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Gradient flow methods for matrix completion with prescribed eigenvalues

Moody T. Chu [a,*,1], Fasma Diele [b], Ivonne Sgura [c]

[a] *Department of Mathematics, North Carolina State University, Raleigh, NC 27695-8205, USA*
[b] *Istituto per Ricerche di Matematica Applicata—IRMA-CNR, via Amendola 122/I, 70126 Bari, Italy*
[c] *Dipartimento di Matematica "E. De Giorgi", Università di Lecce, via Arnesano, 73100 Lecce, Italy*

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Abstract

Matrix completion with prescribed eigenvalues is a special type of inverse eigenvalue problem. The goal is to construct a matrix subject to both the structural constraint of prescribed entries and the spectral constraint of prescribed spectrum. The challenge of such a completion problem lies in the intertwining of the cardinality and the location of the prescribed entries so that the inverse problem is solvable. An intriguing question is whether matrices can have arbitrary entries at arbitrary locations with arbitrary eigenvalues and how to complete such a matrix. Constructive proofs exist to a certain point (and those proofs, such as the classical Schur–Horn theorem, are amazingly elegant enough in their own right) beyond which very few theories or numerical algorithms are available. In this paper the completion problem is recast as one of minimizing the distance between the isospectral matrices with the prescribed eigenvalues and the affined matrices with the prescribed entries. The gradient flow is proposed as a numerical means to tackle the construction. This approach is general enough that it can be used to explore the existence question when the prescribed entries are at arbitrary locations with arbitrary cardinalities.

* Corresponding author. Tel.: +1-919-515-3265; fax: +1-919-515-3798.
E-mail addresses: chu@math.ncsu.edu (M.T. Chu), irmafd03@area.ba.cnr.it (F. Diele), ivonne.sgura@unile.it (I. Sgura).

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1. Introduction

An inverse eigenvalue problem (IEP) concerns the reconstruction of a structured matrix from prescribed spectral data. Such an inverse problem arises in many disciplines of science. A collection of important applications can be found in a recent survey article [5]. Generally speaking, such an application involves determining parameters of a certain physical system from the knowledge or expectation of its dynamical behavior. Since the dynamical behavior often is governed by the underlying natural frequencies and normal modes, spectral information is entailed in the inverse problem. On the other hand, designated structural stipulation is also involved in the formulation because the desired physical system often is subject to some feasibility constraints. The spectral data involved may consist of complete or only partial information of eigenvalues or eigenvectors. The required structure of the matrices can take many forms, ranging from linear form to implicit qualification. The objective of an inverse eigenvalue problem is to construct a matrix that maintains both the specific structure as well the given spectral property.

Depending on the application, inverse eigenvalue problems appear in many different forms. Thirty-nine types of IEPs are reviewed in [3]. Twenty-one structures of IEPs are discussed in [5]. This paper deals with one special kind of IEPs where, under the circumstances that a portion of the physical system is known *a priori*, a portion of the matrix to be constructed has fixed entries. The prescribed entries are used to characterize the underlying structure. The task is to specify values for the remaining entries so that the *completed* matrix has prescribed eigenvalues.

For convenience, let $\sigma(X)$ denote henceforth the spectrum of a given matrix X. The most general setting of an inverse eigenvalue problem with prescribed entries (PEIEP) can be delineated as follows [18]: Given a certain subset $\mathscr{L} = \{(i_\nu, j_\nu)\}_{\nu=1}^{\ell}$ of pairs of subscripts, $1 \leqslant i_\nu, j_\nu \leqslant n$, a certain set of values $\{a_1, \ldots, a_\ell\}$ over a field $\mathbb{F}$, and a set of n values $\{\lambda_1, \ldots, \lambda_n\}$ over the algebraically closed extension of $\mathbb{F}$, find a matrix $X \in \mathbb{F}^{n\times n}$ such that

$$\begin{cases} \sigma(X) = \{\lambda_1, \ldots, \lambda_n\}, \\ X_{i_\nu, j_\nu} = a_\nu, \quad \text{for } \nu = 1, \ldots, \ell. \end{cases} \tag{1.1}$$

Let $|\mathscr{L}|$ denote the cardinality ℓ of the index set $\mathscr{L}$ in general. The PEIEP is to determine (complete) the values for the $n^2 - |\mathscr{L}|$ positions of X that do not belong to $\mathscr{L}$ so as to satisfy the spectral constraint.

By comparing the coefficients in the characteristic polynomial $\det(\lambda I - X)$ with the symmetric functions of the prescribed eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$, it is clear that solving a PEIEP can be cast as solving a polynomial system. The prescribed data, that is, entries, locations, and spectrum of each PEIEP, determine a polynomial system. At the first glance, if $|\mathscr{L}| < n^2 - n$, such a polynomial system would be under-

determined and should always be solvable. It turns out that whether a matrix can have arbitrary entries at arbitrary locations with arbitrary eigenvalues is not as easy as it seems for two reasons: one is that there are situations where the intertwining of the cardinality, the values, and the location of the prescribed entries affects whether such a polynomial system is consistent and, hence, whether an inverse problem can be completed; and the other is that to complete such a construction numerically, after knowing its existence, is a fairly challenging task.

A very large class of inverse problems can be described as PEIEPs. The additive inverse eigenvalue problem (AIEP), for example, concerns adding a diagonal matrix D to a given matrix A so that $\sigma(A + D)$ agrees with prescribed set of eigenvalues. In the context of PEIEP, the AIEP is equivalent to the condition that all $n^2 - n$ off-diagonal entries of $A + D$ are prescribed. The Jacobi inverse eigenvalue problem (JIEP), on the other hand, concerns constructing a symmetric tridiagonal matrix with prescribed eigenvalues. The Jacobi structure can be considered as a special case of the PEIEP where, in addition to the desired symmetry of the band, all $(n - 1)(n - 2)$ elements outside the tridiagonal band are required to be zero. Both problems have been extensively studied in the literature. Readers are referred to [1,3,5] and the extensive bibliography contained therein for more details.

In the first part of this paper, we want to demonstrate that several other classical results, such as the Schur–Horn theorem, the Mirsky theorem, the Sing–Thompson theorem, and the London–Minc theorem, can also be characterized as relating to PEIEPs. These developments will be briefly reviewed in Section 2. Most of the existence proofs in the literature are elegantly done by mathematical induction. In the environment where a programming language allows a routine to invoke itself, an inductive proof can often be converted into a constructive proof and, hence, a rational algorithm can be developed. Unfortunately, results as such exist only to a certain point beyond which little understanding is known and very few numerical algorithms are available.

Our main contribution in this paper is that we propose a general approach that can handle various kinds of PEIEPs under the same framework. The solution of the PEIEP is equivalent to finding the intersection of two geometric entities, characterized respectively by the spectral constraints and the structural constraints. Our idea is to recast the problem of finding the intersection as a minimization of the distance between these two entities. We show in Section 3 that the gradient of the objective function can be explicitly calculated. A steepest descent gradient flow therefore can be formulated. By integrating this gradient flow numerically, we have developed a reasonable means to tackle the PEIEP. In this paper, we further exploit a restart procedure to stabilize the calculation.

It is worthy to point out that there is another way to formulate a PEIEP. This is the case where the completion only requires a one-to-one correspondence between the ℓ positions in $\mathscr{L}$ and the ℓ prescribed values $\{a_1, \ldots, a_\ell\}$, but this correspondence does not need to be in any specific order. Clearly, the order-specified PEIEP as we defined in (1.1) is only one of the $\ell!$ many possible permutations of this correspondence. This interesting yet more general formulation will not be studied in the present paper.

We should also mention that most inverse eigenvalue problems have multiple solutions. The PEIEPs are of no exception. Therefore, it might be desirable to seek the solution that is least sensitive to perturbations. This interesting subject requires additional attention and will be discussed in a separate paper. In view of the lack of a general theory even for the basic PEIEPs, this paper addresses the fundamental issue of first solving the PEIEPs.

This paper is organized as follows: We begin in Section 2 with a brief review of some classical IEPs. We chronicle how these results have been developed one after another in the literature. This historic recount shows us that there has been a long and outstanding interest in PEIEPs. It also illustrates the difficulties and limitations of our current understanding. In Section 3, we formulate our minimization problem and calculate the gradient. The formulation is generic enough that it can handle PEIEPs with arbitrary entries and arbitrary cardinalities at arbitrary locations. In the event that a solution does not exist, the formulation enables us to find a least squares solution. In Section 4, we propose two numerical procedures to follow the gradient flows. The restart scheme, in particular, seems to stabilize the calculation. Finally, in the last section we report some of our numerical experiments.

2. Classical IEPs

Many different classical IEPs may be classified as PEIEPs with various cardinality $|\mathscr{L}|$ and various prescribed locations $\mathscr{L}$. Most of the existing studies in the literature thus far seem to have been focusing on cases of fairly limited cardinality and specific locations of the prescribed entries. In contrast, our goal of this paper is to propose a framework for the most general PEIEP setting. To emphasize our point we briefly describe the chronological development of some of the classical IEPs in this section. This review not only unifies the formulations of these classical IEPs under the context of PEIEPs, but also provides a motivation for our general computational framework.

2.1. Prescribed entries along the diagonal

Perhaps a natural starting point to consider the PEIEP is the construction of a matrix with prescribed diagonal entries and eigenvalues. We begin with Hermitian matrices and then extend the discussion to general cases.

Recall first that a vector $\mathbf{a} \in \mathbb{R}^n$ is said to majorize $\boldsymbol{\lambda} \in \mathbb{R}^n$ if, assuming the ordering $a_{j_1} \leqslant \cdots \leqslant a_{j_n}$ and $\lambda_{m_1} \leqslant \cdots \leqslant \lambda_{m_n}$ of their elements, the following relationships hold:

$$\begin{cases} \displaystyle\sum_{i=1}^{k} \lambda_{m_i} \leqslant \sum_{i=1}^{k} a_{j_i}, \quad k = 1, \ldots, n, \\ \displaystyle\sum_{i=1}^{n} \lambda_{m_i} = \sum_{i=1}^{n} a_{j_i}. \end{cases} \tag{2.1}$$

The notion of majorization is related to doubly stochastic matrices and many other applications (see [17,20]). For a Hermitian matrix, there is a majorization relationship between its diagonal elements and its eigenvalues, as asserted by the well-known Schur–Horn theorem [16].

Theorem 2.1 (Schur–Horn). *A Hermitian matrix H with eigenvalues $\boldsymbol{\lambda}$ and diagonal entries* **a** *exists if and only if* **a** majorizes $\boldsymbol{\lambda}$.

It turns out that the direction of sufficiency is harder to prove than the direction of necessity. Such a corroboration of existence is precisely the heart of the Schur–Horn inverse eigenvalue problem (SHIEP): Given two vectors **a** and $\boldsymbol{\lambda}$ where **a** majorizes $\boldsymbol{\lambda}$, construct a Hermitian matrix with diagonal **a** and spectrum $\boldsymbol{\lambda}$. Note that the SHIEP is the opposite extreme of the AIEP. For the former, the prescribed entries lie entirely along the diagonal (without any specific ordering). For the latter, the prescribed entries take possession of all off-diagonal positions. On the other hand, it should also be pointed out that the SHIEP is not exactly in the same class of PEIEPs as we have defined earlier because there is an additional Hermitian structure in the SHIEP. The existence of a solution to the SHIEP was originally proved by induction [16]. Numerical construction can be done by using either a continuous method [2] or a finite iterative method [29].

Without the Hermitian structure, the connection between eigenvalues and diagonal entries of a general matrix is characterized by the Mirsky theorem [21].

Theorem 2.2 (Mirsky). *A square matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ and main diagonal elements $a_1, \ldots, a_n$ exists if and only if*

$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} \lambda_i. \tag{2.2}$$

Again, the sufficient condition in the Mirsky theorem leads to a PEIEP with cardinality $|\mathscr{L}| = n$ where the prescribed entries are located precisely on the diagonal. It can be shown that such an inverse problem has a closed-form solution for all n [18], though the algebraic expressions involved could be quite complicated. As an example, the following formulas define a sufficient condition for the 5×5 matrix

$$A = \begin{bmatrix} a_{11} & 1 & 0 & -1 & 0 \\ a_{21} & a_{22} & a_{23} & a_{24} & 1 \\ 0 & 0 & a_{33} & 1 & 0 \\ 0 & a_{42} & a_{43} & a_{44} & 1 \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{bmatrix}$$

to have (arbitrarily) prescribed diagonal elements $\{a_{11}, \ldots, a_{55}\}$ and eigenvalues $\{\lambda_1, \ldots, \lambda_5\}$ where necessarily $\lambda_5 = \sum_{i=1}^{5} a_{ii} - \sum_{j=1}^{4} \lambda_j$, if

$$
\begin{aligned}
a_{21} &= (\lambda_1 - a_{11})\lambda_1 - (\lambda_1 - a_{11})(\lambda_2 + \lambda_1 - a_{11}),\\
a_{23} &= 1 + (\lambda_3 - a_{33})\lambda_3 - (\lambda_3 - a_{33})(\lambda_4 + \lambda_3 - a_{33}),\\
a_{24} &= -\lambda_2 - \lambda_1 + a_{11} + a_{44},\\
a_{42} &= -\lambda_2 - \lambda_1 + a_{11} + a_{22},\\
a_{43} &= (\lambda_3 - a_{33})\lambda_3 - (\lambda_3 - a_{33})(\lambda_4 + \lambda_3 - a_{33}),\\
a_{51} &= (\lambda_2 + \lambda_1 - a_{11} - a_{22})\big((\lambda_1 - a_{11})\lambda_1 - (\lambda_1 - a_{11})(\lambda_2 + \lambda_1 - a_{11})\big),\\
a_{52} &= (\lambda_2 + \lambda_1 - a_{11} - a_{22})(\lambda_2 + \lambda_1 - a_{11}) - (\lambda_2 + \lambda_1 - a_{11} - a_{22})\\
&\quad \times (a_{33} + a_{44} + a_5 - \lambda_3 - \lambda_4) + (\lambda_4 + \lambda_3 - a_{33} - a_{44})\\
&\quad \times (-\lambda_2 - \lambda_1 + a_{11} + a_{22}),\\
a_{53} &= (\lambda_2 + \lambda_1 - a_{11} - a_{22})\big(1 + (\lambda_3 - a_{33})\lambda_3 - (\lambda_3 - a_{33})(\lambda_4 + \lambda_3 - a_{33})\big)\\
&\quad + (\lambda_4 + \lambda_3 - a_{33} - a_{44})\big((\lambda_3 - a_{33})\lambda_3 - (\lambda_3 - a_{33})(\lambda_4 + \lambda_3 - a_{33})\big),\\
a_{54} &= (\lambda_2 + \lambda_1 - a_{11} - a_{22})(\lambda_4 + \lambda_3 - a_{33} - \lambda_2 - \lambda_1 + a_{11})\\
&\quad + (\lambda_4 + \lambda_3 - a_{33} - a_{44})(\lambda_4 + \lambda_3 - a_{33}) - (\lambda_4 + \lambda_3 - a_{33} - a_{44})a_{55}.
\end{aligned}
$$

A PEIEP in general has $n^2 - |\mathscr{L}|$ entries to be determined. If $|\mathscr{L}| < n^2 - n$, then there are more than n entries to be determined, and this is more than the number of equations the prescribed eigenvalues can specify. As such, the PEIEPs usually have multiple solutions. Among these multiple solutions, the one that is *least* sensitive to perturbations of problem data perhaps is most critical from a practical point of view. Such a solution, called the *robust solution* in the literature, usually is found by minimizing the condition number associated with the solution. This issue is in addition to the task of just finding a solution to a PEIEP. In order that we can first focus on the general framework we are about to propose in this paper, we will address the problem of finding the robust solution in another paper. The 5×5 example given above, for instance, is not the unique expression of a solution and might very well not be the robust solution.

To move to the next level of PEIEPs, we mention the notion of *ϱ-diagonal* introduced in [7–9]. Given a permutation ϱ, the positions in a matrix corresponding to the index set $\mathscr{L} = \{(i, \varrho(i))\}_{i=1}^n$ is referred to as the ϱ-diagonal of that matrix. The following de Oliveira theorem generalizes the Mirsky theorem and allows us to discuss a PEIEP with entries prescribed at non-principal diagonals.

Theorem 2.3 (de Oliveira). *Let $\{\lambda_1, \ldots, \lambda_n\}$ and $\{a_1, \ldots, a_n\}$ be two sets of arbitrary numbers over a field $\mathbb{F}$ and let σ be a given permutation. Suppose that at least one of the disjoint cycles in the product representation $\varrho = \varrho_1 \cdots \varrho_s$ of σ has length > 2. Then there exists a matrix $X \in \mathbb{F}^{n\times n}$ such that $\sigma(X) = \{\lambda_1, \ldots, \lambda_n\}$ and $X_{i,\varrho(i)} = a_i$ for $i = 1, \ldots, n$.*

Note that the assumption in the above theorem that at least once cycle has length great than 2 precludes the case that ϱ is the identity and, hence, the equality (2.2)

is not needed. If no cycle is of length > 2, then a similar result holds under some additional restrictions [8, Theorem 2]. We shall see latter that the de Oliveira theorem becomes obsolete because the existence theory for a PEIEP can be further generalized.

The PEIEP concerns the completion of a (square) matrix with prescribed eigenvalues. A natural generalization is to consider the completion of a (rectangular) matrix with prescribed singular values (PEISVP). This PEISVP might well be an open question in the field because we are not aware much discussion in the literature. We note that a PEISVP can be converted into a PEIEP because eigenvalues of the *structured* symmetric matrix

$$C = \begin{bmatrix} 0 & A \\ A^{\mathrm{T}} & 0 \end{bmatrix} \tag{2.3}$$

are precisely the pluses and minuses of singular values of A. The PEIEP for C has the fixed structure of zero diagonal blocks plus whatever prescribed entries inherited from A. The PEISVP for a structured A is solvable if and only if the PEIEP for C with structure defined in (2.3) is solvable.

We mention only the special case of the Sing–Thompson theorem [25,27] to illustrate the notion of PEISVP. The theorem characterizes the relationship between singular values and diagonal entries of a general matrix in the following way.

Theorem 2.4 (Sing–Thompson). *Assume that elements in two given vectors* $\mathbf{d}, \mathbf{s} \in \mathbb{R}^n$ *satisfy* $s_1 \geqslant s_2 \geqslant \cdots \geqslant s_n$ *and* $|d_1| \geqslant |d_2| \geqslant \cdots \geqslant |d_n|$. *Then a real matrix with singular values* $\mathbf{s}$ *and main diagonal entries* $\mathbf{d}$ (*possibly in different ordering*) *exists if and only if*

$$\begin{cases} \displaystyle\sum_{i=1}^{k} |d_i| \leqslant \sum_{i=1}^{k} s_i, \quad \textit{for } k = 1, \ldots, n, \\ \displaystyle\left(\sum_{i=1}^{n-1} |d_i| \right) - |d_n| \leqslant \left(\sum_{i=1}^{n-1} s_i \right) - s_n. \end{cases} \tag{2.4}$$

As for the SHIEP, the sufficient condition in the Sing–Thompson theorem gives rise to an inverse singular value problem (STISVP): construct a square matrix with prescribed diagonals and singular values, if (2.4) is satisfied. The original inductive proof can translated into a divide-and-conquer algorithm [4] that, in return, can conveniently be implemented in a programming environment that supports recursion.

2.2. Prescribed entries at arbitrary locations

The cardinality and the location of the prescribed entries are not totally independent of each other in the description of a PEIEP. The specified locations sometimes imply inadvertently additional constraints on the problem. The PEIEP involved in the Mirsky theorem is one such instance. The fact that the prescribed entries are on the diagonal imply that the condition (2.2) must hold so that only $n - 1$ prescribed

entries $a_1, \ldots, a_{n-1}$ are involved in the (Mirsky) PEIEP. The value for a_n is necessarily determined from (2.2) due to the specific location of diagonal entries. We thus wonder whether matrices can have $n-1$ arbitrary prescribed entries at $n-1$ arbitrary locations with n arbitrary eigenvalues. The affirmative answer comes from the London–Minc theorem [19] that was also proved in [8].

Theorem 2.5 (London–Minc). *Let* $\{\lambda_1, \ldots, \lambda_n\}$ *and* $\{a_1, \ldots, a_{n-1}\}$ *be two sets of arbitrary numbers over a field* $\mathbb{F}$. *Suppose* $\mathscr{L} = \{(i_\nu, j_\nu)\}_{\nu=1}^{n-1}$ *is a set of arbitrary but distinct positions. Then there exists a matrix* $X \in \mathbb{F}^{n\times n}$ *such that* $\sigma(X) = \{\lambda_1, \ldots, \lambda_n\}$ *and* $X_{i_\nu, j_\nu} = a_\nu$ *for* $\nu = 1, \ldots, n-1$.

An immediate follow-up question of the London–Minc theorem is this: how many more entries of a matrix can be specified, and the associated PEIEP still be solvable. Obviously, we must be cautious that the locations of these prescribed entries might affect the solvability. With $|\mathscr{L}| = n-1$, the London–Minc theorem asserts that the PEIEP is always solvable without any location constraints. With $|\mathscr{L}| = n$, the following theorem generalizes both the Mirsky and the de Oliveira results. Furthermore, it nicely characterizes the only possible cases where the location of prescribed entries will affect the solvability of a PEIEP. The proof can be found in [18, Section 3b].

Theorem 2.6 (Ikramov–Chugunov). *Suppose that the field* $\mathbb{F}$ *is algebraically closed and that* $|\mathscr{L}| = n$. *Assume that* $\mathscr{L}$ *is arbitrary, but if* $\mathscr{L}$ *happens to contain all positions of the principal diagonal or all positions of a certain row, then the following two conditions must be met, respectively*:

$$\begin{cases} \text{That (2.2) is satisfied}, & \text{if } \mathscr{L} = \{(i,i)\}_{i=1}^{n}, \text{ or} \\ \text{That } a_i = \lambda_j \text{ for some } j, & \text{if } \mathscr{L} = \{(i,\nu)\}_{\nu=1}^{n} \text{ and } a_\nu = 0 \text{ for all } \nu \neq i. \end{cases} \tag{2.5}$$

Then the PEIEP is solvable via rational algorithms in $\mathbb{F}$.

When $|\mathscr{L}| < n^2 - n$, there are more unknowns than the number of equations that the prescribed eigenvalues can defined. It is conceivable that this situation will leave more degrees of freedom for manipulation in constructing such a matrix. So how much further can $|\mathscr{L}|$ go in a PEIEP without suffering from much restriction on the location of prescribed entries? To help better grasp the scope of this complicated issue, we reconsider the subclass AIEP of the PEIEP before we return to this question in Section 2.3.

The classical AIEP exemplifies the other extreme of PEIEPs where there is no room left for free locations. Recall that the AIEP concerns adding a diagonal matrix D to a given matrix A so that $\sigma(A+D)$ has a prescribed spectrum. (In a more general sense, any PEIEP is an AIEP in that the matrix D to be added needs not be a diagonal. Rather, the entries to be added are located at those positions that are

complement to the given index set $\mathscr{L}$.) In the classical AIEP, the prescribed entries consist of all elements not on the principal diagonal. Thus, $|\mathscr{L}| = n^2 - n$. In this case, only n positions (along the diagonal) are left to be determined from the n eigenvalues. It is remarkable to have the following result due to Friedland [10] (see also [11]).

Theorem 2.7 (Friedland). *The AIEP over any algebraically closed field is always solvable. If n is the order of the problem, then there exist at most $n!$ solutions. For almost all given $\{\lambda_1, \ldots, \lambda_n\}$, there are exactly $n!$ solution.*

In contrast to Theorem 2.6, the AIEP in general cannot be solved in finitely many steps. The AIEP in which all off-diagonal entries are 1, for example, is not solvable in radicals for $n \geqslant 5$. The AIEP for a Jacobi matrix with subdiagonal (and superdiagonal) entries 1 is not solvable in radicals even for $n = 4$ [18]. The AIEP has to be solved by other types of numerical methods [5,12,22].

It is critical to observe that the solvability assured in both Theorems 2.6 and 2.7 requires that the underlying field $\mathbb{F}$ be algebraically closed. In [3], such an AIEP was referred to as AIEP3. The AIEP over the field $\mathbb{R}$ of real numbers was referred to as the AIEP1, and AIEP2 if the matrix A is real symmetric. All these problems are different. The AIEP is not always solvable over $\mathbb{R}$. It is easy to see, for example, that a necessary condition for the real solvability of the AIEP1 is that

$$\sum_{i \neq j} (\lambda_i - \lambda_j)^2 \geqslant 2n \sum_{i \neq j} a_{ij} a_{ji}.$$

On the other hand, let $\pi(X) := \|X - \mathrm{diag}(X)\|_\infty$ denote a measure of the size of the off-diagonal entries of a given matrix X. The following theorem demonstrate that enough separation of prescribed eigenvalues relative to the size of the (prescribed) off-diagonal entries of A leads to some sufficient conditions for the real solvability [6,13] for the AIEP.

Theorem 2.8. *Given a set $\boldsymbol{\lambda} = \{\lambda_1, \ldots, \lambda_n\}$ of eigenvalues, define*

$$d(\boldsymbol{\lambda}) := \min_{i \neq j} |\lambda_i - \lambda_j|. \tag{2.6}$$

Then,

1. (Hadeler) *If $d(\boldsymbol{\lambda}) > 2\sqrt{3}(\pi(A \circ A))^{1/2}$, then the AIEP2 is solvable.*
2. (de Oliveira) *If $d(\boldsymbol{\lambda}) > 4\pi(A)$, then the AIEP1 is solvable.*

This theorem offers no clue on what will happen when the separation $d(\boldsymbol{\lambda})$ is too small. Consider the example where $A = \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix}$ and $\boldsymbol{\lambda} = \{3, -3\}$. Note that $d(\boldsymbol{\lambda}) = 6$ and does not satisfy Hadeler's sufficient condition. Yet, $D = \mathrm{diag}\{0, 0\}$ does solve the AIEP2. At the extreme case, Shapiro [24] and Sun and Qiang [26] proved that

the problem would be unsolvable almost everywhere. This is very different from the complex solvability guaranteed by the Friedland theorem.

Theorem 2.9 (Shapiro–Sun–Qiang). *Both AIEP1 and AIEP2 are unsolvable almost everywhere if multiple eigenvalues are present in* $\boldsymbol{\lambda}$.

In addition, the symmetric problem AIEP2 enjoys a complete set of orthogonal eigenvectors that, in turn, facilitates a sensitivity analysis for the AIEP2 as was done in [28, Corollary 4.5.5].

Theorem 2.10 (Xu). *Suppose D is a solution to the AIEP2 with symmetric matrix A and eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$. Let the spectral decomposition of $A + D$ be written as $A + D = Q(D)^{\mathrm{T}} \operatorname{diag}\{\lambda_1, \ldots, \lambda_n\} Q(D)$ with $Q(D) = [q_{ij}(D)] = [\mathbf{q}_1, \ldots, \mathbf{q}_n]$. Define*

$$\Omega(D) := \left[q_{ji}^2(D)\right],$$
$$b(D) := \left[\mathbf{q}_1(D)^{\mathrm{T}} A \mathbf{q}_1(D), \ldots, \mathbf{q}_n(D)^{\mathrm{T}} A \mathbf{q}_n(D)\right]^{\mathrm{T}}.$$

Assume that the matrix $\Omega(D)$ is nonsingular and that the perturbation

$$\delta = \|\lambda - \tilde{\lambda}\|_\infty + \|A - \tilde{A}\|_2$$

is sufficiently small. Then the AIEP2 associated with the perturbed data $\tilde{A}$ and $\tilde{\lambda}$ is solvable. Furthermore, there is a solution $\tilde{D}$ near to X in the sense that

$$\frac{\|D - \tilde{D}\|_\infty}{\|D\|_\infty} \leqslant \kappa_\infty(\Omega(D)) \left(\frac{\delta}{\|\lambda - b\|_\infty}\right) + \mathrm{O}(\delta^2),$$

where $\kappa_\infty(M)$ stands for the condition number of the matrix M in the infinity norm.

In view of these classical results and the advance made for the class of AIEPs, it is reasonable to ask how far the existence theory can be extended beyond the AIEPs, particular for $n \leqslant |\mathscr{L}| \leqslant n^2 - n$. It is equally important to develop some numerical schemes to complete the matrix construction for a given PEIEP.

2.3. Cardinality and locations

The prescribed entries in both the SHIEP and the STISVP are located along the diagonal; this means certain inequalities involving the prescribed eigenvalues and entries must be satisfied (Theorems 2.1 and 2.4). The prescribed entries in an AIEP being located at the off-diagonal, the complex solvability is answered in Theorem 2.7. In all these cases, the prescribed entries are located at special positions. Theorem 2.6 relaxes the specification to arbitrary locations and, under very mild conditions, asserts the existence of a solution to the PEIEP when $|\mathscr{L}| = n$. It is clear that we somehow need to strike a balance between how freely we want the prescribed entries to be located and how many prescribed entries we want to impose upon a PEIEP.

Suppose we insist on the freedom of location with restrictions no more than the obvious stipulation such as (2.5). Then possibly the strongest result known today about $|\mathscr{L}|$ in the class of PEIEPs is the work presented in the M.Sc. thesis by Hershkowitz [14] that was proved again in [15,18].

Theorem 2.11 (Hershkowitz). *Suppose that the field $\mathbb{F}$ is algebraically closed and that $|\mathscr{L}| = 2n - 3$. Assume that $\mathscr{L}$ is arbitrary, but if $\mathscr{L}$ happens to contain as a subset all positions of the principal diagonal or all positions of a certain row, then the following two conditions must be met, respectively*:

$$\begin{cases} \text{That (2.2) is satisfied}, & \text{if } \mathscr{L} \supseteq \{(i,i)\}_{i=1}^{n}, \text{ or} \\ \text{That } a_i = \lambda_j \text{ for some } j, & \text{if } \mathscr{L} \supseteq \{(i,v)\}_{v=1}^{n} \text{ and } a_v = 0 \text{ for all } v \neq i. \end{cases} \tag{2.7}$$

Then the PEIEP is solvable in $\mathbb{F}$.

Note that the effect of locations in $\mathscr{L}$ is limited to the two conditions (2.7) stated in the theorem. These conditions are quite straightforward, so we might say that the Hershkowitz theorem has the least restriction on the locations of prescribed entries. The proof of the Hershkowitz theorem was established by induction. In principle, it was declared in [18] that the construction could be done by a rational algorithm. The seven basic cases plus the many subcases of analysis in the 15-page proof might make a computer implementation quite a challenge. It would be interesting to see if other numerical algorithms could be developed. Even beyond, we are curious to know what can be said and done for the case when $|\mathscr{L}| > 2n - 3$. Toward that goal, we propose our optimization formulation.

3. Optimization formulation

The approach proposed below can easily be generalized to the complex case, but we shall limit our discussion to the real matrices henceforth. Consequently, the prescribed eigenvalues $\lambda_1, \ldots, \lambda_n$ are necessarily closed under complex conjugation.

Let $\Lambda \in \mathbb{R}^{n\times n}$ denote a matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$. The simplest choice of Λ would be either $\Lambda = \operatorname{diag}\{\lambda_1, \ldots, \lambda_n\}$, if all eigenvalues are real, or the block diagonal matrix with one 2×2 real-valued block for every complex conjugate pair of eigenvalues. However, we shall see in Section 4 that during the restart procedure the matrix Λ will be replaced step by step by new matrices with the same prescribed eigenvalues. If necessary, we could also consider Λ as the (real) Jordan canonical form or the Schur form to include the geometric multiplicities. Let $\mathscr{G}l(n)$ denote the general group of $n \times n$ nonsingular matrices in $\mathbb{R}^{n\times n}$. The set

$$\mathscr{M}(\Lambda) = \left\{ V\Lambda V^{-1} \mid V \in \mathscr{G}l(n) \right\} \tag{3.1}$$

consists of all matrices that are isospectral to (and with the same kind of geometric multiplicity as) Λ. Given an index subset of locations $\mathscr{L} = \{(i_\nu, j_\nu)\}_{\nu=1}^{\ell}$ and the prescribed values $\mathbf{a} = \{a_1, \ldots, a_\ell\}$, the set

$$\mathscr{S}(\mathscr{L}, \mathbf{a}) = \left\{A \in \mathbb{R}^{n\times n} \mid A_{i_\nu j_\nu} = a_\nu, \quad \nu = 1, \ldots, \ell\right\} \tag{3.2}$$

contains all matrices with the prescribed entries at the desired locations. Solving the PEIEP is equivalent to finding the intersection of $\mathscr{M}(\Lambda)$ and $\mathscr{S}(\mathscr{L}, \mathbf{a})$. Toward that end, we propose a least squares approximation.

3.1. Descent flow

For convenience, split any given matrix X as the sum

$$X = X_{\mathscr{L}} + X_{\mathscr{L}^c} \tag{3.3}$$

where entries in $X_{\mathscr{L}}$ are the same as X, except those that do not correspond to positions in $\mathscr{L}$ are set identically zero and $\mathscr{L}^c$ is simply the index subset complementary to $\mathscr{L}$. The drawing in Fig. 1, though only symbolic, indicates the various situations in our approach. With respect to the Frobenius inner product

$$\langle A, B\rangle = \sum_{i,j=1}^{n} a_{ij}b_{ij},$$

the projection $P(X)$ of any matrix X onto the affine subspace $\mathscr{S}(\mathscr{L}, \mathbf{a})$ is given by

$$P(X) = A_{\mathscr{L}} + X_{\mathscr{L}^c}, \tag{3.4}$$

where $A_{\mathscr{L}}$ is the constant matrix in $\mathscr{S}(\mathscr{L}, \mathbf{a})$ with zero entries at all locations corresponding to $\mathscr{L}^c$. Note that the Fréchet derivative of the projection P at X acting on a general $H \in \mathbb{R}^{n\times n}$ is simply

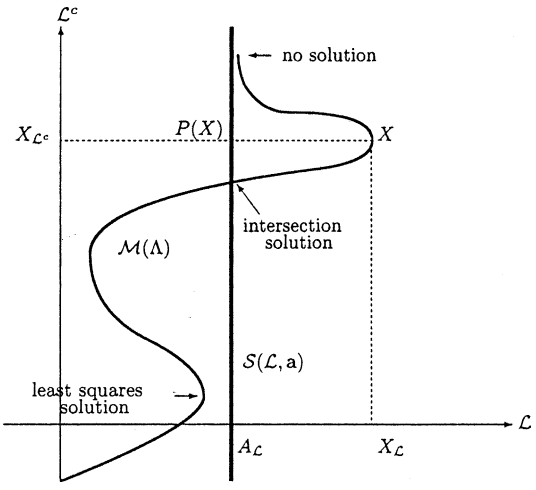

Fig. 1. Representation of splitting, intersection, and least squares solution.

$$P'(X) \cdot H = H_{\mathscr{L}^c}. \tag{3.5}$$

For each given $X \in \mathscr{M}(\Lambda)$, we intend to minimize the distance between X and $\mathscr{S}(\mathscr{L}, \mathbf{a})$. Equivalently, we want to minimize the function defined by

$$f(X) = \tfrac{1}{2}\langle X - P(X), X - P(X)\rangle. \tag{3.6}$$

We can rewrite this minimization more conveniently as an unconstrained optimization problem in terms of V in the open set $\mathscr{G}l(n)$. Let $X = V\Lambda V^{-1}$. The objective function $f(X)$ can be written as

$$g(V) = \tfrac{1}{2}\langle V\Lambda V^{-1} - P(V\Lambda V^{-1}), V\Lambda V^{-1} - P(V\Lambda V^{-1})\rangle. \tag{3.7}$$

Since the action of the derivative of P (see (3.5)) is perpendicular to the residual $X - P(X) = X_{\mathscr{L}} - A_{\mathscr{L}}$, the Fréchet derivative of g at $V \in \mathscr{G}l(n)$ acting on $H \in \mathbb{R}^{n\times n}$ is given by

$$g'(V) \cdot H = \langle H\Lambda V^{-1} - V\Lambda V^{-1}HV^{-1}, V\Lambda V^{-1} - P(V\Lambda V^{-1})\rangle. \tag{3.8}$$

By the Riesz representation theorem and the fact that

$$\langle A, BC\rangle = \langle B^{\mathrm{T}}A, C\rangle = \langle AC^{\mathrm{T}}, B\rangle,$$

we find that the gradient ∇g of the objective function g is given by

$$\begin{aligned}\nabla g(V) = {} & \big(V\Lambda V^{-1} - P(V\Lambda V^{-1})\big)V^{-\mathrm{T}}\Lambda \\ & - (V\Lambda V^{-1})^{\mathrm{T}}\big(V\Lambda V^{-1} - P(V\Lambda V^{-1})\big)V^{-\mathrm{T}}.\end{aligned} \tag{3.9}$$

Equivalently, the equation

$$\nabla g(V)V^{\mathrm{T}} = [X - P(X), X^{\mathrm{T}}] \tag{3.10}$$

with $[M, N] = MN - NM$ denoting the Lie bracket commutator is true. It follows that the vector field

$$\frac{\mathrm{d}V}{\mathrm{d}t} = k(X)V^{-\mathrm{T}}, \tag{3.11}$$

where

$$k(X) = [X^{\mathrm{T}}, X - P(X)], \tag{3.12}$$

defines a flow in the open set $\mathscr{G}l(n)$ and moves in the steepest descent direction to reduce the value of $g(V)$. Likewise, the vector field

$$\frac{\mathrm{d}X}{\mathrm{d}t} = [h_V(X), X] \tag{3.13}$$

with

$$h_V(X) = k(X)V^{-\mathrm{T}}V^{-1} \tag{3.14}$$

defines the steepest descent flow on the manifold $\mathscr{M}(\Lambda)$ for $f(X)$. The system (3.13) is not particularly important in practice since it is known that its solution is $X(t) = V(t)\Lambda V(t)^{-1}$.

The differential equation (3.11) can be readily integrated from a starting point, say, $V(0) = I$. This forms the basis of our numerical algorithm. We shall discuss its numerical implementation and, particularly, a restart strategy to avoid ill-conditioning of V in more detail in Section 4. At this moment, it is appropriate to point out that the framework of our gradient flow (3.11) applies to general PEIEPs with any kind of index subset $\mathscr{L}$. Different specifications of $\mathscr{L}$ simply mean different projections $P(X)$. It is important to note that, except for the AIEPs, all the classical IEPs discussed in the previous section can be solved (over the complex field) by using rational algorithms. The theory developed hitherto, however, only supports $|\mathscr{L}| \leqslant 2n - 3$. In contrast, our differential equation offers a continuous approach that has no limitation on either the locations $\mathscr{L}$ or the cardinality $|\mathscr{L}|$. If a PEIEP is not solvable, our approach finds a least squares solution.

Finally, we point out that the formulation above is for general matrices in $\mathbb{R}^{n\times n}$. If we are interested only in symmetric matrices, the group action by $\mathscr{G}l(n)$ is replaced by the group $\mathscr{O}(n)$ of $n \times n$ orthogonal matrices, $V^{-1} = V^{\mathrm{T}}$, and many of the expressions can be simplified.

3.2. Convergence

Along the solution flow $V(t)$ of (3.11), it is clear that

$$\frac{\mathrm{d}g(V(t))}{\mathrm{d}t} = -\|\nabla g(V(t))\|_F^2 \leqslant 0.$$

The functional value $g(V(t))$ will continue to decrease until one of only two possible events happens. The first is that $V(t)$ becomes undefined in finite time. This is the case when $V(t)$ converges to a singular matrix. The restart strategy proposed in the next section can remedy this failure. The second is that $\nabla g(V(t))$ converges to zero as t goes to infinity, implying that we have found a *local* minimum for $g(V)$.

We characterize the local minimum solution further. The first order optimality condition follows directly from the right-hand side of (3.11).

Theorem 3.1. *Suppose that $\hat{V}$ is a stationary point at which $\nabla g(\hat{V}) = 0$. Then the corresponding $\hat{X}^{\mathrm{T}} = \hat{V}\Lambda\hat{V}^{-1}$ and $\hat{X} - P(\hat{X})$ commutes.*

Observe that for any given nonsingular matrix $V \in \mathbb{R}^{n\times n}$, the set $\mathscr{I}(V) = \{VDV^{-1} \mid D = \mathrm{diag}\{d_1, \ldots, d_n\}$ arbitrary$\}$ of all real matrices having columns of V as eigenvectors form an n-dimensional subspace in $\mathbb{R}^{n\times n}$. Given any index subset $\mathscr{L}$, the subset $\mathscr{I}_{\mathscr{L}}(V) = \{X \in \mathscr{I}(V) \mid X_{\mathscr{L}^{\mathrm{c}}} = 0\}$, containing at least the zero matrix, is an even smaller subspace of $\mathscr{I}(V)$. Consider the generic case that all prescribed eigenvalues $\lambda_1, \ldots, \lambda_n$ in the PEIEP have linear elementary divisor. In this case, it is well known that $\hat{X} - P(\hat{X})$ is a simple matrix and that $\hat{X} - P(\hat{X})$ and $\hat{X}^{\mathrm{T}}$ must have a set of n linearly independent eigenvectors in common. Additionally, note that the matrix

$$\hat{X} - P(\hat{X}) = \hat{X}_{\mathscr{L}} - A_{\mathscr{L}}$$

has zeros at all locations in $\mathscr{L}^{\mathrm{c}}$, i.e., $\hat{X}_{\mathscr{L}} - A_{\mathscr{L}} \in \mathscr{I}_{\mathscr{L}}(\hat{V})$. Clearly, if $\hat{X}_{\mathscr{L}} - A_{\mathscr{L}} = 0$, then we have solved the PEIEP; otherwise, what this optimality condition suggests is that the critical point $\hat{X}$ must be quite "peculiar" in that changes by zeroing out its $\mathscr{L}^{\mathrm{c}}$ components and by subtracting $A_{\mathscr{L}}$ from its $\mathscr{L}$ components remain to have the same eigenvectors as $\hat{X}^{\mathrm{T}}$. Our descent flow moves to find such a peculiar stationary point. We are hoping that this peculiarity happens at $\hat{X} = P(\hat{X})$ and hence the PEIEP is solved. We shall see some numerical examples at the end of this paper.

4. Numerical methods

We can rewrite the differential equation (3.11) as a self-sustaining system

$$\frac{\mathrm{d}V}{\mathrm{d}t} = k(V\Lambda V^{-1})V^{-\mathrm{T}}, \tag{4.1}$$

where we recall that Λ is a constant real matrix with prescribed eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$. Since $V(t)$ generally has no additional structure, the system can be integrated by any available ODE solver starting with initial value $V(0) = I$. This naturally constitutes a reasonable numerical method for solving the PEIEP.

Clearly, the initial value at $t = 0$ is perfectly conditioned. By continuity, the conditioning of $V(t)$ will remain reasonable good for at least small values of t. As the integration continues, however, concerns about $V(t)$ converging to singularity or becoming ill-conditioned may arise. We thus propose an alternative numerical algorithm. We begin to outline a restart strategy with the following theorem.

Theorem 4.1. *Let $B : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ be a given piecewise continuous function. Then the solution $X(t)$ to the initial value problem*

$$\begin{cases} \dfrac{\mathrm{d}X}{\mathrm{d}t} = [B(X), X], \quad t \geqslant 0, \\ X(0) = X_0, \end{cases} \tag{4.2}$$

satisfies the relationship

$$X(t) = U(t)X_0U^{-1}(t) \tag{4.3}$$

if and only if the nonsingular matrix $U(t)$ is the solution to the initial value problem

$$\begin{cases} \dfrac{\mathrm{d}U}{\mathrm{d}t} = B(UX_0U^{-1})U, \quad t \geqslant 0, \\ U(0) = I. \end{cases} \tag{4.4}$$

To emphasize the dependence on the initial value X_0, it is convenient to denote the solution of problem (4.2) by $X(t; X_0)$. Observe that

$$X(t; X(s; X_0)) = X(t+s; X_0), \tag{4.5}$$

at whichever t and s where the pertaining solution is defined. Let $\{t_i\}_{i=0}^{\infty}$ be a sequence of positive numbers whose values will be defined later. For each $i = 0, 1, \ldots,$ let $U_i(t)$ denote the solution to the initial value problem

$$\begin{cases} \dfrac{\mathrm{d}U_i}{\mathrm{d}t} = B\big(U_i X_i U_i^{-1}\big)U_i, \quad t \in [0, t_i], \\ U_i(0) = I, \end{cases} \tag{4.6}$$

where, starting with X_0, we recursively define

$$X_{i+1} = U_i(t_i) X_i U_i^{-1}(t_i). \tag{4.7}$$

It follows from Theorem 4.1 that $U_i(t)$ implicitly defines a flow

$$X(t; X_i) = U_i(t) X_i U_i^{-1}(t) \tag{4.8}$$

for $t \in [0, t_i]$, although such a flow is never needed in real calculation. Furthermore, by (4.5), we know that

$$\begin{aligned} X(t; X_i) &= X(t; X(t_{i-1}; X_{i-1})) = X(t + t_{i-1}; X_{i-1}) \cdots \\ &= X\left(t + \sum_{j=0}^{i-1} t_j; X_0\right). \end{aligned} \tag{4.9}$$

On the other hand, applying (4.7) to (4.8) recursively, we obtain a different representation of $X(t; X_i)$, i.e.,

$$X(t; X_i) = U_i(t) U_{i-1}(t_{i-1}) \cdots U_0(t_0) X_0 U_0^{-1}(t_0) \cdots U_{i-1}^{-1}(t_{i-1}) U_i^{-1}(t). \tag{4.10}$$

Comparing (4.9) with (4.10) and using Theorem 4.1, we find an alternative way to compute the flow $U(t)$ of (4.4) via factorization as follows.

Theorem 4.2. *Suppose that the point $t + t_0 + \cdots + t_{i-1}$ belongs to the interval of existence for the initial value problem* (4.4). *Then*

$$U\left(t + \sum_{j=0}^{i-1} t_j\right) = U_i(t) U_{i-1}(t_{i-1}) \cdots U_0(t_0), \tag{4.11}$$

where each $U_j(t)$ is the solution of (4.6).

Theorem 4.2 is remarkable on two fronts: First, with sufficiently small t_i, the solution $U_i(t)$ should be well conditioned because $U_i(0)$ is perfectly conditioned. Secondly, even if the solution $U(t)$ becomes ill conditioned for large t, we can circumvent this situation by computing $U(t)$ via its decomposition into the product of a sequence of well conditioned matrices. The formulation of this sequence conforms naturally with a restart numerical algorithm which we now describe.

Clearly, the value of t_i for each i must be within the maximal interval of existence for each system (4.6). In theory, we can continue to integrate (4.6) within that interval until the condition number of $U_i(t)$ reaches some predesignated threshold. The value of t at which that threshold is reached is the *maximal* t_i we can define. We then update X_{i+1} according to (4.7) and switch to solve a new initial value problem (4.6). We call this process a *restart*. The choice of t_i can be quite flexible. In the extreme case, we can restart after *each* single integration step.

Applying this notion to our problem of solving the PEIEP, we note that $B(X) = h_V(X) = k(X)V^{-\mathrm{T}}V^{-1}$ and that the right-hand side of (4.4) reduces to that of (4.1). Because our ultimate goal is to find the stationary point for the objective function $g(V)$, and also because (4.1) defines a steepest descent flow, the issue of following the analytic solution $V(t)$ closely does not seem to be critically important, so long as we stay in a neighborhood of $V(t)$ where $g(V)$ is going downhill. Thus it seems likely that by using a (low-order) fixed-step size method, we could jump over the singularity when it occurs.

5. Test results

We have found success in solving many PEIEPs by using our schemes. Some experimental results on the various behavior of the dynamical system (4.1) are reported in this section. To avoid overrunning by the display of large matrices we shall limit our presentation to the case $n = 5$ only. Similar behavior has been observed for higher n, but will not be reported. For the ease of running text, we shall exhibit all numerics in only 5 digits, although all computations are done using complex double precision.

For the purpose of demonstration, we shall employ existing routines in Matlab as the ODE integrators. It is understood that any other available ODE solver can be used as well. The ODE Suite [23] in Matlab contains in particular a Klopfenstein-Shampine, quasi-constant step size, stiff system solver *ode15s*. To control the integration, we set local tolerance AbsTol $= 10^{-10}$ and RelTol $= 10^{-9}$ while maintaining all other parameters at the default values of the Matlab codes.

In what follows, the matrix $A_{\mathscr{L}}$ contains all information needed about the prescribed index subset $\mathscr{L} = \{(i_\nu, j_\nu)\}_{\nu=1}^{\ell}$ and the corresponding prescribed values $\{a_\nu\}_{\nu=1}^{\ell}$. To check how the optimality conditions are met, we plot for each example the history of the residual

$$R(t) = \|X(t) - P(X(t))\|_F \quad \left(= \sqrt{2g(V(t))}\right)$$

and the commutativity

$$C(t) = \|k(X(t))\|_F.$$

In an ideal situation when the PEIEP is solved, $R(t)$ should be monotone decreasing and both $R(t)$ and $C(t)$ should converge to zero. To compare the limiting behavior, we sample values of solution $V(t)$ of (4.1) and $U_i(t)$ of (4.6) at predesignated

output points $t_k = kh_{\text{restart}}$, $k = 0, 1, \ldots$ whereas h_{restart} is a user-specified value. For convenience, let

$$X^*(X_0) = \lim_{t\to\infty} X(t; X_0) \quad \text{and} \quad R^*(X_0) = \lim_{k\to\infty} X_k,$$

denote, respectively, the (numerical) limit points of the flow $X(t; X_0)$ and the sequence $\{X_k\}$ generated by the restart scheme that starts with initial values X_0. We terminate the integration (and likewise, the iteration) when the stop criteria

$$\min\left\{\|X(t_k) - P(X(t_k))\|_F, \|V(t_k) - V(t_{k-1})\|_F, \|k(X(t_k))\|_F\right\} \leqslant 10^{-8}$$

are satisfied. The choice of threshold 10^{-8} for stopping is based on the heuristic assumption that the global error is usually one or two orders less accurate than the local error.

We shall assume throughout the testing that the prescribed eigenvalues are the well separated integers $\lambda_i = i$, $i = 1, \ldots, 5$. In the first three examples below, both the index set $\mathscr{L}$ and the prescribed entries for experiment are randomly generated. In the fourth example, the prescribed entries, also randomly generated, take positions at all off-diagonal locations. We shall use the diagonal matrix $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$ as well as other kinds of isospectral matrices as the initial values.

Example 1. We first consider the case where $|\mathscr{L}| = n$. Recall that the Ikramov–Chugunov theorem asserts that such a problem can be solved via a rational algorithm. A Maple code based on the proof in [18] has been implemented by Chugunov. Using our gradient flow approach, we have observed some additional interesting behavior.

Case 1a. We first demonstrate that different (isospectral) initial values often lead to different limit points. We further demonstrate that even starting with the same initial value, the scheme with restart often ends at a limit point that is different from the scheme with continuation. The latter is partly due to the fact that the solutions of a PEIEP normally form an algebraic variety. A PEIEP with continuum solutions is ill-posed. A small perturbation can easily drive the numerical calculation to follow a different trajectory. Nevertheless, because we are following descent flows, the objective value $g(V(t))$ will continue to descend even if we are not following a certain trajectory precisely. It will be interesting to see how the ill-posedness can be regulated and we will discuss this issue in a separate paper.

Assume that

$$A_{\mathscr{L}} = \begin{bmatrix} \mathbf{-6.9178} & 0 & \mathbf{-13.3618} & 0 & \mathbf{16.2356} \\ 0 & 0 & \mathbf{2.9441} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{7.1432} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

at $t \approx 0.15$, the continuation scheme with integrator *ode15s* converges to the approximate limit point

$$X^*(\Lambda) \approx \begin{bmatrix} \mathbf{-6.9178} & 0.3827 & \mathbf{-13.3618} & 0 & \mathbf{16.2356} \\ 1.8749 & 1.9146 & \mathbf{2.9441} & 0 & -3.7011 \\ 0.1940 & -0.0070 & 3.0274 & 0 & 0.1342 \\ 0 & 0 & 0 & \boxed{4.0000} & 0 \\ -5.6139 & 0.2481 & -7.9730 & 0 & 12.9758 \end{bmatrix}.$$

Observe that values at the $(4, 4)$ position remains at the constant λ_4 throughout the integration. This limit point certainly does not solve our PEIEP. Rather, it is a least squares solution. Likewise, from the restart technique with the same initial value Λ and $h_{\text{restart}} = 10^{-2}$, a different least squares solution

$$R^*(\Lambda) \approx \begin{bmatrix} \mathbf{-6.9178} & 0.2354 & \mathbf{-13.3618} & 0 & \mathbf{16.2356} \\ 1.9519 & 1.9477 & \mathbf{2.9441} & 0 & -3.6421 \\ 0.2259 & -0.0045 & 3.0740 & 0 & 0.0764 \\ 0 & 0 & 0 & \boxed{4.0000} & 0 \\ -5.5727 & 0.1519 & -7.9087 & 0 & 12.8962 \end{bmatrix}$$

is found at $t \approx 0.1$. The history of $R(t)$, $C(t)$, the condition numbers, and the smallest singular values for both methods are recorded in Fig. 2.

Changing the initial value also changes the behavior. Suppose $X_0 = M\Lambda M^{-1}$ where M is a random matrix, say,

$$M = \begin{bmatrix} -10.1063 & -6.4360 & 0.0004 & 8.9564 & 5.6890 \\ 6.1446 & 3.8034 & -3.1786 & 7.3096 & -2.5565 \\ 5.0774 & -10.0912 & 10.9500 & 5.7786 & -3.7747 \\ 16.9243 & -0.1951 & -18.7399 & 0.4031 & -2.9589 \\ 5.9128 & -0.4822 & 4.2818 & 6.7709 & -14.7513 \end{bmatrix}.$$

We find that at $t \approx 0.09$, *ode15s* converges to a limit point

$$X^*(M\Lambda M^{-1}) \approx \begin{bmatrix} \mathbf{-6.9178} & 19.2148 & \mathbf{-13.3618} & -4.6997 & \mathbf{16.2356} \\ 3.4353 & -1.6506 & \mathbf{2.9441} & 0.9474 & -4.5826 \\ -2.9331 & 5.9434 & -3.1618 & -1.5972 & 6.1474 \\ 9.3879 & -16.5491 & 8.7526 & \mathbf{7.1432} & -12.1577 \\ -9.4871 & 16.8368 & -13.9850 & -3.9572 & 19.5870 \end{bmatrix}$$

that can be considered as an approximate solution to the PEIEP. In the meantime, the restart technique with $h_{\text{restart}} = 10^{-2}$ converges to a different approximate solution

$$R^*(M\Lambda M^{-1}) \approx \begin{bmatrix} \mathbf{-6.9178} & 12.0373 & \mathbf{-13.3618} & -4.3334 & \mathbf{16.2356} \\ 2.7285 & 1.4068 & \mathbf{2.9441} & 0.2729 & -4.0318 \\ 1.1038 & -2.1165 & 2.5431 & 0.8937 & -1.3976 \\ 7.9886 & -10.1791 & 7.8301 & \mathbf{7.1432} & -11.7856 \\ -4.4404 & 3.3882 & -8.1308 & -0.6934 & 10.8247 \end{bmatrix},$$

at $t \approx 0.13$. The history of convergence is recorded in Fig. 3.

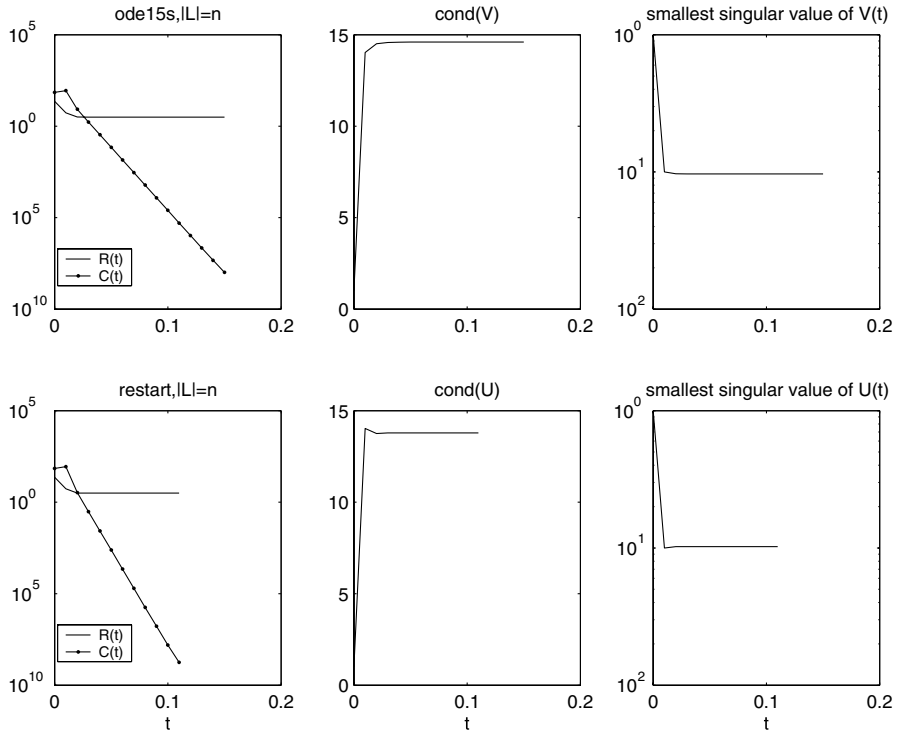

Fig. 2. Case 1a. History of $R(t)$ and $C(t)$ with $X_0 = \Lambda$ and a least square solution.

It is necessary to make one interesting and important remark. While the solution $X(t_k; X_0)$ from the continuation scheme and the iterates X_k from the restart scheme should be identical in theory, we have observed time and again that their numerical results behave very differently. They give rise to entirely different limit points. Computations for ill-posed problems such as this would have been difficult, but it seems that the descent property inherent in our flow approach has the advantage that it is able to track down multiple solutions of the PEIEPs without much trouble.

Case 1b. In theory, the flow $V(t)$ of the dynamical system (4.1) could evolve to become ill-conditioned. In this example, we experiment with a hybrid method that caps the condition number under a given threshold and automatically applies a restart when the condition number of V violates the given threshold. This is not the most robust way to control the condition number, but we demonstrate in this example that this primitive composite scheme offers an interesting way to control the condition number.

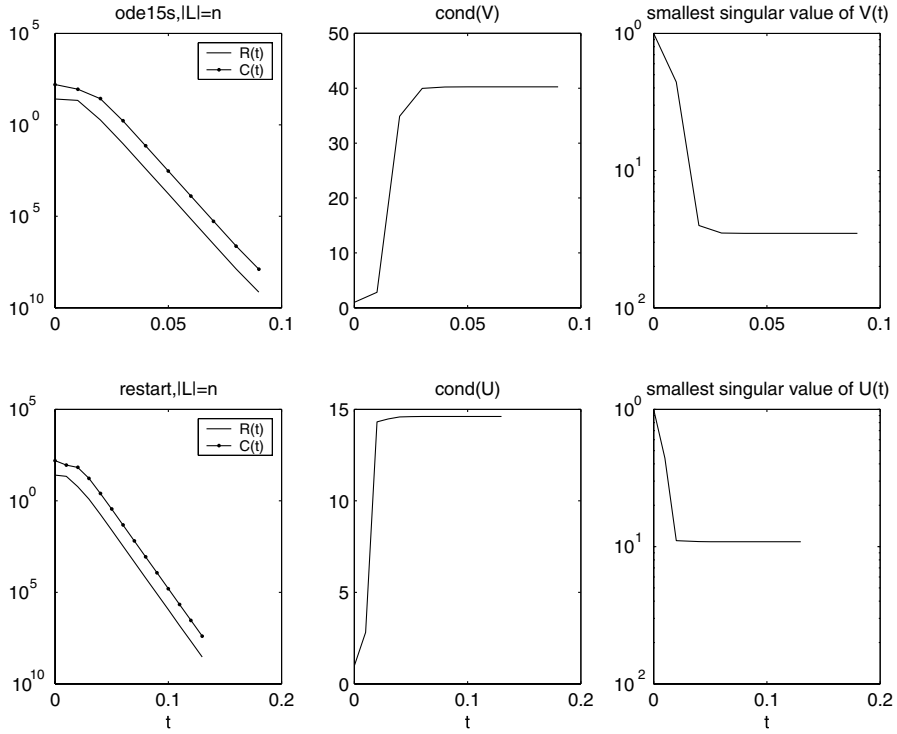

Fig. 3. Case 1a. History of $R(t)$ and $C(t)$ with $X_0 = M\Lambda M^{-1}$ and convergence.

Assume that

$$A_{\mathscr{L}} = \begin{bmatrix} 0 & 0 & 0 & 0 & \mathbf{5.7980} \\ 0 & 0 & \mathbf{2.2594} & \mathbf{4.3290} & 0 \\ 0 & \mathbf{5.2982} & \mathbf{7.6036} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

at $t \approx 0.5$, the integrator *ode15s* reports an approximate solution

$$X^*(\Lambda) \approx \begin{bmatrix} -0.0603 & 0 & 0 & 0 & \mathbf{5.7980} \\ 0 & 4.7356 & \mathbf{2.2594} & \mathbf{4.3290} & 0 \\ 0 & \mathbf{5.2982} & \mathbf{7.6036} & \boxed{-2153.4} & 0 \\ 0 & 0.0206 & 0.0201 & -3.3392 & 0 \\ -0.9254 & 0 & 0 & 0 & 6.0603 \end{bmatrix}.$$

Note that this equilibrium point has a much larger component at the $(3,4)$-position, making this matrix extremely unbalanced. Indeed, nearby this equilibrium we find that smallest singular value of the matrix V has dropped to approximately 1.2985×10^{-4}, and the condition number $\mathrm{cond}(X^*(\Lambda)) \approx 1.0165 \times 10^6$ is fairly high.

On the other hand, by applying *ode15s* with auto-restart and setting the threshold at, say, 40, we find that only one restart is enough. After one restart at approximately $t = 0.25$, we obtain an approximate limit point

$$X_R^*(\Lambda) \approx \begin{bmatrix} -0.0603 & 0 & 0 & 0 & \mathbf{5.7980} \\ 0 & 3.4828 & \mathbf{2.2594} & \mathbf{4.3290} & 0 \\ 0 & \mathbf{5.2982} & \mathbf{7.6036} & -50.0641 & 0 \\ 0 & 0.6370 & 0.7466 & -2.0864 & 0 \\ -0.9254 & 0 & 0 & 0 & 6.0603 \end{bmatrix},$$

at $t \approx 1.3$. In this case, $\mathrm{cond}(X_R^*(\Lambda)) \approx 528$, representing a significant improvement over $\mathrm{cond}(X^*(\Lambda))$. We plot the history of convergence in Fig. 4. If the threshold is lowered, it is possible that several restarts will be needed. This way of controlling the condition number via restart is quite interesting.

In addition to the hybrid method described above, we could also affect the conditioning behavior by using different initial values for the flows. For instance, if we start with $X_0 = M\Lambda M^{-1}$, the continuation method finds an approximate limit point

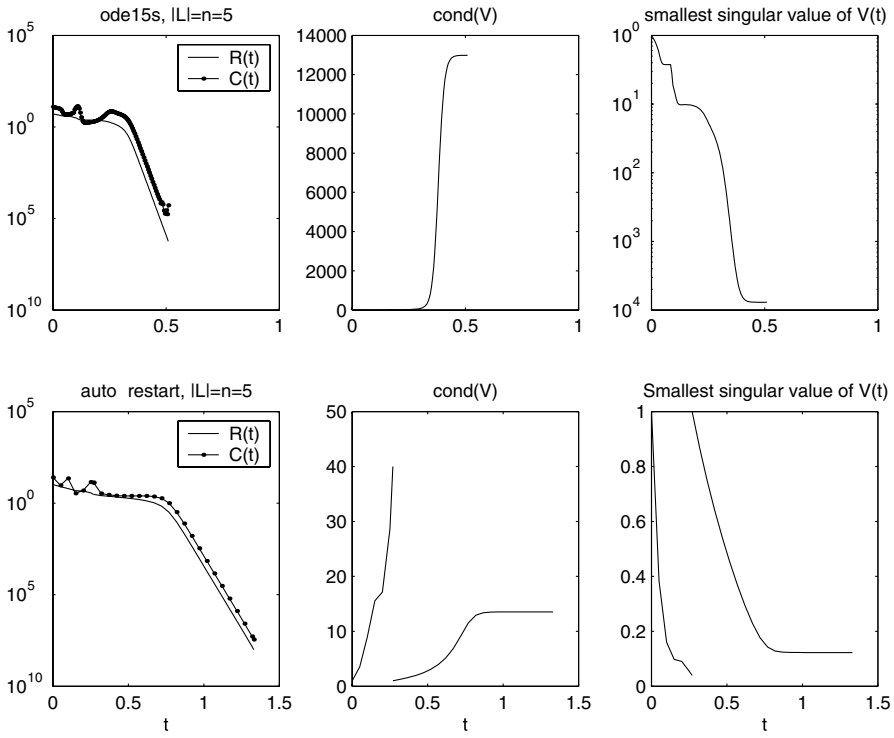

Fig. 4. Case 1b. History of $R(t)$ and $C(t)$ with $X_0 = \Lambda$. Ill-conditioning and restart-control.

$$X^*(M\Lambda M^{-1}) \approx \begin{bmatrix} 0.6191 & 3.8118 & 2.7968 & 6.8257 & \mathbf{5.7980} \\ -1.4213 & 4.6582 & \mathbf{2.2594} & \mathbf{4.3290} & 1.9759 \\ -4.2898 & \mathbf{5.2982} & \mathbf{7.6036} & 10.0042 & 7.1066 \\ 2.0087 & -1.3904 & -2.0907 & -1.8566 & -3.4188 \\ 0.2537 & -0.7727 & -0.5389 & -0.1400 & 3.9756 \end{bmatrix}$$

that has condition number around 105. If we start with the diagonal matrix $\Lambda_r = \mathrm{diag}(5, 4, 3, 2, 1)$, we find an approximate limit point

$$X^*(\Lambda_r) \approx \begin{bmatrix} 6.0603 & 0 & 0 & 0 & \mathbf{5.7980} \\ 0 & 5.3552 & \mathbf{2.2594} & \mathbf{4.3290} & 0 \\ 0 & \mathbf{5.2982} & \mathbf{7.6036} & 8.3259 & 0 \\ 0 & -4.0560 & -3.7227 & -3.9588 & 0 \\ -0.9254 & 0 & 0 & 0 & -0.0603 \end{bmatrix}$$

that has condition number around 28.

Example 2. In this example, we demonstrate the Hershkowitz theorem where $|\mathscr{L}| = 2n - 3$. This is the maximal cardinality under which the PEIEP is known to be solvable. No rational algorithm for find this matrix is known to exist at the writing of this paper.

Assume that

$$A_{\mathscr{L}} = \begin{bmatrix} \mathbf{4.6478} & 0 & 0 & 0 & \mathbf{3.0769} \\ 0 & 0 & \mathbf{5.4142} & 0 & 0 \\ 0 & \mathbf{3.4176} & \mathbf{4.0180} & 0 & \mathbf{9.4233} \\ 0 & 0 & \mathbf{5.0605} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Using $X_0 = M\Lambda M^{-1}$ as the starting value, we find an approximate PEIEP solution

$$X^*(M\Lambda M^{-1}) \approx \begin{bmatrix} \mathbf{4.6478} & 1.4747 & 0.2769 & -2.1889 & \mathbf{3.0769} \\ 5.8775 & 7.6853 & \mathbf{5.4142} & -8.5807 & 25.4085 \\ 2.7866 & \mathbf{3.4176} & \mathbf{4.0180} & -4.4980 & \mathbf{9.4233} \\ 5.4620 & 4.2719 & \mathbf{5.0605} & -5.2338 & 24.1737 \\ -0.2689 & -0.4190 & 0.4635 & 0.3473 & 3.8826 \end{bmatrix},$$

at $t \approx 1.01$, while the restart technique with $h_{\mathrm{restart}} = 1 \times 10^{-2}$ leads to the limit point

$$R^*(M\Lambda M^{-1}) \approx \begin{bmatrix} \mathbf{4.6478} & 2.1026 & -0.7122 & -1.6803 & \mathbf{3.0769} \\ 7.6563 & 8.4636 & \mathbf{5.4142} & -4.2439 & 21.9336 \\ 2.7615 & \mathbf{3.4176} & \mathbf{4.0180} & -2.1595 & \mathbf{9.4233} \\ 5.5707 & 3.2899 & \mathbf{5.0605} & 0.3565 & 16.6768 \\ -2.2464 & -2.1997 & -0.3955 & 1.8084 & -2.4859 \end{bmatrix},$$

at $t \approx 1.3$. The history of $R(t)$ and $C(t)$ for both method is plotted in Fig. 5.

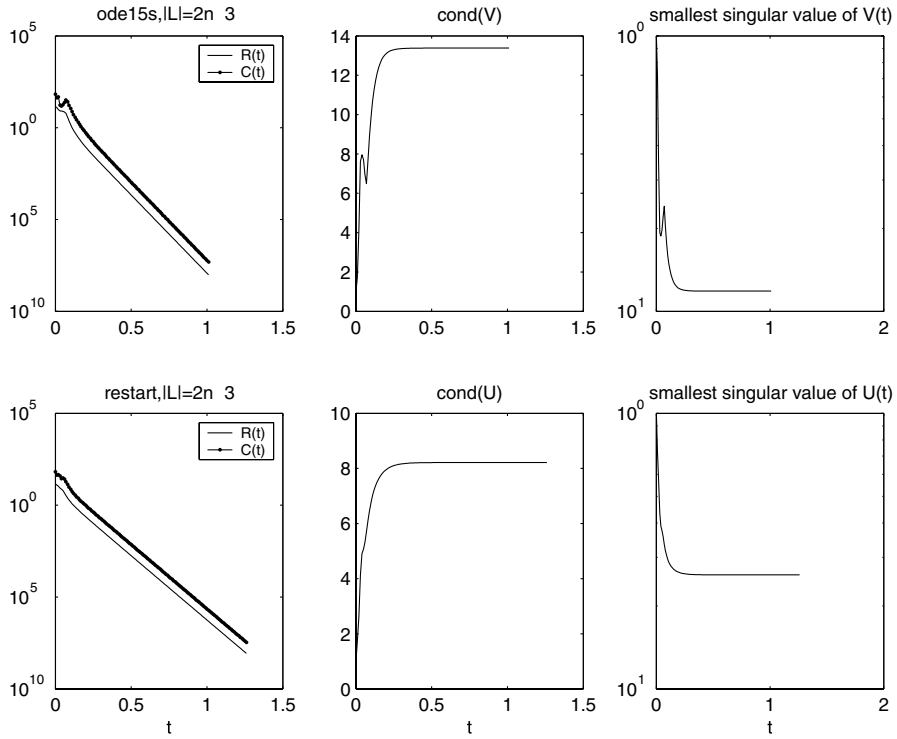

Fig. 5. Example 2. History of $R(t)$ and $C(t)$ with $X_0 = M\Lambda M^{-1}$ and convergence.

Example 3. In this example, we examine the convergence behavior when $|\mathscr{L}| = 2n$. This is a situation that goes beyond existing theory, so it is particularly interesting to see that our gradient flow can still find a solution.

Assume

$$A_{\mathscr{L}} = \begin{bmatrix} 0 & \mathbf{2.2191} & \mathbf{2.3114} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{2.2804} \\ \mathbf{1.7220} & \mathbf{7.1335} & \mathbf{5.2206} & \mathbf{7.0368} & \mathbf{9.6882} \\ 0 & 0 & 0 & 0 & \mathbf{4.4964} \\ 0 & 0 & 0 & 0 & \mathbf{9.7709} \end{bmatrix}.$$

By means of the integrator *ode15s* and starting with $X_0 = M\Lambda M^{-1}$, we obtain an approximate solution at $t \approx 0.63$

$$X^*(M\Lambda M^{-1}) \approx \begin{bmatrix} -6.3069 & \mathbf{2.2191} & \mathbf{2.3114} & -40.5399 & -63.9824 \\ 0.1834 & 0.5859 & -0.7024 & 1.3190 & \mathbf{2.2804} \\ \mathbf{1.7220} & \mathbf{7.1335} & \mathbf{5.2206} & \mathbf{7.0368} & \mathbf{9.6882} \\ 1.1973 & -0.1793 & -0.5067 & 5.7296 & \mathbf{4.4964} \\ 0.5877 & -0.0496 & 0.0617 & 4.3919 & \mathbf{9.7709} \end{bmatrix},$$

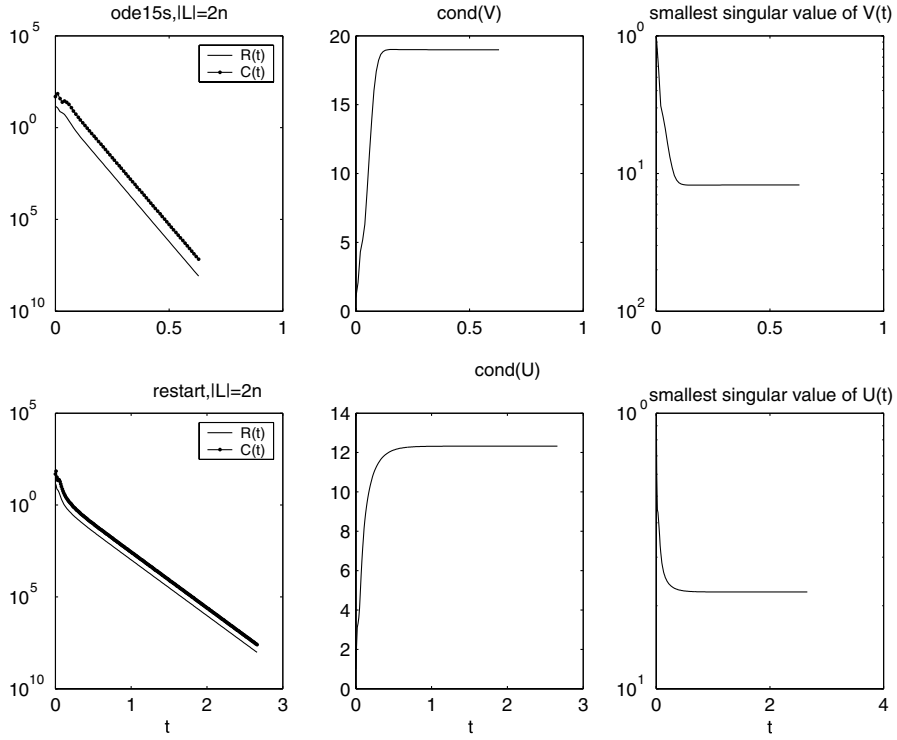

Fig. 6. Example 3. $|\mathscr{L}| = 2n$ with $X_0 = M\Lambda M^{-1}$.

while the restart method with $h_{\text{restart}} = 10^{-2}$ gives rise to another limit point

$$R^*(M\Lambda M^{-1}) \approx \begin{bmatrix} -5.7391 & \mathbf{2.2191} & \mathbf{2.3114} & -35.4855 & -68.9824 \\ 0.0699 & 0.5252 & -0.71140 & 0.6925 & \mathbf{2.2804} \\ \mathbf{1.7220} & \mathbf{7.1335} & \mathbf{5.2206} & \mathbf{7.0368} & \mathbf{9.6882} \\ 0.8172 & -0.2678 & -0.3317 & 5.2224 & \mathbf{4.4964} \\ 0.6835 & -0.0095 & -0.0580 & 3.5772 & \mathbf{9.7709} \end{bmatrix},$$

at $T \approx 2.66$. The history of convergence for both methods is reported in Fig. 6.

Example 4. In this example, we consider solving the AIEP where $|\mathscr{L}| = n^2 - n$ and the prescribed entries are located along the off-diagonal. Recall that the Friedland theorem guarantees only that the AIEP is solvable over the complex field. Thus the problem of AIEP1 where the desired solution is real-valued imposes particular challenges. To generate feasible test data, we take $A_{\mathscr{L}} = \text{off-diag}(Q\Lambda Q^{-1})$ where Q is a random matrix, and attempts to recover the diagonal entries. Note that the AIEP for such an $A_{\mathscr{L}}$ is real solvable, since $Q\Lambda Q^{-1}$ is a already solution. We are curious to know what the gradient flow will find.

Case 4a. Suppose Q is a random orthogonal matrix, say,

$$Q = \begin{bmatrix} 0.5246 & 0.2559 & -0.6663 & -0.4108 & 0.0396 \\ 0.3078 & 0.7086 & 0.4825 & -0.4108 & 0.0396 \\ -0.6242 & 0.5398 & -0.4824 & -0.1286 & -0.2639 \\ -0.0557 & -0.3681 & -0.1963 & -0.8907 & 0.1716 \\ -0.4871 & 0.0737 & 0.2279 & 0.1102 & 0.8326 \end{bmatrix},$$

so that

$$A_{\mathscr{L}} = \begin{bmatrix} 0 & -0.5077 & 0.2648 & 0.2232 & 1.2588 \\ -0.5077 & 0 & 0.0337 & 0.6747 & 0.2683 \\ 0.2648 & 0.0337 & 0 & 0.1531 & -1.1016 \\ 0.2232 & 0.6747 & 0.1531 & 0 & 0.1606 \\ 1.2588 & 0.2683 & -1.1016 & 0.1606 & 0 \end{bmatrix}$$

is symmetric. This is the AIEP2 described earlier. We have observed that the flow starting with $X_0 = \Lambda$ converges slowly to an AIEP solution $X^*(\Lambda)$ with

$$\operatorname{diag}(X^*(\Lambda)) = [2.5076, 2.1235, 2.7057, 3.7506, 3.9126],$$

at an extended $t \approx 304$. For this symmetric problem, note that the solution $V(t)$ should remain orthogonal at all t. Being ideally conditioned all the time, no restart is needed. It is interesting to note that the limit point of our flow differs from the original randomly generated matrix $Q\Lambda Q^{\mathrm{T}}$ in that

$$\operatorname{diag}(Q\Lambda Q^{\mathrm{T}}) = [2.8054, 2.4803, 2.0852, 3.7106, 3.9186].$$

Case 4b. Suppose matrix Q is just random but not orthogonal, say,

$$Q = \begin{bmatrix} -10.1063 & -6.4360 & 0.0004 & 8.9564 & 5.6890 \\ 6.1446 & 3.8034 & -3.1786 & 7.3096 & -2.5565 \\ 5.0774 & -10.0912 & 10.9500 & 5.7786 & -3.7747 \\ 16.9243 & -0.1951 & -18.7399 & 0.4031 & -2.9589 \\ 5.9128 & -0.4822 & 4.2818 & 6.7709 & -14.7513 \end{bmatrix},$$

so that

$$A_{\mathscr{L}} = \begin{bmatrix} 0 & 3.2301 & 1.2912 & -0.2463 & -1.9822 \\ 1.0504 & 0 & -0.6282 & -0.0002 & 1.0437 \\ 0.3767 & 0.3454 & 0 & -0.5026 & 0.9917 \\ 1.6297 & -2.1979 & -1.9755 & 0 & 1.9820 \\ 1.1029 & -2.0588 & -1.6816 & 0.0962 & 0 \end{bmatrix}.$$

We find that the flow starting with $X_0 = \Lambda$ converges to a least square solution, while the flow starting with $X_0 = \Lambda_r$, defined previously in Example 1 (Case 1b), solves the AIEP with an approximate solution

$$\operatorname{diag}(X^*(\Lambda_r)) = [0.8456, 0.6733, 1.8224, 2.9538, 4.7049],$$

at $t \approx 136$ after about 1374 internal steps. In contrast, we also have observed that the flow starting with $X_0 = M\Lambda M^{-1}$ can be more effectively followed in computation. At $t \approx 116$, the flow converges to another AIEP solution with

$$\operatorname{diag}(X^*(M\Lambda M^{-1})) = [2.0403, 2.2429, 1.8523, 2.6714, 6.1931]$$

that happen to coincide with the diagonal of the original randomly generated matrix $Q\Lambda Q^{-1}$. This time, however, much larger step sizes are taken since *ode15s* requires only about 131 internal steps to accomplish the integration.

6. Conclusion

Matrix completion with prescribed spectrum has been a classical yet quite challenging problem both theoretically and computationally. In the first part of this paper, we have chronicled some major developments on this subject in the literature. Starting with the well-known Schur–Horn theorem, we point out in particular that the attention has been centering around special locations of the prescribed entries and that the cardinality has usually been low. We also point out that the Hershkowitz theorem where $|\mathscr{L}| = 2n - 3$ appears to be the most general result at present under which the PEIEP is ensured to be solvable. In the second part of this paper, we have proposed a dynamical system of which the trajectory allows us to complete the construction of a matrix numerically even under the situation when no existence theory is available at all. Extensive numerical experiments seem to suggest that our idea of gradient flow approach can serve as a reasonable means to tackle the most general PEIEPs where the prescribed entries are at arbitrary locations with arbitrary cardinalities.

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