

INVERSE MODE PROBLEM FOR REAL AND SYMMETRIC QUADRATIC MODELS

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Abstract. Many natural phenomena can be modeled by a second-order dynamical system $M\ddot{\mathbf{y}} + C\dot{\mathbf{y}} + K\mathbf{y} = f(t)$, where $\mathbf{y}(t)$ stands for an appropriate state variable and M, C, K are time-invariant, real and symmetric matrices. In contrast to the classical inverse vibration problem where a model is to be determined from natural frequencies corresponding to various boundary conditions, the inverse mode problem concerns the reconstruction of the coefficient matrices (M, C, K) from a prescribed or observed subset of natural modes. This paper set forth a mathematical framework for the inverse mode problem and resolves some open questions raised in the literature. In particular, it shows that, given merely the desirable structure of the spectrum, namely, given the cardinalities of real or complex eigenvalues but not the actual eigenvalues, the set of eigenvectors can be completed via solving an under-determined nonlinear system of equations. This completion suffices to construct symmetric coefficient matrices (M, C, K) whereas the underlying system can have arbitrary eigenvalues. Generic conditions under which the real symmetric quadratic inverse mode problem is solvable are discussed. Applications to important tasks such as updating models without spill-over or constructing models with positive semi-definite coefficient matrices are discussed.

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1. Introduction. It is well known that the dynamical behavior of the differential system

$$M\ddot{\mathbf{y}} + C\dot{\mathbf{y}} + K\mathbf{y} = f(t), \quad (1.1)$$

arising from many important applications, is directly related to the eigenpairs (λ, \mathbf{u}) associated with the quadratic eigenvalue problem (QEP)

$$\Omega(\lambda)\mathbf{u} = 0, \quad (1.2)$$

where $\Omega(\lambda)$ denotes the quadratic matrix polynomial

$$\Omega(\lambda) := \Omega(\lambda; M, C, K) = \lambda^2 M + \lambda C + K. \quad (1.3)$$

The process of retrieving and analyzing the spectral information (λ, \mathbf{u}) has received considerable attention because of its significant consequences in various disciplines of sciences and engineering including electrical oscillation, applied mechanics, vibro-acoustics, fluid mechanics, and signal processing [36]. Typically, the dynamical behavior of a physical system depends on a prior known physical parameters which are embedded in the coefficient matrices M, C and K . Solving the QEP is considered as a forward problem. On the other hand, there are situations where we wish to confirm, decide, or validate the parameters of the system through its expected or observed behavior. This way of determining physical parameters from complete or partial eigeninformation constitutes the notion of quadratic inverse eigenvalue problems (QIEP). This paper is concerned with a special class of QIEPs.

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There are various ways to characterize the specifics of QIEPs. To set forth the discussion in the sequel, we define a general QIEP as follows:

(QIEP) Construct a nontrivial quadratic matrix polynomial $\mathfrak{Q}(\lambda) = \lambda^2 M + \lambda C + K$ so that its matrix coefficients (M, C, K) are of a specified structure and $\mathfrak{Q}(\lambda)$ has a prescribed subset of either eigenvalues or eigenvectors, or both, as part of its spectral data.

The two key ingredients in formulating a QIEP are the partially available eigeninformation and the desirable structure of the matrix coefficients. The former is due to the fact that it is impossible to retrieve the whole and reliable spectral information for a large or complicated physical system. For example, quantities associated with high frequency terms are generally inclined to measurement errors owing to the finite bandwidth of measuring devices [1, 12, 13, 14, 25]. The latter is due to the reality that the underlying physical systems often impose inherent structures such as the inner-connectivity among elements within the physical configuration or the mandatory nonnegativity of physical parameters.

Research interest in the inverse eigenvalue problems (IEP) has been quite extensive in recent years, with applications ranging from theoretical Sturm-Liouville problems to applied mechanics and discourses from mathematical abstraction to engineering implementation. Some general discussions on IEPs can be found in the review treatises [4, 5, 14, 19]. Most of the studies in the literature thus far are geared toward linear problems, even though the quadratic problems are perhaps the most important in practice and challenging in theory. A partial list illustrating the kaleidoscopic research activities for QIEPs include the work by Ram and Elhay [32] where a damped oscillatory system with symmetric tridiagonal coefficient matrices is determined from two sets of prescribed eigenvalues, by Starek and Inman [35] for nonproportional underdamped systems, by Lancaster and Prells [29] for a solution with positive definite M (denoted henceforth by $M \succ 0$), positive semi-definite C (denoted by $C \succeq 0$), and $K \succ 0$ from complete spectral information about eigenvalues and eigenvectors, by Lancaster [27] where all eigenvalues and only partial eigenvectors corresponding to real eigenvalues are given, by Chu, Kuo and Lin [6] for a special solution with $M \succ 0$ and $K \succeq 0$ from partially prescribed eigenpairs, and by Kuo, Lin and Xu [26] for general solutions $M \succ 0$. See also [9, 10, 31] from the feedback control aspect to reassign the eigenstructure. Despite the numerous endeavors, many QIEP questions remain open [6].

Our main thrust in this paper is the inverse mode problem (IMP), that is, under the mild structural constraint that all coefficient matrices (M, C, K) be real-valued and symmetric, solve the QIEP with a prescribed subset of eigenvectors. Although the formulation under our current consideration appears admittedly at the low end of difficulty and in the simpler discrete form, the notion of an IMP perhaps can be best exemplified by this counter-analogy — In the famous paper entitled “Can one hear the shape of a drum?” [24], the question was, “If one has perfect pitch (to hear the natural frequencies), could one find the shape of a drum?” — The IMP asks, “Can one see the sound of a string?”, that is, “If one has perfect vision (to see the natural modes), could one tell the tone of the string?”

The IMP is yet another special type of QIEPs. First considered by Gladwell [17] for the finite difference model of a vibrating rod, it was shown that, apart from a scale factor, the discrete system could be constructed uniquely from two eigenvalues and corresponding eigenvectors. Other related works on IMPs include the reconstruction of the unique, up to a scale factor, cross-sectional area of an axially vibrating non-uniform rod by Ram and Elishakoff [33] from one eigenvector, a general discussion of a simple chain-like finite element models by Gladwell [18], which naturally involves tridiagonal structure, and a numerical procedure by Ram and Gladwell [34] from a single eigenvalue, two eigenvectors and the total mass of the rod. This paper offers a mathematical framework for the general IMPs subject to the mild constraint of coefficients being merely real and symmetric.

Of particular interest in our results is that, given simply the desirable cardinalities of real or

complex eigenvalues but not the actual eigenvalues, we solve the IMP by first completing the set of eigenvectors via resolving a certain under-determined nonlinear system of equations. The completion of eigenvectors alone, which could be accomplished by numerical optimization, suffices to affirm the existence of real and symmetric coefficient matrices (M, C, K) for the IMP whereas the unspecified eigenvalues can be arbitrary. As an application, the challenging yet important task of updating an existent quadratic model without spill-over can easily be addressed effectually by our theory.

Only if additional structures are imposed upon (M, C, K) , the information of eigenvalues becomes essential. For example, if it is further desirable that $M, K \succ 0$ and $C \succeq 0$, then the eigenvalues must play a critical role in determining whether the completed set of eigenvectors is feasible at all. We have numerical evidence showing that the eigenvectors for the IMP with symmetric and positive definite coefficient matrices cannot be arbitrary.

It is important to point out that our approach in this paper differs from those proposed in the literature in two aspects. First, we exploit the fact that eigenvectors alone are sufficient to address the solvability of real and symmetric IMPs. In this way, the role of eigenvectors is separated from that of eigenvalues in the analysis of QIEPs. Secondly, we exploit the importance of a special block diagonal matrix H whose structure depends only upon the numbers of real and complex eigenvalues. Had all eigenvectors been known, then columns in the adjoint of the eigenvector matrix would be H -orthogonal¹ to themselves (see (2.5)). In this case, this H matrix may be normalized to a canonical form [8, 21]. In an IMP, however, only partial eigenvectors are known, implying that the “rows” of the eigenvector matrix are short of their full length. To remedy this shortage, we insist that part of H must be treated as unknown. We employ the homogeneous nonlinear relationship of H -orthogonality to complete the unprescribed eigenvectors. This approach not only simplifies the numerical procedure for solving a symmetric QIEP [2, 26], but also more importantly manifests the entire eigenstructure. This way of thinking to “relax” some designated entries of the matrix H as free variables unifies different approaches in the literature and resolves some difficult issues encountered in [8] and [27].

This paper is organized as follows. We begin in Section 2 with some background information. Two critical facts relating (M, C, K) to the notion of standard pair [21] in a way analogous to the spectral decomposition [8] serve as the foundation of our theory. For QIEP applications, we cannot alter the prescribed eigeninformation. Our focus therefore turns to the special standard pair part of which is composed directly of the given eigenvalues and eigenvectors. In Section 3 we study the structure of the matrix H mentioned above. The key point in our approach is that, unlike (and not allowed to make) the typical assumption for convenience in the literature that H is a constant matrix in its canonical form, the entries of H are part of the unknowns in the IMP formulation. In this setting, we show a solvability condition in terms of only eigenvectors. In Section 4, we exploit a homogeneous, under-determined, nonlinear system of equations which provides the basis of eigenvector completion. An algorithm based on the nonlinear least squares technique is proposed in Section 5 as a numerical means to accomplish eigenvector completion. An application of our theory to the model updating problem with no spill-over phenomenon is discussed in Section 6. This application alone should be of significance by itself. For IMPs, very little information about eigenvalues is needed, except for the cardinalities of their being real or complex numbers. We demonstrate the role of eigenvalues for the case when (M, C, K) are required further to be positive semi-definite in Section 7.

2. Background. In this section, we briefly review a fundamental relationship between the real symmetric coefficient matrices (M, C, K) and the spectral data of the associated quadratic matrix polynomial $\mathfrak{Q}(\lambda)$. We want to bring forth the fact that such a relationship depends on a specially formulated matrix H characterized by the corresponding spectral data.

¹A column vector \mathbf{x} is H -orthogonal to a column vector \mathbf{y} if and only if $\mathbf{x}^*H\mathbf{y} = 0$.

In order to facilitate the distinction among matrices with different attributes, we adopt the following notation system in the subsequent discussion.

- Roman capital letters such as M or H denote real-valued matrices of general sizes.
- German capital letters such as \mathfrak{X} or \mathfrak{J} denote complex-valued matrices of general sizes.
- Script capital letters such as \mathcal{B} or \mathcal{H} denote some specially defined $2n \times 2n$ matrices.
- Calligraphic capital letters such as \mathcal{X} or \mathcal{J} denote those matrices associated specifically to the Jordan canonical decomposition of \mathcal{C} .

Given a quadratic matrix polynomial $\mathfrak{Q}(\lambda)$ with nonsingular leading coefficient matrix M , let \mathcal{B} and \mathcal{C} in $\mathbb{R}^{2n \times 2n}$ be defined by

$$\mathcal{B} := \begin{bmatrix} C & M \\ M & 0 \end{bmatrix}, \quad \mathcal{C} := \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix}. \quad (2.1)$$

The notion of a standard pair introduced in the GLR theory [20, 21] is an elegant way to encapsulate the general eigenstructure of $\mathfrak{Q}(\lambda)$.

DEFINITION 2.1. *A pair of matrices $(\mathfrak{X}, \mathfrak{J}) \in \mathbb{C}^{n \times 2n} \times \mathbb{C}^{2n \times 2n}$ is called a standard pair for the quadratic matrix polynomial $\mathfrak{Q}(\lambda)$ if and only if the pair $(\mathfrak{X}, \mathfrak{J})$ satisfies the equation*

$$M\mathfrak{X}\mathfrak{J}^2 + C\mathfrak{X}\mathfrak{J} + K\mathfrak{X} = 0 \quad (2.2)$$

and the square matrix $\begin{bmatrix} \mathfrak{X} \\ \mathfrak{X}\mathfrak{J} \end{bmatrix} \in \mathbb{C}^{2n \times 2n}$ is nonsingular.

Associated with a standard pair $(\mathfrak{X}, \mathfrak{J})$ for a given real symmetric quadratic matrix polynomial $\mathfrak{Q}(\lambda)$, define

$$\mathcal{H}(\mathfrak{X}, \mathfrak{J}) := \begin{bmatrix} \mathfrak{X} \\ \mathfrak{X}\mathfrak{J} \end{bmatrix}^* \mathcal{B} \begin{bmatrix} \mathfrak{X} \\ \mathfrak{X}\mathfrak{J} \end{bmatrix}. \quad (2.3)$$

This matrix $\mathcal{H}(\mathfrak{X}, \mathfrak{J}) \in \mathbb{C}^{2n \times 2n}$ plays a central role in our analysis from two complementary aspects. First, the matrix coefficients (M, C, K) of the quadratic pencil $\mathfrak{Q}(\lambda)$ enjoys a spectral decomposition in the following sense [8, 20].

THEOREM 2.2. *Suppose $(\mathfrak{X}, \mathfrak{J})$ is a standard pair for the quadratic matrix polynomial $\mathfrak{Q}(\lambda)$ and $\mathcal{H} = \mathcal{H}(\mathfrak{X}, \mathfrak{J})$. Then*

$$\begin{cases} M = (\mathfrak{X}\mathfrak{J}\mathcal{H}^{-1}\mathfrak{X}^*)^{-1}, \\ C = -M\mathfrak{X}\mathfrak{J}^2\mathcal{H}^{-1}\mathfrak{X}^*M, \\ K = -M\mathfrak{X}\mathfrak{J}^3\mathcal{H}^{-1}\mathfrak{X}^*M + CM^{-1}C. \end{cases} \quad (2.4)$$

Secondly, the existence of a matrix $\mathcal{H} \in \mathbb{C}^{2n \times 2n}$ qualifies a pair of matrices $(\mathfrak{X}, \mathfrak{J})$ as a standard pair for some real symmetric quadratic matrix polynomial if the following criteria are met [8].

THEOREM 2.3. *Let $(\mathfrak{X}, \mathfrak{J}) \in \mathbb{C}^{n \times 2n} \times \mathbb{C}^{2n \times 2n}$ be some given matrices. If there exists a nonsingular matrix $\mathcal{H} \in \mathbb{C}^{2n \times 2n}$ such that $\mathfrak{X}\mathfrak{J}\mathcal{H}^{-1}\mathfrak{X}^*$ is nonsingular and \mathcal{H} satisfies the three equalities*

$$\begin{cases} \mathfrak{X}\mathcal{H}^{-1}\mathfrak{X}^* = 0, \\ \mathcal{H}\mathfrak{J} = (\mathcal{H}\mathfrak{J})^*, \\ \mathcal{H} = \mathcal{H}^*, \end{cases} \quad (2.5)$$

then $(\mathfrak{X}, \mathfrak{J})$ is a standard pair for the real symmetric quadratic matrix polynomial $\mathfrak{Q}(\lambda)$ whose matrix coefficients (M, C, K) are defined according to (2.4). Moreover, the relationship

$$\mathcal{C} \begin{bmatrix} \mathfrak{X} \\ \mathfrak{X}\mathfrak{J} \end{bmatrix} = \begin{bmatrix} \mathfrak{X} \\ \mathfrak{X}\mathfrak{J} \end{bmatrix} \mathfrak{J}. \quad (2.6)$$

holds and the nonsingular matrix \mathcal{H} is identical to the matrix $\mathcal{H}(\mathfrak{X}, \mathfrak{J})$ defined via (2.3).

The relationship (2.6) strongly suggests one special standard pair which, by its explicit reference to eigenvalues and eigenvectors, is particularly useful for studying the inverse problem. Let the Jordan canonical decomposition of \mathcal{C} be denoted by

$$\mathcal{C} = \mathcal{Q}\mathcal{J}\mathcal{Q}^{-1}, \quad (2.7)$$

where $\mathcal{J} \in \mathbb{C}^{2n \times 2n}$ stands for the a block diagonal matrix consisting of Jordan blocks associated with eigenvalues of \mathcal{C} and columns of $\mathcal{Q} \in \mathbb{C}^{2n \times 2n}$ are comprised of the corresponding generalized eigenvectors. Evenly partition the matrix \mathcal{Q} across rows into two blocks of size $n \times 2n$,

$$\mathcal{Q} := \begin{bmatrix} \mathcal{X} \\ \mathcal{Z} \end{bmatrix}.$$

Rewriting (2.7) as

$$\begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix} \begin{bmatrix} \mathcal{X} \\ \mathcal{Z} \end{bmatrix} = \begin{bmatrix} \mathcal{X} \\ \mathcal{Z} \end{bmatrix} \mathcal{J}, \quad (2.8)$$

we find the two relationships

$$\mathcal{Z} = \mathcal{X}\mathcal{J}, \quad (2.9)$$

$$M\mathcal{X}\mathcal{J}^2 + C\mathcal{X}\mathcal{J} + K\mathcal{X} = 0, \quad (2.10)$$

justifying that the pair $(\mathcal{X}, \mathcal{J})$ is a *standard pair* for the quadratic matrix polynomial $\mathcal{Q}(\lambda)$. It follows that the corresponding

$$\mathcal{H} := \mathcal{H}(\mathcal{X}, \mathcal{J}) = \mathcal{Q}^* \mathcal{B} \mathcal{Q} \quad (2.11)$$

must satisfy the relationships $\mathcal{X}\mathcal{J}\mathcal{H}^{-1}\mathcal{X}^* = M^{-1}$, $\mathcal{X}\mathcal{H}^{-1}\mathcal{X}^* = 0$, $\mathcal{H} = \mathcal{H}^*$, and $\mathcal{H}\mathcal{J} = (\mathcal{H}\mathcal{J})^*$, from which the structure of \mathcal{H} can be characterized.

3. Structure of \mathcal{H} matrix. For simplicity, we shall assume henceforth that all eigenvalues of $\mathcal{Q}(\lambda)$ are simple. The congruence relationship

$$\begin{bmatrix} I_n & -\frac{1}{2}CM^{-1} \\ 0 & I_n \end{bmatrix} \mathcal{B} \begin{bmatrix} I_n & -\frac{1}{2}CM^{-1} \\ 0 & I_n \end{bmatrix}^T = \begin{bmatrix} 0 & M \\ M & 0 \end{bmatrix} \quad (3.1)$$

asserts that the matrix \mathcal{B} must have equal numbers of positive and negative eigenvalues. Likewise, the matrix $\mathcal{H}(\mathfrak{X}, \mathfrak{J})$ in general and \mathcal{H} in particular should also have equal number of positive and negative eigenvalues. Suppose that the Jordan matrix \mathcal{J} along with the corresponding matrix \mathcal{X} of eigenvectors are expressed as

$$\begin{cases} \mathcal{J} &= \text{diag}\{\lambda_1, \bar{\lambda}_1, \lambda_2, \bar{\lambda}_2, \dots, \lambda_t, \bar{\lambda}_t, \lambda_{2t+1}, \dots, \lambda_{2n}\}, \\ \mathcal{X} &= [\mathbf{x}_1, \bar{\mathbf{x}}_1, \mathbf{x}_2, \bar{\mathbf{x}}_2, \dots, \mathbf{x}_t, \bar{\mathbf{x}}_t, \mathbf{x}_{2t+1}, \dots, \mathbf{x}_{2n}], \end{cases} \quad (3.2)$$

where t is the number of distinct complex-conjugate pairs of eigenvalues. Rearranging the ordering of real eigenvalues in \mathcal{J} if necessary, from the facts that $\mathcal{H} = \mathcal{H}^*$ and $\mathcal{H}\mathcal{J} = \mathcal{J}^*\mathcal{H}$, we see that \mathcal{H} must be a block diagonal of the form [15, Chapter VIII, Theorem 1]

$$\mathcal{H} = \text{diag} \left\{ \begin{bmatrix} 0 & \bar{h}_1 \\ h_1 & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & \bar{h}_t \\ h_t & 0 \end{bmatrix}, h_{2t+1}, \dots, h_{2t+r}, -h_{2t+r+1}, \dots, -h_{2n} \right\} \quad (3.3)$$

where $r := n - t$; for $j = 1, \dots, t$, h_j is a complex number; and for $j = 2t + 1, \dots, 2n$, h_j is a positive real number. The values of h_i 's depend on the pair $(\mathcal{X}, \mathcal{J})$. Those real eigenvalues in \mathcal{J} whose corresponding diagonal entries in \mathcal{H} are positive (or negative) are said to have a positive (or negative) *sign characteristic*².

We can convert the standard pair $(\mathcal{X}, \mathcal{J})$ into a real-valued standard pair (X, J) by defining

$$\begin{cases} J & := \mathcal{R}\mathcal{J}\mathcal{R}^* = \text{diag}\{\lambda_1^{[2]}, \dots, \lambda_t^{[2]}, \lambda_{2t+1}, \dots, \lambda_{2n}\} \in \mathbb{R}^{2n \times 2n}, \\ X & := \mathcal{X}\mathcal{R}^* = [\sqrt{2}\mathbf{x}_{1R}, \sqrt{2}\mathbf{x}_{1I}, \dots, \sqrt{2}\mathbf{x}_{tR}, \sqrt{2}\mathbf{x}_{tI}, \mathbf{x}_{2t+1}, \dots, \mathbf{x}_{2n}] \in \mathbb{R}^{n \times 2n}, \end{cases} \quad (3.4)$$

with

$$\mathcal{R} := \text{diag} \left\{ \underbrace{\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}, \dots, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}}_{t \text{ copies}}, I_{2r} \right\}, \quad (3.5)$$

where $i = \sqrt{-1}$ and for $j = 1, \dots, t$,

$$\begin{cases} \lambda_j^{[2]} & = \begin{bmatrix} \alpha_j & \beta_j \\ -\beta_j & \alpha_j \end{bmatrix} \in \mathbb{R}^{2 \times 2}, \quad \text{if } \lambda_j = \alpha_j + i\beta_j, \\ \mathbf{x}_j & = \mathbf{x}_{jR} + i\mathbf{x}_{jI}. \end{cases}$$

The corresponding $H = \mathcal{H}(X, J)$ should have a similar block structure as in \mathcal{H} ,

$$H = \mathcal{R}\mathcal{H}\mathcal{R}^* = \text{diag} \left\{ \begin{bmatrix} a_1 & b_1 \\ b_1 & -a_1 \end{bmatrix}, \dots, \begin{bmatrix} a_t & b_t \\ b_t & -a_t \end{bmatrix}, h_{2t+1}, \dots, h_{2t+r}, -h_{2t+r+1}, \dots, -h_{2n} \right\} \quad (3.6)$$

with $a_j, b_j \in \mathbb{R}$.

In the forward problem, with appropriate scaling and rotations of the eigenvectors, the following canonical form is true even for the case of semi-simple eigenvalues [8, Corollary 3.5].

THEOREM 3.1. *Suppose that all eigenvalues of a given real symmetric quadratic pencil $\mathcal{Q}(\lambda)$ are semi-simple but not necessarily distinct. Then there exists a real standard pair $(\mathfrak{X}, \mathfrak{J})$ such that*

$$\begin{cases} \begin{bmatrix} \mathfrak{X} \\ \mathfrak{X}\mathfrak{J} \end{bmatrix}^\top \begin{bmatrix} C & M \\ M & 0 \end{bmatrix} \begin{bmatrix} \mathfrak{X} \\ \mathfrak{X}\mathfrak{J} \end{bmatrix} & = \Gamma := \text{diag} \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \dots, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\}, \\ \begin{bmatrix} \mathfrak{X} \\ \mathfrak{X}\mathfrak{J} \end{bmatrix}^\top \begin{bmatrix} -K & 0 \\ 0 & M \end{bmatrix} \begin{bmatrix} \mathfrak{X} \\ \mathfrak{X}\mathfrak{J} \end{bmatrix} & = \Gamma\mathfrak{J}. \end{cases} \quad (3.7)$$

The matrix Γ is precisely the so called sip (standard involutory permutation) matrix repeatedly referred by Lancaster [27, 29]. In the IMPs, however, we only have a partial list of eigenvectors in hand. There is not enough information to know how these given eigenvectors should be scaled or rotated. We have to use the eigenvector information as is given, which may not give rise to the sip form as in (3.7). The corresponding blocks in H should be considered as part of the unknowns to be determined. This thinking is fundamentally different from those approaches considered in [27, 29] where the sip form is assumed and thus significantly delimits the solvability because some rank conditions may not be satisfied.

²More details about the concept of sign characteristics can be found in the book [21] and their usages for QIEPs in the two recent articles [27, 29]. We only need the fact that real eigenvalues of \mathcal{J} are divided into two mutually exclusive groups in our discussion.

4. Eigenvector completion. Theorem 2.2 provides a basis of solving a QIEP in terms of a standard pair (X, J) . Nonetheless, we do not know (X, J) in its entirety. Theorem 2.3 gives us a clue that maybe the partial eigeninformation can be extended into a standard pair, so long as a nonsingular matrix H satisfying (2.5) can be found. To solve an IMP, our strategy is to first determine the structure of $H = \mathcal{H}(X, J)$ from the (guessed or assigned) structure, but not values, of J . This would automatically satisfy the last conditions in (2.5). We then use the equation

$$XH^{-1}X^T = 0. \quad (4.1)$$

to build H and to complete X simultaneously. The values of J , which can be assigned almost arbitrarily, are needed only to ensure the invertibility of the matrix $XJH^{-1}X^T$.

Suppose that a prescribed subset of k eigenpairs, closed under complex conjugation, has been given. To fix the idea, suppose that the desirable number t of complex conjugate pairs of eigenvalues or, equivalently, the desirable number $r = n - t$ of real eigenvalues with positive (or negative) sign characteristic in the constructed quadratic pencil is also given³. For convenience, we partition the columns of X as

$$X = [\underbrace{C_0, C_1}_{2t \text{ columns}}, \underbrace{P_0, P_1}_r, \underbrace{N_0, N_1}_r] \quad (4.2)$$

where $[C_0, P_0, N_0]$ is a submatrix of size $n \times k$ whose columns represent the k prescribed eigenvectors. Assume further that the matrices C_0 , P_0 , and N_0 are of sizes $n \times 2k_C$, $n \times k_P$, and $n \times k_N$, respectively, with $k = 2k_C + k_P + k_N$. That is, among the prescribed eigeninformation there are $2k_C$ complex eigenvalues closed under conjugation, k_P real eigenpairs with positive characteristics, and another k_N real eigenpairs with negative characteristics⁴. Columns of $[C_1, P_1, N_1]$ denote the unknown eigenvectors that are to be completed.

It is easy to see that H^{-1} has exactly the same structure as H . We might be as well working on H^{-1} directly. Partition the inverse of the matrix H in (3.6) into blocks of sizes compatible with those in (4.2),

$$H^{-1} = \text{diag} \{ H_0^C, H_1^C, H_0^P, H_1^P, -H_0^N, -H_1^N \}, \quad (4.3)$$

where each block has its own structure, e.g., H_0^C is a $k_C \times k_C$ block diagonal matrix consisting of 2×2 submatrices, H_0^P and H_0^N respectively are $k_P \times k_P$ and $k_N \times k_N$ diagonal matrices with positive diagonal entries, and so on. We rewrite (4.1) as

$$C_0 H_0^C C_0^T + P_0 H_0^P P_0^T - N_0 H_0^N N_0^T = N_1 H_1^N N_1^T - P_1 H_1^P P_1^T - C_1 H_1^C C_1^T. \quad (4.4)$$

Taking into account the fact that H_1^P and H_1^N are diagonal matrices with positive entries and P_1 and N_1 are indeterminate, we can rename the products $P_1(H_1^P)^{1/2}$ and $N_1(H_1^N)^{1/2}$ as the new variables P_1 and N_1 , respectively. Likewise, by the identity

$$\begin{bmatrix} a & b \\ b & -a \end{bmatrix} = U \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} U^T,$$

with

$$U := \underbrace{\begin{bmatrix} a + \sqrt{a^2 + b^2} & -b \\ b & a + \sqrt{a^2 + b^2} \end{bmatrix}}_{\text{orthogonal}} \underbrace{\begin{bmatrix} \frac{1}{\sqrt{b^2 + (a + \sqrt{a^2 + b^2})^2}} & 0 \\ 0 & \frac{1}{\sqrt{b^2 + (a + \sqrt{a^2 + b^2})^2}} \end{bmatrix}}_{\text{scaling}} \sqrt{a^2 + b^2},$$

³This pair of nonnegative integers (t, r) is what we refer to as the structure of J .

⁴In practice, it appears that the choice of k_P and k_N is immaterial for IMPs so long as $k_P + k_N = k - 2k_C$. See the discussion in Section 5.

we can properly rotate and scale the columns of C_1 in (4.4) and, after renaming the new variable the same as C_1 , reduce the system (4.4) to

$$\Omega := C_0 H_0^C C_0^\top + P_0 H_0^P P_0^\top - N_0 H_0^N N_0^\top + C_1 \Upsilon C_1^\top + P_1 P_1^\top - N_1 N_1^\top = 0, \quad (4.5)$$

where, for convenience, we have denoted the constant matrix

$$\Upsilon := \text{diag} \left\{ \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \dots, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}_{t - k_C \text{ copies}} \right\}.$$

It is important to note that this reduction process cannot take place at the left hand side of (4.4) because $[C_0, P_0, N_0]$ are fixed matrices which cannot assimilate the unknown scalings or rotations. The eigenvector completion problem amounts to finding a real matrix $[C_1, P_1, N_1]$ of size $n \times (2n - k)$, $2k_C$ real numbers for the block diagonal matrix H_0^C , and $k_P + k_N$ positive numbers for the diagonal matrices H_0^P and H_0^N so that the equation (4.5) is satisfied. Totally there are $n(2n - k) + k$ unknowns in $\frac{n(n+1)}{2}$ equations. If

$$k < \frac{n(3n - 1)}{2(n - 1)}, \quad (4.6)$$

then the system (4.5) is under-determined. Generically, the algebraic solutions to (4.5) form a nontrivial smooth manifold [23], but for our IMPs we also need positive H_0^P and H_0^N from this solution manifold. The following result can easily be verified from (4.6).

THEOREM 4.1. *The maximal allowable number of prescribed eigenvectors so that the system (4.5) is generically solvable is given by*

$$k_{max} = \begin{cases} 3\ell + 1, & \text{if } n = 2\ell, \\ 3\ell + 2, & \text{if } n = 2\ell + 1. \end{cases} \quad (4.7)$$

It is interesting to note that exactly the same condition (4.7) has been proved in [3, Theorem 3.5] by using an entirely different approach. That is, suppose we are given k eigenpairs $\{(\sigma_j, \mathbf{y}_j)\}_{j=1}^k$ which are closed under complex conjugation. Convert this eigenpair information into real-valued matrices (Σ, Y) in the same way as we did in (3.4). Then the coefficient matrices (M, C, K) for the QIEP with eigenpair (Σ, Y) are solutions to the linear system

$$[M, C, K] \begin{bmatrix} \Sigma Y^2 \\ \Sigma Y \\ \Sigma \end{bmatrix} = 0, \quad (4.8)$$

which is nontrivial if $k \leq k_{max}$. The collection of all possible solutions (M, C, K) to the QIEP forms a *linear* subspace, but no information about the remaining eigenstructure in the reconstructed matrix polynomial is known. In contrast, our current approach tackles the QIEP by first assigning the eigenvalues structure in J and then solving the nonlinear system (4.5) for the remaining eigenvectors $[C_1, P_1, N_1]$. Once the partial eigenvectors Y is fully extended to a complete set X of eigenvectors, the remaining eigenvalues can be almost arbitrarily assigned and the coefficient matrices (M, C, K) are obtainable from the formula (2.4). Observe that the collection of all possible remaining eigenvectors to the QIEP form a *nonlinear* algebraic variety characterized by the polynomial system (4.5).

An example might be more informative to demonstrate our point.

Example 1. Consider the simple case when $n = 2$ and $k = 2k_C = 2$. Suppose

$$C_0 = \begin{bmatrix} 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

is given. There are only two ways to complete the eigenstructure. Assuming first that the remaining two eigenvectors are real, that is, $r = 1$, then we need to determine $H_0^C = \begin{bmatrix} a_1 & b_1 \\ b_1 & -a_1 \end{bmatrix}$, $P_1 = [p_1, p_2]^\top$ and $N_1 = [n_1, n_2]^\top$ from the equation

$$C_0 H_0^C C_0^\top + P_1 P_1^\top - N_1 N_1^\top = \begin{bmatrix} -\frac{1}{2}a_1 + p_1^2 - n_1^2 & -\frac{1}{2}b_1 + p_1 p_2 - n_1 n_2 \\ -\frac{1}{2}b_1 + p_1 p_2 - n_1 n_2 & \frac{1}{2}a_1 + p_2^2 - n_2^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

This amounts to the under-determined system of three equations in six variables

$$\begin{cases} p_1^2 - n_1^2 & = & \frac{1}{2}a_1, \\ p_2^2 - n_2^2 & = & -\frac{1}{2}a_1, \\ p_1 p_2 - n_1 n_2 & = & \frac{1}{2}b_1, \end{cases}$$

whose solution can be expressed as

$$P_1 = \begin{bmatrix} \frac{\sqrt{(n_1+n_2)^2+b_1} \pm \sqrt{(n_1-n_2)^2-b_1}}{2} \\ \frac{\sqrt{(n_1+n_2)^2+b_1} \mp \sqrt{(n_1-n_2)^2-b_1}}{2} \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \frac{-\sqrt{(n_1+n_2)^2+b_1} \pm \sqrt{(n_1-n_2)^2-b_1}}{2} \\ \frac{-\sqrt{(n_1+n_2)^2+b_1} \mp \sqrt{(n_1-n_2)^2-b_1}}{2} \end{bmatrix},$$

with n_1 , n_2 and $b_1 \neq 0$ as free variables. Similarly, assuming the remaining two eigenvectors are complex, that is, $r = 0$, then we need to determine $H_0^C = \begin{bmatrix} a_1 & b_1 \\ b_1 & -a_1 \end{bmatrix}$ and $C_1 = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$ from the equation

$$C_0 H_0^C C_0^\top + C_1 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} C_1^\top = \begin{bmatrix} -\frac{1}{2}a_1 + c_{11}^2 - c_{12}^2 & -\frac{1}{2}b_1 + c_{11}c_{21} - c_{12}c_{22} \\ -\frac{1}{2}b_1 + c_{11}c_{21} - c_{12}c_{22} & \frac{1}{2}a_1 + c_{21}^2 - c_{22}^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

or, equivalently,

$$\begin{cases} c_{11}^2 - c_{12}^2 & = & \frac{1}{2}a_1, \\ c_{21}^2 - c_{22}^2 & = & -\frac{1}{2}a_1, \\ c_{11}c_{21} - c_{12}c_{22} & = & \frac{1}{2}b_1. \end{cases}$$

The solution can be expressed as

$$C_1 = \begin{bmatrix} \frac{\sqrt{(c_{12}+c_{22})^2+b_1} \pm \sqrt{(c_{12}-c_{22})^2-b_1}}{2} & c_{12} \\ \frac{\sqrt{(c_{12}+c_{22})^2+b_1} \mp \sqrt{(c_{12}-c_{22})^2-b_1}}{2} & c_{22} \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \frac{-\sqrt{(c_{12}+c_{22})^2+b_1} \pm \sqrt{(c_{12}-c_{22})^2-b_1}}{2} & c_{12} \\ \frac{-\sqrt{(c_{12}+c_{22})^2+b_1} \mp \sqrt{(c_{12}-c_{22})^2-b_1}}{2} & c_{22} \end{bmatrix},$$

with c_{12} , c_{22} , and $b_1 \neq 0$ arbitrary. These two scenarios are essentially the same.

We conclude this section with two special cases of our theory. The first result bypasses the peculiar rank condition for the QIEP considered in [27] because we reckon H_0^P and H_0^N as additional variables which are more general than the sip matrix. The second result answers an open question raised in the same paper by bringing in H_0^C as an additional variable. In both cases, we offer more flexibility than just using the sip matrix as in [27].

THEOREM 4.2. *Suppose that all eigenvalues are simple. Let $n = t + r$.*

1. *Suppose that all real eigenpairs are given, that is, only $P_0 \in \mathbb{R}^{n \times r}$ and $N_0 \in \mathbb{R}^{n \times r}$ are specified. Then, a necessary condition for the IMP to be solvable is that the system*

$$P_0 H_0^P P_0^\top - N_0 H_0^N N_0^\top + C_1 \Upsilon C_1^\top = 0. \quad (4.9)$$

has a nontrivial solution for nonnegative diagonal matrices $H_0^P, H_0^N \in \mathbb{R}^{r \times r}$ and $C_1 \in \mathbb{R}^{n \times 2t}$. The maximal allowable number r of columns for each P_0 and N_0 is bounded by

$$r < \frac{n(3n-1)}{4(n-1)}. \quad (4.10)$$

2. *Suppose that all the complex eigenpairs are given, that is, only $C_0 \in \mathbb{R}^{n \times 2t}$ is specified. Then, a necessary condition for the IMP to be solvable is that the system*

$$C_0 H_0^C C_0^\top = N_1 N_1^\top - P_1 P_1^\top. \quad (4.11)$$

has a nontrivial solution for a block diagonal matrix $H_0^C \in \mathbb{R}^{2t \times 2t}$ with 2×2 symmetric blocks and $P_1, N_1 \in \mathbb{R}^{n \times r}$. The maximal allowable number t of complex conjugate eigenvectors in C_0 is bounded by

$$t < \frac{n(3n-1)}{4(n-1)}. \quad (4.12)$$

5. Numerical method. In our theory, the unspecified eigenvalues can be arbitrary. To carry out a specific eigenvector completion for an IMP of size n , we will assume that there are t pairs of complex conjugate eigenvalues and r ($= n - t$) pairs of real eigenvalues with opposite sign characteristics in the constructed quadratic matrix polynomial. Out of the k prescribed eigenvectors, it is easy to identify the $2k_C$ complex conjugate eigenvectors. Obviously, it is necessary that $k_C \leq t$. Among the remaining $k - 2k_C$ prescribed real eigenvectors, however, we generally do not know a priori the associated sign characteristics of their corresponding eigenvalues. The splitting of $k - 2k_C$ as the sum $k_P + k_N$ and, hence, the assignment of sign characteristics to the corresponding real eigenvalues therefore are again at random. It is intriguing to study whether the IMP is solvable for arbitrary splitting $n = t + r$ and distribution $k = 2k_C + k_P + k_N \leq k_{max}$. In this discussion, we assume that (t, r) and (k_C, k_P, k_N) are given.

Our theory asserts the algebraic solvability of (4.5) so long as $k \leq k_{max}$. However, we must take into account that the diagonal matrices H_0^P and H_0^N should be positive. One possible approach is to consider the constrained nonlinear least squares optimization problem

$$\begin{cases} \text{Minimize} & f(H_0^C, H_0^P, H_0^N, C_1, P_1, N_1), \\ \text{Subject to} & H_0^P \geq 1 \text{ and } H_0^N \geq 1, \end{cases} \quad (5.1)$$

where

$$f(H_0^C, H_0^P, H_0^N, C_1, P_1, N_1) := \frac{1}{2} \langle \Omega, \Omega \rangle, \quad (5.2)$$

with Ω being defined in (4.5), $\langle \cdot, \cdot \rangle$ denoting the Frobenius inner product, and to avoid the trivial solution, we have scaled ‘‘upward’’ the positivity of H_0^P and H_0^N . Ideally, we would like to see a zero objective value at an optimal solution.

Of particular advantage in our formulation is that the derivatives of f are readily available in closed form, which would help to enhance the efficiency in the optimization process. For example,

by identifying the objective functional as a quadruple map $f : \mathbb{R}^{2k_C} \times \mathbb{R}^{k_P} \times \mathbb{R}^{k_N} \times \mathbb{R}^{n(2t-2k_C)} \times \mathbb{R}^{n(r-k_P)} \times \mathbb{R}^{n(r-k_N)} \rightarrow \mathbb{R}$, the first-order partial derivatives of f with respect to each group of variables are given below.

LEMMA 5.1. Let $\frac{\partial f}{\partial \Phi}$ denote the partial gradient of f in (5.2) with respect to Φ where the symbol Φ stands for any of the six variables $(H_0^C, H_0^P, H_0^N, C_1, P_1, N_1)$. Then

$$\frac{\partial f}{\partial H_0^C} = [\gamma_{1,1} - \gamma_{2,2}, 2\gamma_{21}, \dots, \gamma_{2k_C-1, 2k_C-1} - \gamma_{2k_C, 2k_C}, 2\gamma_{2k_C, 2k_C-1}]^\top, \quad (5.3)$$

with $\gamma_{i,j}$ denoting the (i, j) entry of the matrix $C_0^\top \Omega C_0$,

$$\begin{cases} \frac{\partial f}{\partial H_0^P} = \mathbf{diag}(P_0^\top \Omega P_0), \\ \frac{\partial f}{\partial H_0^N} = -\mathbf{diag}(N_0^\top \Omega N_0), \end{cases} \quad (5.4)$$

with $\mathbf{diag}(A)$ denoting the column vector of the diagonal of the matrix A , and

$$\begin{cases} \frac{\partial f}{\partial C_1} = \mathbf{vec}(2\Omega C_1 \Upsilon), \\ \frac{\partial f}{\partial P_1} = \mathbf{vec}(2\Omega P_1), \\ \frac{\partial f}{\partial N_1} = -\mathbf{vec}(2\Omega N_1), \end{cases} \quad (5.5)$$

with $\mathbf{vec}(B)$ denoting the vectorization of the matrix B by stacking the columns of B into a single column vector.

There are readily available software packages to solve (5.1). For example, the MATLAB routine `fmincon` that implements a subspace trust-region approach based on the interior-reflective Newton method and the preconditioned conjugate gradients method seems capable of finding a solution to (4.5) with high precision. We also have experimented with other optimization packages such as SNOPT [16] with similar success.

6. Model updating with no spill-over. One challenge of practical importance in engineering applications is to update an existent model while keeping vibration parameters not related to the newly measured parameters invariant. The model updating problem can be described as follows;

(MUP) Given a real symmetric quadratic model with coefficient matrices $(\widetilde{M}, \widetilde{C}, \widetilde{K})$ and a few of its associated eigenpairs $\{(\lambda_j, \mathbf{x}_j)\}_{j=1}^k$ with $k < n$, assume that new eigenpairs $\{(\sigma_j, \mathbf{y}_j)\}_{j=1}^k$ have been measured. Update matrices $(\widetilde{M}, \widetilde{C}, \widetilde{K})$ to a new real symmetric quadratic model (M, C, K) such that

- (i) The newly measured $\{(\sigma_j, \mathbf{y}_j)\}_{j=1}^k$ form k eigenpairs of the new model (M, C, K) .
- (ii) The remaining $2n - k$ eigenpairs of (M, C, K) are kept the same as those of the original $(\widetilde{M}, \widetilde{C}, \widetilde{K})$.

The second condition above is known as the *no spill-over phenomenon*. Model updating with no spill-over has been studied extensively. See, for example, [3, 7, 8, 14, 28]. It is most desirable to construct the update (M, C, K) without the knowledge of the remaining $2n - k$ eigeninformation. Our IMP approach can help to resolve the MUP.

The following formulation appears to be a slight modification of the previous result for the eigenvalue embedding problems (EEP) where all eigenvectors are kept invariant [8]. But the analogy is really due to our recent discovery about a necessary condition that the updated eigenvectors $\{\mathbf{y}\}_{j=1}^k$ must satisfy [7, Theorem 4.1]. The easy generalization from EEPs to MUPs therefore is significant because the EEPs are usually regarded as “locum tenentes” in the literature for the much harder MUPs, and now we have almost identical recipes for the solutions.

Assume as before that all eigenvalues of the original model $(\widetilde{M}, \widetilde{C}, \widetilde{K})$ are simple. Let the $2n$ eigenpairs $\{(\lambda_j, \mathbf{x}_j)\}_{j=1}^k$ and $\{(\lambda_i, \mathbf{x}_i)\}_{i=k+1}^{2n}$ of the original model be denoted in real-valued form by

$$\begin{cases} \Lambda_1 & := \text{diag} \left\{ \begin{bmatrix} \alpha_1 & \beta_1 \\ -\beta_1 & \alpha_1 \end{bmatrix}, \dots, \begin{bmatrix} \alpha_{\ell_1} & \beta_{\ell_1} \\ -\beta_{\ell_1} & \alpha_{\ell_1} \end{bmatrix}, \lambda_{2\ell_1+1}, \dots, \lambda_k \right\}, \\ X_1 & := [\mathbf{x}_{1R}, \mathbf{x}_{1I}, \dots, \mathbf{x}_{\ell_1 R}, \mathbf{x}_{\ell_1 I}, \mathbf{x}_{2\ell_1+1}, \dots, \mathbf{x}_k] \end{cases} \quad (6.1)$$

and

$$\begin{cases} \Lambda_2 & := \text{diag} \left\{ \begin{bmatrix} \alpha_{k+1} & \beta_{k+1} \\ -\beta_{k+1} & \alpha_{k+1} \end{bmatrix}, \dots, \begin{bmatrix} \alpha_{k+\ell_2} & \beta_{k+\ell_2} \\ -\beta_{k+\ell_2} & \alpha_{k+\ell_2} \end{bmatrix}, \lambda_{k+2\ell_2+1}, \dots, \lambda_{2n} \right\}, \\ X_2 & := [\mathbf{x}_{(k+1)R}, \mathbf{x}_{(k+1)I}, \dots, \mathbf{x}_{(k+\ell_2)R}, \mathbf{x}_{(k+\ell_2)I}, \mathbf{x}_{k+2\ell_2+1}, \dots, \mathbf{x}_{2n}], \end{cases} \quad (6.2)$$

respectively. Since X_1 is to be updated, we may regard $X_2 = [C_0, P_0, N_0]$ in reference to (4.2). Recall that the corresponding $\widetilde{H} = \mathcal{H}([X_1, X_2], \text{diag}\{\Lambda_1, \Lambda_2\})$ is block diagonal (See (3.6)). The two submatrices

$$\begin{cases} \widetilde{H}_1 & := \begin{bmatrix} X_1 \\ X_1 \Lambda_1 \end{bmatrix}^\top \begin{bmatrix} \widetilde{C} & \widetilde{M} \\ \widetilde{M} & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_1 \Lambda_1 \end{bmatrix}, \\ \widetilde{H}_2 & := \begin{bmatrix} X_2 \\ X_2 \Lambda_2 \end{bmatrix}^\top \begin{bmatrix} \widetilde{C} & \widetilde{M} \\ \widetilde{M} & 0 \end{bmatrix} \begin{bmatrix} X_2 \\ X_2 \Lambda_2 \end{bmatrix}, \end{cases}$$

are also block diagonal and satisfy

$$X_1 \widetilde{H}_1^{-1} X_1^\top + X_2 \widetilde{H}_2^{-1} X_2^\top = 0. \quad (6.3)$$

By Theorem 2.2, we know that

$$\begin{cases} \widetilde{M}^{-1} & = X_1 \Lambda_1 \widetilde{H}_1^{-1} X_1^\top + X_2 \Lambda_2 \widetilde{H}_2^{-1} X_2^\top, \\ \widetilde{C} & = -\widetilde{M} (X_1 \Lambda_1^2 \widetilde{H}_1^{-1} X_1^\top + X_2 \Lambda_2^2 \widetilde{H}_2^{-1} X_2^\top) \widetilde{M}, \\ \widetilde{K} & = -\widetilde{M} (X_1 \Lambda_1^3 \widetilde{H}_1^{-1} X_1^\top + X_2 \Lambda_2^3 \widetilde{H}_2^{-1} X_2^\top) \widetilde{M} + \widetilde{C} \widetilde{M}^{-1} \widetilde{C}. \end{cases}$$

Assume that the structure of the newly measured eigeninformation $\{(\sigma_j, \mathbf{y}_j)\}_{j=1}^k$ is conformal to that of $\{(\lambda_j, \mathbf{x}_j)\}_{j=1}^k$. Let (Σ, Y_1) denote the corresponding real-valued representation of $\{(\sigma_j, \mathbf{y}_j)\}_{j=1}^k$. One one hand, it is known that for the MUP to be solvable we must have

$$Y_1 = X_1 T. \quad (6.4)$$

for some nonsingular matrix $T \in \mathbb{R}^{k \times k}$ [7, Theorem 4.1]. On the other hand, to avoid spill-over in the model updating, our theory demands a nonsingular matrix $\widehat{H} = \text{diag}\{\widehat{H}_1, \widehat{H}_2\}$, with \widehat{H}_1 and \widehat{H}_2 having the same block structures respectively as those of \widetilde{H}_1 and \widetilde{H}_2 , such that

$$Y_1 \widehat{H}_1^{-1} Y_1^\top + X_2 \widehat{H}_2^{-1} X_2^\top = 0, \quad (6.5)$$

even before the eigenvalues are updated. Upon substituting (6.4) into (6.5) and comparing with (6.3), we find an obvious solution \widehat{H} for (4.1) by choosing

$$\begin{cases} \widehat{H}_1 & := T^\top \widetilde{H}_1 T \\ \widehat{H}_2 & := \widetilde{H}_2. \end{cases} \quad (6.6)$$

By Theorem 2.3, we only need to make sure that T is such that

$$\begin{cases} T^\top \tilde{H}_1 T \Sigma \text{ is symmetric,} \\ X_1 T \Sigma T^{-1} \tilde{H}_1^{-1} X_1^\top + X_2 \Lambda_2 \tilde{H}_2^{-1} X_2^\top \text{ in nonsingular,} \end{cases} \quad (6.7)$$

then the MUP is solvable. In this case, the recipe in Theorem 2.2 gives rise to one particular solution to the MUP by

$$\begin{cases} M^{-1} &= X_1 T \Sigma T^{-1} \tilde{H}_1^{-1} X_1^\top + X_2 \Lambda_2 \tilde{H}_2^{-1} X_2^\top, \\ C &= -M(X_1 T \Sigma^2 T^{-1} \tilde{H}_1^{-1} X_1^\top + X_2 \Lambda_2^2 \tilde{H}_2^{-1} X_2^\top)M, \\ K &= -M(X_1 T \Sigma^3 T^{-1} \tilde{H}_1^{-1} X_1^\top + X_2 \Lambda_2^3 \tilde{H}_2^{-1} X_2^\top)M + CM^{-1}C. \end{cases}$$

Combining (6.4) with (6.8), we see that the update takes place in the following way:

$$\begin{cases} M^{-1} &= \tilde{M}^{-1} + X_1(T \Sigma T^{-1} - \Lambda_1) \tilde{H}_1^{-1} X_1^\top, \\ C &= M[\tilde{M}^{-1} \tilde{C} \tilde{M}^{-1} - X_1(T \Sigma^2 T^{-1} - \Lambda_1^2) \tilde{H}_1^{-1} X_1^\top]M, \\ K &= M[\tilde{M}^{-1}(\tilde{K} - \tilde{C} \tilde{M}^{-1} \tilde{C}) \tilde{M}^{-1} - X_1(T \Sigma^3 T^{-1} - \Lambda_1^3) \tilde{H}_1^{-1} X_1^\top]M + CM^{-1}C, \end{cases} \quad (6.8)$$

whereas it is critically essential in formula (6.8) that the update from $(\tilde{M}, \tilde{C}, \tilde{K})$ to (M, C, K) does not involve any information about (Λ_2, X_2) at all.

7. Role of eigenvalues. Thus far, we have shown that eigenvalues play a very small role in the real symmetric IMPs. Only the structure (t, r) of eigenvalues in J is needed for the eigenvector completion process. The reconstructed (M, C, K) literally can have arbitrary eigenvalues. In other words, one cannot “see” the sound of a string⁵! What happens is that the structural constraint of (M, C, K) being merely real and symmetric is too loose. Only if additional constraints are imposed upon (M, C, K) , the information of eigenvalues might become essential.

The subject of structured QIEPs is too complicated to be covered in a single study. In this section we illustrate the role of eigenvalues by considering only the case when M and K are required to be positive definite and C positive semi-definite. An IMP with this kind of structure becomes a much harder problem.

Assume that zero is not an eigenvalue of the desirable quadratic matrix polynomial. Define the moments Γ_j , $j = -1, 0, 1, 2$, by

$$\Gamma_j := X J^j H^{-1} X^\top. \quad (7.1)$$

By the fact that $HJ = (HJ)^\top$, all moments Γ_j are symmetric. We have already seen in Theorem 2.2 that

$$\begin{cases} \Gamma_0 &= 0, \\ \Gamma_1 &= M^{-1}, \\ \Gamma_2 &= -M^{-1}CM^{-1}. \end{cases} \quad (7.2)$$

Post-multiplying both sides of (2.10) by $J^{-1}H^{-1}X$ and using (7.2), we obtain the relationship

$$\Gamma_{-1} = -K^{-1}. \quad (7.3)$$

The following theorem characterizes the positive semi-definiteness for (M, C, K) in terms of moments which, in turn, relate to the eigenvalue matrix J [27, 37].

THEOREM 7.1. *Given (X, J) , let (M, C, K) be the symmetric coefficient matrices constructed from (2.4). Then*

⁵Likewise, twenty-six years after [24], it was answered that one cannot hear the shape of a drum [22].

1. If $M > 0$, $K > 0$, and $C \succeq 0$, then all eigenvalues of J have non-positive real part, Γ_1, Γ_{-1} are nonsingular, and $\Gamma_2 \preceq 0$.
2. If all eigenvalues of J have negative real part, Γ_1, Γ_{-1} are nonsingular, and $\Gamma_2 \preceq 0$, then $M > 0$, $K > 0$, and $C \succeq 0$.

Solving the IMP with real, symmetric, and positive semi-definite (M, C, K) therefore means that the eigenvalues of J , including those already prescribed and those to be completed, must be such that the matrix $XJ^2H^{-1}X^\top$ is negative semi-definite. Completing both the eigenvectors and the eigenvalues simultaneously for structured IMPs is rather challenging task. To our knowledge, this area is still open for further research. Some numerical experiments using truncated QR and semi-definite programming techniques are reported in [11, 30]. We only illustrate below that the completed eigenpair (X, J) must work together in order that an IMP has positive semi-definite coefficient matrices (M, C, K) .

Example 2. Consider the scenarios described in Example 1 where the complex eigenvectors are prescribed through the matrix C_0 . Assume the prescribed eigenvalues are given by $J_0^C = \begin{bmatrix} -2 & 6 \\ -6 & -2 \end{bmatrix}$. Consider the first case $r = 1$ where H_0^C and the two real eigenvectors P_1 and N_1 are to be constructed. Taking advantage of the free parameters already established in Example 1, we assume $n_1 = 2$, $n_2 = -1$ and $b_1 = 4$ so that the completed eigenvectors are given by

$$X = \frac{1}{2} \begin{bmatrix} 0 & -\sqrt{2} & 2\sqrt{5} & 4 \\ \sqrt{2} & 0 & 0 & -2 \end{bmatrix}.$$

Let the eigenvalues corresponding to P_1 and N_1 be noted as λ_3 and λ_4 , respectively. Certainly, λ_3 and λ_4 must be real and negative. Additionally, in order that $XJ^2H^{-1}X^\top$ be negative semi-definite, its principal minors must alternate signs, leading to the inequalities

$$\begin{aligned} 80 + 5\lambda_3^2 - 4\lambda_4^2 &\leq 0, \\ -8000 + 80\lambda_4^2 - 400\lambda_3^2 - 5\lambda_3^2\lambda_4^2 &\geq 0. \end{aligned}$$

The curves where these minors vanish are sketched in Figure 7.1. It can be checked that all points (λ_3, λ_4) below the solid curve satisfy the inequalities and, therefore, can be used to complete the spectrum J .

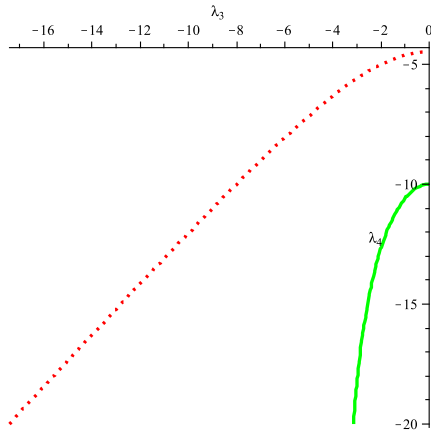


FIG. 7.1. Curves where the principle minors of $XJ^2H^{-1}X^\top$ vanish.

On the other hand, if we change to $n_2 = -3$ while keeping other parameters the same, the corresponding matrix

$$X = \frac{1}{2} \begin{bmatrix} 0 & -\sqrt{2} & \sqrt{5} + \sqrt{21} & 4 \\ \sqrt{2} & 0 & \sqrt{5} - \sqrt{21} & -6 \end{bmatrix}.$$

remains to be a solution to (4.5), but the determinant of $XJ^2H^{-1}X^\top$ is given by

$$-58400 - 4000\sqrt{105} - (1648 + 224\sqrt{105})\lambda_3^2 - (688 - 64\sqrt{105})\lambda_4^2 - \frac{1}{2}(5\sqrt{105} + 73)\lambda_3^2\lambda_4^2,$$

which obviously is always negative, implying that the spectrum J can never be completed with this choice of P_1 and N_1 to make $XJ^2H^{-1}X^\top$ negative semi-definite.

8. Conclusion. By exploiting a special block diagonal matrix H whose structure is pre-determined by the desirable numbers (t, r) of complex and real eigenvalues, but not by the actual eigenvalues, we have proposed a general framework to solve the real symmetric IMPs. We reduce the IMP to a problem of solving an under-determined homogeneous equation (4.5).

Our approach has the advantages that any possible splitting $n = t + r$ and distribution $k = 2k_c + k_p + k_n \leq k_{max}$, where k_{max} is given in (4.7), $k_c \leq t$, $k_p, k_n \leq r$, of complex and real eigenpairs with sign characteristic are allowable. We find that eigenvectors alone are sufficient to determine a solution whereas eigenvalues literally can be arbitrary. Eigenvalues come into play in the inverse problem only when (M, C, K) are required to have additional structures. As an important application of our framework, the difficult task of updating an existent model without spill-over can now easily be accomplished.

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