\ell=2}^{k}\left(\lambda_{[p+1]}^{(\ell)}\left\|\delta\mathbf{u}_{[p]}^{(\ell)}\right\|_2^2\right)+\left\langle\sum_{j=2}^{k}\Delta_{[p]}^{(j)},\sum_{\ell=2}^{k}\Delta_{[p]}^{(\ell)}\right\rangle.
\end{aligned}\tag{3.11}
$$

In contrast to the estimate in [28, Theorem 4.1] for order-3 tensors, the equation (3.11) gives exact account on how the generalized Rayleigh quotient varies quadratically as the perturbations for general order-k tensors. Taking advantage of the equality, we are now able to gain insight into a sequence of important facts which collectively completes our convergence analysis.

LEMMA 3.3. *The sequence* $\left\{ g\left(U_{[p]}^{\{2..k\}} \right) \right\}$ *is monotone nondecreasing and converges.*

Proof. The inequalities in (2.13) assure that all $\lambda_{[p+1]}^{(\ell)}$ are positive and bounded away from zero. We also note that the two summations on the right side of (3.11) are always nonnegative. Comparing the two sides of (3.11), the sequence $\left\{ g\left(U_{[p]}^{\{2..k\}} \right) \right\}$ is monotone nondecreasing. By the boundedness of g given in (3.3), the monotone sequence must converge. □

LEMMA 3.4. *The increment* $\left\{ \delta \mathbf{u}_{[p]}^{(\ell)} \right\}$ *converges to zero for all* $\ell = 2, \ldots k$.

Proof. Taking the limit on both side of (3.11), we find that the sequence $\left\{ \delta \mathbf{u}_{[p]}^{(\ell)} \right\}$, for each $\ell = 2, \ldots, k$, is necessarily squeezed to zero. □

Note that, thought expected, the convergence of $\left\{ \delta \mathbf{u}_{[p]}^{(1)} \right\}$ has not been proved yet. Also, the convergence of $\left\{ \delta \mathbf{u}_{[p]}^{(\ell)} \right\}$ only shows that the gap between two consecutive iterates $\mathbf{u}_{[p]}^{(\ell)}$ and $\mathbf{u}_{[p+1]}^{(\ell)}$ is getting smaller, but does not guarantee that $\left\{ \mathbf{u}_{[p]}^{(\ell)} \right\}$ is forming a Cauchy sequence. The remedy comes from a slight modification of the following result already proved in [3, Lemma 4.3].

LEMMA 3.5. *Let* $\{a_n\}$ *be a bounded sequence of real numbers with the property that the difference* $a_{n+1} - a_n$ *converges to zero as* n *goes to infinity. If there are only finitely many limit points for the sequence, then the sequence* $\{a_n\}$ *converges to a unique limit point.*

The original statement of Lemma 3.5 requires the knowledge of finitely many limit points. It turns out that all we need is the assumption that the limit points are isolated, which is exactly a fact in our application as we now justify below.

LEMMA 3.6. *If a sequence generated by Algorithm 1 has any limit points at all, then these limit points are isolated. That is, around any limit point there is a neighborhood such that the sequence of iterates from Algorithm 1 has no other limit point.*

Proof. The limit points, if there is any, necessarily satisfy the system of equations (2.9). So it suffices to prove that the system (2.9) has only isolated solutions. Toward this end, we shall employ a well known result from the theory of dynamical systems.

Consider the function

$$\phi\left(\mathbf{u}^{(2)}, \ldots, \mathbf{u}^{(k)} \right) := \left\langle T - \mathbf{w} \otimes \mathbf{u}^{(2)} \otimes \ldots \otimes \mathbf{u}^{(k)}, T - \mathbf{w} \otimes \mathbf{u}^{(2)} \otimes \ldots \otimes \mathbf{u}^{(k)} \right\rangle \tag{3.12}$$

where $\mathbf{w} = \mathbf{w}\left(\mathbf{u}^{(2)}, \ldots, \mathbf{u}^{(k)} \right)$ is defined in (3.1). The (partial) Fréchet derivative of ϕ with respect to $\mathbf{u}^{(\ell)}$ acting on $\mathbf{h}^{(\ell)} \in \mathbb{R}^{I_\ell}$ is given by

$$\begin{aligned} \frac{\partial \phi}{\partial \mathbf{u}^{(\ell)}} . \mathbf{h}^{(\ell)} = & -2 \left\langle \left(T \circledast_1 \left(\mathbf{u}^{(2)} \otimes \ldots \otimes \mathbf{h}^{(\ell)} \otimes \ldots \otimes \mathbf{u}^{(k)} \right) \right) \otimes \mathbf{u}^{(2)} \otimes \ldots \otimes \mathbf{u}^{(k)}, T - \mathbf{w} \otimes \mathbf{u}^{(2)} \otimes \ldots \otimes \mathbf{u}^{(k)} \right\rangle \\ & -2 \left\langle \mathbf{w} \otimes \mathbf{u}^{(2)} \otimes \ldots \otimes \mathbf{h}^{(\ell)} \otimes \ldots \otimes \mathbf{u}^{(k)}, T - \mathbf{w} \otimes \mathbf{u}^{(2)} \otimes \ldots \otimes \mathbf{u}^{(k)} \right\rangle . \end{aligned} \tag{3.13}$$

The first term in (3.13) is precisely zero because

$$\begin{aligned} & \left\langle \left(T \circledast_1 \left(\mathbf{u}^{(2)} \otimes \ldots \otimes \mathbf{h}^{(\ell)} \otimes \ldots \otimes \mathbf{u}^{(k)} \right) \right) \otimes \mathbf{u}^{(2)} \otimes \ldots \otimes \mathbf{u}^{(k)}, T \right\rangle \\ & - \left\langle \left(T \circledast_1 \left(\mathbf{u}^{(2)} \otimes \ldots \otimes \mathbf{h}^{(\ell)} \otimes \ldots \otimes \mathbf{u}^{(k)} \right) \right) \otimes \mathbf{u}^{(2)} \otimes \ldots \otimes \mathbf{u}^{(k)}, \mathbf{w} \otimes \mathbf{u}^{(2)} \otimes \ldots \otimes \mathbf{u}^{(k)} \right\rangle \\ = & \left\langle T \circledast_1 \left(\mathbf{u}^{(2)} \otimes \ldots \otimes \mathbf{h}^{(\ell)} \otimes \ldots \otimes \mathbf{u}^{(k)} \right), T \circledast_1 \left(\mathbf{u}^{(2)} \otimes \ldots \otimes \mathbf{u}^{(k)} \right) \right\rangle \\ & - \left\langle T \circledast_1 \left(\mathbf{u}^{(2)} \otimes \ldots \otimes \mathbf{h}^{(\ell)} \otimes \ldots \otimes \mathbf{u}^{(k)} \right), \mathbf{w} \right\rangle = 0. \end{aligned}$$

In the second equation above, the first term is obtained by using (2.5), whereas the second term is by the fact that $\mathbf{u}^{(2)}, \ldots, \mathbf{u}^{(k)}$ are unit vectors. By using (2.5) again, we can rewrite

$$\begin{aligned}&\left\langle \mathbf{w}\otimes\mathbf{u}^{(2)}\otimes\ldots\otimes\mathbf{h}^{(\ell)}\otimes\ldots\otimes\mathbf{u}^{(k)}, T-\mathbf{w}\otimes\mathbf{u}^{(2)}\otimes\ldots\otimes\mathbf{u}^{(k)}\right\rangle\\ &=\left\langle \mathbf{h}^{(\ell)}, T\circledast_\ell\left(\mathbf{w}\otimes\mathbf{u}^{(2)}\otimes\ldots\otimes\widehat{\mathbf{h}}^{(\ell)}\otimes\ldots\otimes\mathbf{u}^{(k)}\right)\right\rangle-\left\langle \mathbf{h}^{(\ell)}, \left(\mathbf{w}^\top\mathbf{w}\right)\mathbf{u}^{(\ell)}\right\rangle\end{aligned}$$

and, hence, obtain

$$\nabla^{(\ell)}\phi := 2\left(\left(\mathbf{w}^\top\mathbf{w}\right)\mathbf{u}^{(\ell)}-T\circledast_\ell\left(\mathbf{w}\otimes\mathbf{u}^{(2)}\otimes\ldots\otimes\widehat{\mathbf{h}}^{(\ell)}\otimes\ldots\otimes\mathbf{u}^{(k)}\right)\right) \tag{3.14}$$

as the (partial) gradient of ϕ with respect to the variable $\mathbf{u}^{(\ell)}$. It is not difficult to see that the projected gradient onto the unit ball $\|\mathbf{u}^{(\ell)}\|=1$ is precisely the gradient $\nabla^{(\ell)}\phi$ itself. Thus the dynamical system

$$\frac{d\mathbf{u}^{(\ell)}}{dt}=T\circledast_\ell\left(\mathbf{w}\otimes\mathbf{u}^{(2)}\otimes\ldots\otimes\widehat{\mathbf{h}}^{(\ell)}\otimes\ldots\otimes\mathbf{u}^{(k)}\right)-\left(\mathbf{w}^\top\mathbf{w}\right)\mathbf{u}^{(\ell)},\quad \ell=2,\ldots,k \tag{3.15}$$

gives rise to the so called projected gradient flow for the function ϕ. Because this is a gradient flow, and also because the vector field on the right hand side of (3.15) is analytic in $\mathbf{u}^{(2)}, \ldots, \mathbf{u}^{(k)}$, it is known by the Łojasiewicz inequality that the solution flow $\mathbf{u}^{(\ell)}(t)$ must converge to an isolated limit point. In other words, the equilibria to the vector field must be isolated. The system of equations describing the vector field in (3.15) happens to be the same as that in (2.9). The assertion is proved. □

Finally, we complete the last stage of convergence analysis as follows.

THEOREM 3.7. *For each $\ell=1,\ldots,k$, the sequence $\left\{\mathbf{u}_{[p]}^{(\ell)}\right\}$ generated by Algorithm 1 converges.*

Proof. Starting with any initial value $\mathbf{u}_{[0]}^{(\ell)}$, it is clear that the iterates $\left\{\mathbf{u}_{[p]}^{(\ell)}\right\}$ is bounded. By Lemma 3.6, its limit points are isolated. Because $\left\{\delta\mathbf{u}_{[p]}^{(\ell)}\right\}$ converges to zero for $\ell=2,\ldots,k$, we see by Lemma 3.5 that $\left\{\mathbf{u}_{[p]}^{(\ell)}\right\}$, $\ell=2,\ldots k$, generated by Algorithm 1 converges to a single limit point. The convergence of $\left\{\mathbf{u}_{[p]}^{(1)}\right\}$ follows from the simultaneous convergence of $\mathbf{u}_{[p]}^{(1)},\ldots,\mathbf{u}_{[p]}^{(k)}$. □

Two points are worth noting. First, we have showed in the discussion above that the higher-order power method, that is, Algorithm 1, converges globally. This result is much stronger than the local convergence argued in [28]. Second, the objective function f defined in (2.7) generally has multiple local minimizers (and thus multiple local maximizers for λ). A limit point obtained by the high-order power method depends on the starting value and is not guaranteed to be the best rank-1 approximation to T [23].

4. Starting strategy. Obviously, it is a concern that the high-order power method might not find the best rank-1 approximation. Questions then arise on

1. How many local minimizers could the objective function f possibly possess?
2. Could suitable starting strategies help to improve the rank-1 approximation?

For order-2 tensors, the answer to the first question is explicitly known. Stationary points to (2.7) are precisely those triplets of singular values and associated singular vectors of the matrix T. Furthermore, the minimizer (or maximizer) of the functional f is typically unique, that is, the triplet corresponding to the largest (or smallest) singular value, whereas all other triplets are saddle points. But for tensors of higher order, the answer is indefinite. Taking 200 random tensors with i.i.d. Gaussian distribution in $\mathbb{R}^{10\times5\times3}$, for example, we find that the high-order power method can find one to four limit points. How the cardinalities of limit points vary in accordance with the size of the tensors is an open question. Table 4.1 depicts a distribution of numbers of limit points out of this particular random test. Since the ALS algorithm is an ascent method for the generalized Rayleigh quotient λ, these limit points are local minimizers for f. Figure 4.1 represents the probability distribution of multiple limit points from 10 random tensors. It clearly illustrates that, when multiple limit points are present, there is a high possibility that the best rank-1 approximation is not found.

# of limit points	1	2	3	4
Percentage of occurrence	29.50%	52.00%	17.00%	1.50%

TABLE 4.1
Distribution of numbers of limit points for tensors in $\mathbb{R}^{10\times5\times3}$

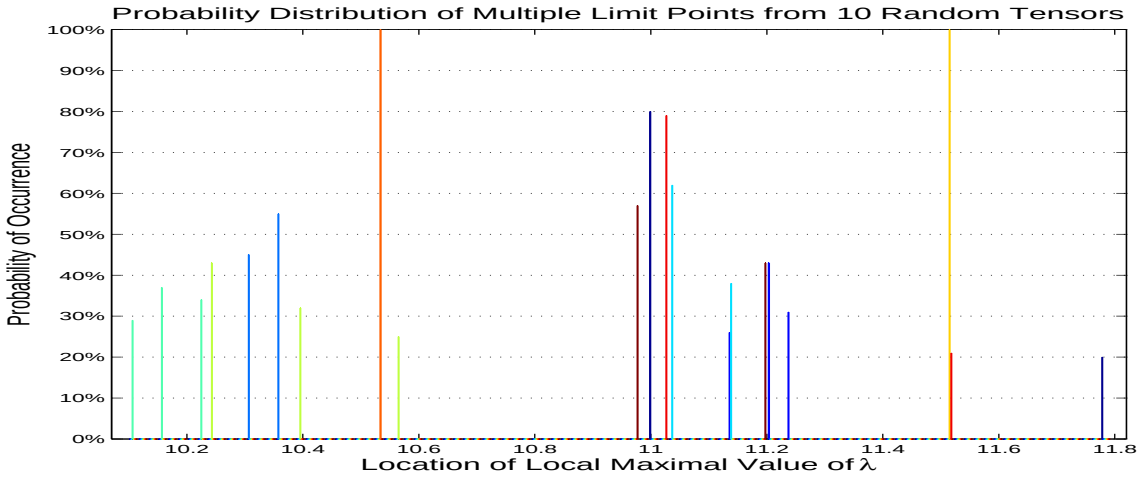


FIGURE 4.1. *Probability distribution of multiple local minimizers for 10 random tensors (identified by distinct colors) in* $\mathbb{R}^{10\times5\times3}$.

Since we are interested in the best rank-1 approximation, it might be sensible to compare the ALS results with those from the global optimization techniques. The objective of global optimization is to find the globally best solution in the presence of multiple local optima. Over the years, many strategies for global optimization have been proposed. Far from being exhaustive, we mention the multistart framework enhanced with sophisticated scatter search algorithm [25]; generic algorithms motivated by survival of the fittest principle [11, 27]; the simulated annealing mimicking the cooling of crystal structures [22, 26]; and pattern search methods comparing instantiations to explore new regions for global search [1, 17]. A comprehensive survey on continuous global optimization can be found in [21]. Motivated by the experiment observed in Figure 4.1, which itself uses multiple random starting points, we choose to use the MULTISTART method with solver FMINCON from the global optimization toolbox in MATLAB to carry out the comparison. As expected, after sufficiently many trials of the ALS algorithm with random starting points for a fixed tensor, the resulting best solution is "compatible" with that obtained from the MULTISTART method. The difference is that the number of trials needed for the ALS algorithm to secure the best solution is often much larger than that needed for the MULTISTART method which employs a more systematic scatter search, whereas the MULTISTART is a much more computationally involved technique.

We thus wonder whether a suitable starting strategy could help to improve the performance of the ALS algorithm. We use order-3 tensors to demonstrate a starting strategy. The idea is based on rewriting the generalized Rayleigh quotient (2.8) via the identity (2.2) as

$$\lambda = \langle T, \mathbf{u}^{(1)} \otimes \mathbf{u}^{(2)} \otimes \mathbf{u}^{(3)} \rangle = \langle \sum_{i=1}^{p} u_i^{(1)} \tau_{i,:}, \mathbf{u}^{(2)} {\mathbf{u}^{(3)}}^\top \rangle,$$

where $\tau_{i,:} \in \mathbb{R}^{m\times n}$ is the i-th slice of T in the first direction. (In practice, we choose the direction associated with the highest p, m, or n.) For fixed $\mathbf{u}^{(1)} \in \mathbb{R}^p$, it is clear that $(\lambda, \mathbf{u}^{(2)}, \mathbf{u}^{(3)})$ should be the singular triplet corresponding to the largest singular value of the matrix $\sum_{i=1}^{p} u_i^{(1)} \tau_{i,:}$. The question of finding the best rank-1

approximation therefore is equivalent to the singular value maximization problem

$$\max_{\mathbf{u}^{(1)}\in\mathbb{R}^p,\|\mathbf{u}^{(1)}\|\leq 1} \quad \sigma_1\left(\sum_{i=1}^{p} u_i^{(1)}\mathcal{T}_{i,:}\right). \tag{4.1}$$

Note that the objective function $\sigma_1(\cdot)$, standing for the largest singular value, is convex over the convex set of unit ball in $\mathbb{R}^p$. The absolute maximum therefore exists. However, maximizing a convex function does not fit a semi-definite programming model and has inherent difficulties. Instead, we propose using MULTISTART with low precision to roughly solve the problem (4.1) with the hope of locating a starting point in the vicinity of a global solution. We then turn on the ALS algorithm with this starting point and begin the iteration process.

It is important to note that the MULTISTART method (or any global minimization technique) can only "suggest" a possible global solution, unless it is validated by further testing. It is especially the case in our application since we have solved (4.1) only loosely to estimate a starting point. For this reason, we design an experiment to assess the starting strategy in the following way. We consider five groups of square tensors $p = m = n$ with dimensions 10, 20, . . ., 50, respectively. For each group, we generate 200 random tensors with i.i.d. Gaussian distribution. For each fixed tensor, we start with the estimated solution to (4.1) and terminate the ALS method in precisely 20 iterations. The result, referred to as the potential solution, is then used as the par that the ALS method without the starting strategy must meet. For each fixed tensor, we restart the ALS method with multiple random starting points and search for the best solution per iteration across the trends. The iteration stops only when its best solution differs from the potential solution by an amount less than 10^{-2}. Test results showed in Figure 4.2 clearly evidence the extra iterations that the ALS method with random starting points must take in order to reach a rank-1 approximation compatible to that obtained by the ALS method with a fixed number of 20 iterations and starting strategy. The advantage of the starting strategy is especially high when the size of the tensor gets larger.

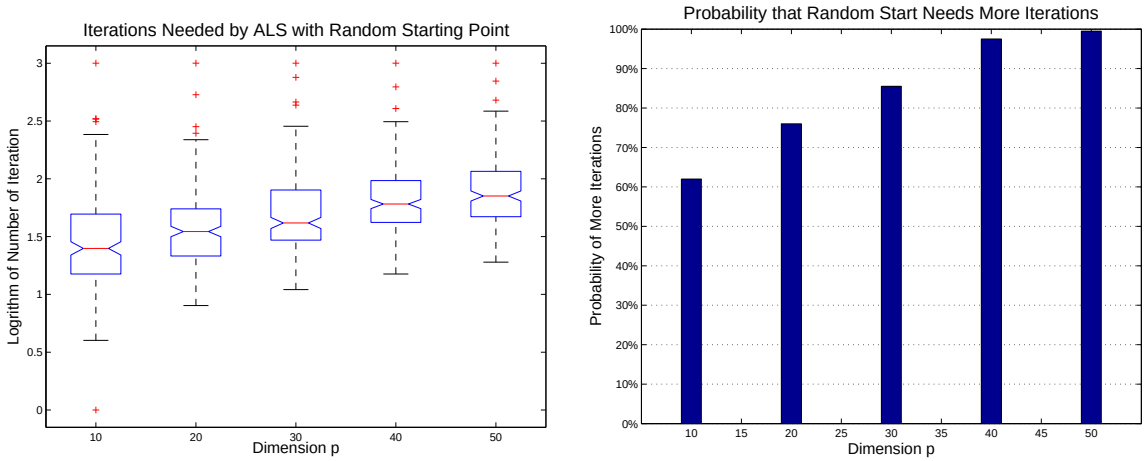


FIGURE 4.2. *Number of ALS iterations needed by random start to be compatible with starting strategy.*

5. Conclusion. Alternating least squares algorithm has been regarded as the workhorse method for computing tensor rank decomposition and low rank approximations. Its application to rank-1 approximation appears as a high-order power method whose computation is quite straightforward. This paper offers a rigorous proof on the global convergence of this method. We also suggest a feasible starting strategy.

High-order tensors may not have the best low-rank approximations. The ALS scheme will also involve inversions. Under specific conditions, such as mutually orthogonal rank-1 components [15], it is possible that the theory developed in this paper can be generalized. We shall report this investigation in a forthcoming paper.

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