REAL SYMMETRIC QUADRATIC MODEL UPDATING THAT PRESERVES POSITIVE DEFINITENESS AND NO SPILL-OVER

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Abstract. Updating a model framed as a real symmetric quadratic eigenvalue problem to match observed spectral information has been a powerful tool for practitioners in different discipline areas. It is often desirable in updating to match only the part of newly measured data without tampering with the other part of unmeasured and often unknown eigenstructure inhering in the original model. Such an updating, known as no spill-over, has been a critical yet challenging task in practice. Only recently, a mathematical theory on updating with no spill-over has begun to be understood. In applications, however, often there is the additional requisite of maintaining positive definiteness in the coefficient matrices. Toward that need, this paper advances one step forward by preserving both no spill-over and positive definiteness of the mass and the stiffness matrices. This investigation establishes some necessary and sufficient solvability conditions for this open problem.

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1. Introduction.

By a *real symmetric quadratic model*, we refer to in this paper any system that leads to a quadratic λ-matrix of the form,

$$Q(\lambda) = \lambda^2 M + \lambda C + K, \tag{1.1}$$

where M, D and K are $n \times n$ real symmetric matrices and, additionally, *both M and K are positive definite*. Real symmetric quadratic models arise frequently in areas such as applied mechanics, electrical oscillation, vibro-acoustics, fluid dynamics, signal processing, and finite element model of some critical PDEs [15, 35]. The specifications of the underlying physical system are embedded (in certain structural ways) in the matrix coefficients (M, C, K). For example, in a vibrating system, M, C, and K often represent the mass, damping, and stiffness, respectively. A typical forward analysis involves, given (M, C, K), finding scalars $\lambda \in \mathbb{C}$ and nonzero vectors $\mathbf{x} \in \mathbb{C}^n$, called the eigenvalues and eigenvectors of the system, respectively, to satisfy the algebraic equation $Q(\lambda)\mathbf{x} = 0$. The spectral information is essential for deducing the dynamical behavior of the underlying physical system. The theoretical framework for matrix polynomials in general and quadratic eigenvalue problems (QEP) in particular can be found in the seminal books by Lancaster [35] and by Gohberg, Lancaster and Rodman [31, 32]. A good survey of applications, mathematical properties, and a variety of numerical algorithms for the QEP can be found in the treatise by Tisseur and Meerbergen [15].

While the forward problem characterizes the dynamical behavior of a system in terms of its physical parameters, an equivalently important topic is the inverse problem of expressing the physical parameters in terms of the dynamical behavior. A quadratic inverse eigenvalue problem (QIEP) is concerned about determining coefficients M, C and K of the system (1.1) from its observed or expected eigeninformation.

Depending on the type of eigeninformation available and the properties to be imposed on the matrix coefficients, there are different ways of formulating a QIEP. We mention, for instance,

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the QIEP where only partial eigenstructure has been prescribed [3, 5] and the QIEP where the complete spectral information is given but M and K are positive definite [6, 7, 11, 12]. Updating the coefficient matrices of an existent model is another class of QIEPs where the idea is to correct errors in the model specifications so that the updated model will have a behavior closely matching the experimental data [21, 23, 25].

The notion of model updating has emerged in the 90's as an important tool for the design, construction, and maintenance of mechanical systems. It has attracted extensive research interests ever since. A real symmetric quadratic model updating problem can be formulated as follows:

(MUP) *Given a real symmetric quadratic model* (M_0, C_0, K_0) *and a few of its associated eigenpairs* $\{(\lambda_j, \mathbf{x}_j)\}_{j=1}^k$, $k \leq n$, *where* λ_j*'s are distinct simple eigenvalues, assume that new eigenpairs* $\{(\sigma_j, \mathbf{y}_j)\}_{j=1}^k$ *have been measured, where* σ_j*'s are distinct and have the same type of number (real or complex) as the corresponding* λ_j*'s. Update the quadratic model* (M_0, C_0, K_0) *to a new real symmetric quadratic model* (M, C, K) *such that*

- *(i) The newly measured* $\{(\sigma_j, \mathbf{y}_j)\}_{j=1}^k$ *form* k *eigenpairs of the new model* (M, C, K).
- *(ii) The remaining* $2n - k$ *eigenpairs of* (M, C, K) *are kept the same as those of the original* (M_0, C_0, K_0).

Note that M and K being positive definite is a requirement inhering in the real symmetric quadratic model by our definition. The second condition above is known as the *no spill-over phenomenon* [2] to the unmeasured or unknown eigenstructure. No spill-over is required in the updating process either because these vibrating parameters are proven to be acceptable in the original model $Q_0(\lambda) = \lambda^2 M_0 + \lambda C_0 + K_0$ and engineers do not wish to introduce new excitements via updating or, more realistically, because engineers simply do not know of any information about these parameters.

Far from being complete, we mention references [17, 26, 27, 29, 30, 34, 33] for the undamped case $C = C_0 = 0$, [1, 9, 17, 18] for the damped problem, and [10, 13, 23, 24, 28] for the low-rank updating on C and K. Needless to say, the theory developed thus far is still fragmentary and the techniques are certainly inadequate. One major difficulty is that all these methods can maintain the symmetry and reproduce measured data, but cannot guarantee no spill-over after the update. Neither can these methods warrant positive definiteness of the mass matrix M and stiffness matrix K which often is critical in applications. A mathematical theory for the modified MUP without the requirement of positive definiteness has been developed in two recent papers [2, 4]. This paper is another step of advance that generalizes the theory to include the important condition of maintaining positive definiteness of M and K. The main thrust in this study is to develop some necessary and sufficient conditions for the solvability of the MUP. We believe that our results are innovative in the field.

To set the notation for later discussion, let

$$\lambda_1, \cdots, \lambda_s,\ \lambda_{s+1}, \cdots, \lambda_t,\ \overline{\lambda}_{s+1}, \cdots, \overline{\lambda}_t$$

denote the portion $(k = 2t - s)$ of the spectrum to be replaced, where $\lambda_1, \cdots, \lambda_s \in \mathbb{R}$ are the distinct real eigenvalues and $\lambda_{s+1}, \cdots, \lambda_t \in \mathbb{C}$ are the distinct complex eigenvalues. Let the corresponding eigenvectors be denoted by

$$\mathbf{x}_1, \cdots, \mathbf{x}_s,\ \mathbf{x}_{s+1}, \cdots, \mathbf{x}_t,\ \overline{\mathbf{x}}_{s+1}, \cdots, \overline{\mathbf{x}}_t.$$

For $j = s+1, \ldots, t$, write $\lambda_j = \alpha_j + \imath\beta_j$ and $\mathbf{x}_j = \mathbf{x}_{jR} + \imath\mathbf{x}_{jI}$ with $\alpha_j, \beta_j \in \mathbb{R}$ and $\mathbf{x}_{jR}$, $\mathbf{x}_{jI} \in \mathbb{R}^n$. Upon introducing

$$\Lambda_i := \begin{cases} \lambda_i, & i = 1, \ldots, s \\ \begin{bmatrix} \alpha_i & \beta_i \\ -\beta_i & \alpha_i \end{bmatrix}, & i = s+1, \ldots, t, \end{cases}$$

and

$$X_i := \begin{cases} \mathbf{x}_i, & i = 1, \ldots, s \\ \left[\begin{array}{cc} \mathbf{x}_{iR} & \mathbf{x}_{iI} \end{array} \right], & i = s+1, \cdots, t, \end{cases}$$

we see that the equation

$$M_0 X \Lambda^2 + C_0 X \Lambda + K_0 X = 0 \tag{1.2}$$

is satisfied by the matrices

$$\Lambda = \operatorname{diag}\{\Lambda_1, \cdots, \Lambda_s, \Lambda_{s+1}, \cdots, \Lambda_t\} \in \mathbb{R}^{k\times k}, \tag{1.3}$$
$$X = [X_1, \cdots, X_s, X_{s+1}, \cdots, X_t] \in \mathbb{R}^{n\times k}. \tag{1.4}$$

Without loss of generality, we shall say that the pair of matrices (Λ, X) represents k eigenpairs of $Q(\lambda)$. In a similar way, let $(\Sigma, Y) \in \mathbb{R}^{k\times k} \times \mathbb{R}^{n\times k}$ and $(\Upsilon, Z) \in \mathbb{R}^{(2n-k)\times(2n-k)} \times \mathbb{R}^{n\times(2n-k)}$ denote the newly measured k eigenpairs and the remaining $2n-k$ eigenpairs of $Q_0(\lambda)$, respectively. The MUP is equivalent to finding $n \times n$ real symmetric matrices $(\Delta M, \Delta C, \Delta K)$ such that

$$(M_0 + \Delta M) Y \Sigma^2 + (C_0 + \Delta C) Y \Sigma + (K_0 + \Delta K) Y = 0, \tag{1.5}$$
$$(M_0 + \Delta M) Z \Upsilon^2 + (C_0 + \Delta C) Z \Upsilon + (K_0 + \Delta K) Z = 0, \tag{1.6}$$

and that $M_0 + \Delta M$ and $K_0 + \Delta K$ are positive definite.

It has been shown in [4, Theorem 4.1] that a necessary condition for the equation (1.5) to hold is that

$$Y = XT \tag{1.7}$$

for some $T \in \mathbb{R}^{k\times k}$. Furthermore, if Y is of full rank as is assumed hereafter, then T is nonsingular. In the subsequent discussion, we shall always assume that (1.7) is satisfied.

2. Solvability under a general structure. It is easy to see that the incremental matrices $(\Delta M, \Delta C, \Delta K)$ of the form,

$$\begin{cases} \Delta M &:= M_0 X \Phi X^\top M_0, \\ \Delta C &:= -M_0 X \Phi \Lambda^{-\top} X^\top K_0 - K_0 X \Lambda^{-1} \Phi X^\top M_0, \\ \Delta K &:= K_0 X \Lambda^{-1} \Phi \Lambda^{-\top} X^\top K_0, \end{cases} \tag{2.1}$$

where $\Phi \in \mathbb{R}^{k\times k}$ is an arbitrary symmetric matrix, are sufficient for solving the equation (1.6). The theory established in [4] also shows that, under some very mild conditions on (Υ, Z) which generally are true, this form is also necessary for solving (1.6). We shall assume this parametric form on $(\Delta M, \Delta C, \Delta K)$ in the discussion of this section. The case of $(\Delta M, \Delta C, \Delta K)$ not in this generic form will be considered in the next section.

We derive a necessary and sufficient condition for the solvability of the MUP in this section. The following three lemmas have already been established in the literature. We list them here as preparatory results for bringing forth our main discussion.

LEMMA 2.1. *[22] Given $\mathcal{A} \in \mathbb{R}^{n\times n}$ and $\mathcal{B} \in \mathbb{R}^{m\times m}$, the equation*

$$\mathcal{A} H - H \mathcal{B} = 0$$

has only the trivial solution $H = 0$ if and only if A and B have no common eigenvalues.

LEMMA 2.2. *[5] Suppose that $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C} \in \mathbb{R}^{k\times k}$ are symmetric and that all eigenvalues of $\Omega \in \mathbb{R}^{k\times k}$ are distinct. Then the equation*

$$\mathcal{A}\Omega^2 + \mathcal{B}\Omega + \mathcal{C} = 0$$

holds if and only if

$$\mathcal{B} = \Pi - \mathcal{A}\Omega - \Omega^\top \mathcal{A},$$
$$\mathcal{C} = \Omega^\top \mathcal{A}\Omega - \Pi\Omega,$$

for some symmetric matrix $\Pi \in \mathbb{R}^{k\times k}$ *satisfying* $\Omega^\top \Pi = \Pi\Omega$. *Moreover, if the matrix* Ω *is block diagonal with the same structure as that of* Λ *described in (1.3), then the matrix* Π *is also block diagonal of the same structure except that its* 2×2 *diagonal blocks are of this special form* $\begin{bmatrix} \mu & \nu \\ \nu & -\mu \end{bmatrix}$.

LEMMA 2.3. *[14] Suppose that* $\mathcal{A}$, $\mathcal{B}$, *and* $\mathcal{C} \in \mathbb{R}^{n\times n}$, $\mathcal{E} = [\mathcal{E}_1, \mathcal{E}_2]$ *and* $\Omega = \operatorname{diag}\{\Omega_1, \Omega_2\}$ *satisfy the equation*

$$\mathcal{A}\mathcal{E}\Omega^2 + \mathcal{B}\mathcal{E}\Omega + \mathcal{C}\mathcal{E} = 0,$$

where $\mathcal{E}_1 \in \mathbb{R}^{n\times m}$, $\mathcal{E}_2 \in \mathbb{R}^{n\times(2n-m)}$, $\Omega_1 \in \mathbb{R}^{m\times m}$, *and* $\Omega_2 \in \mathbb{R}^{(2n-m)\times(2n-m)}$ *with* $m \le n$. *If the two matrices* Ω_1 *and* Ω_2 *have no common eigenvalues, then it is true that*

$$(\mathcal{E}_1\Omega_1)^\top \mathcal{A}\mathcal{E}_2\Omega_2 - \mathcal{E}_1^\top \mathcal{C}\mathcal{E}_2 = 0.$$

Obviously if Φ is positive definite, then we will have the desired positive definiteness for $M_0 + \Delta M$ and $K_0 + \Delta K$, albeit that (1.5) is yet to be satisfied. Our goal is to characterize the more general symmetric matrix Φ for the MUP. Toward that end, let

$$X = [Q_1, Q_2] \begin{bmatrix} R_1 \\ 0 \end{bmatrix}$$

be the QR factorization of X, where $[Q_1, Q_2] \in \mathbb{R}^{n\times n}$ is orthogonal, $Q_1 \in \mathbb{R}^{n\times k}$, and $R_1 \in \mathbb{R}^{k\times k}$ is nonsingular. We shall use the congruence transformation by the nonsingular matrix $[Q_1R_1, Q_2]$ to examine the three equations (1.2), (1.5) and (1.6). This would provide us with a handle to grasp the conditions that Φ must satisfy in order to solve the MUP.

First, write

$$\begin{aligned} \begin{bmatrix} M_1 & M_2 \\ M_2^\top & M_3 \end{bmatrix} &:= \begin{bmatrix} Q_1R_1, & Q_2 \end{bmatrix}^\top M_0 \begin{bmatrix} Q_1R_1, & Q_2 \end{bmatrix}, \\ \begin{bmatrix} C_1 & C_2 \\ C_2^\top & C_3 \end{bmatrix} &:= \begin{bmatrix} Q_1R_1, & Q_2 \end{bmatrix}^\top C_0 \begin{bmatrix} Q_1R_1, & Q_2 \end{bmatrix}, \\ \begin{bmatrix} K_1 & K_2 \\ K_2^\top & K_3 \end{bmatrix} &:= \begin{bmatrix} Q_1R_1, & Q_2 \end{bmatrix}^\top K_0 \begin{bmatrix} Q_1R_1, & Q_2 \end{bmatrix}, \end{aligned} \tag{2.2}$$

where the partitioning is such that the symmetric matrices M_1, C_1 and K_1 are all of size $k \times k$. Then the equation (1.2) is equivalent to

$$\begin{bmatrix} M_1 \\ M_2^\top \end{bmatrix} \Lambda^2 + \begin{bmatrix} C_1 \\ C_2^\top \end{bmatrix} \Lambda + \begin{bmatrix} K_1 \\ K_2^\top \end{bmatrix} = 0.$$

By Lemma 2.2, if we define

$$\Gamma := C_1 + M_1\Lambda + \Lambda^\top M_1, \tag{2.3}$$

then Γ is symmetric, satisfies

$$\Lambda^\top \Gamma = \Gamma\Lambda, \tag{2.4}$$

and is block diagonal with similar (but symmetric) structure as that of Λ. Note the relationships that

$$\left\{\begin{array}{lcl} C_1 &=& \Gamma - M_1\Lambda - \Lambda^\top M_1, \\ K_1 &=& \Lambda^\top M_1\Lambda - \Gamma\Lambda, \\ K_2 &=& -(\Lambda^\top)^2 M_2 - \Lambda^\top C_2. \end{array}\right. \tag{2.5}$$

Likewise, apply the same congruence transformation to $(\Delta M, \Delta C, \Delta K)$ and write

$$\begin{array}{rcl} \left[\begin{array}{cc} \Delta M_1 & \Delta M_2 \\ (\Delta M_2)^\top & \Delta M_3 \end{array}\right] &:=& \left[\begin{array}{cc} Q_1R_1, & Q_2 \end{array}\right]^\top \Delta M \left[\begin{array}{cc} Q_1R_1, & Q_2 \end{array}\right], \\ \left[\begin{array}{cc} \Delta C_1 & \Delta C_2 \\ (\Delta C_2)^\top & \Delta C_3 \end{array}\right] &:=& \left[\begin{array}{cc} Q_1R_1, & Q_2 \end{array}\right]^\top \Delta C \left[\begin{array}{cc} Q_1R_1, & Q_2 \end{array}\right], \\ \left[\begin{array}{cc} \Delta K_1 & \Delta K_2 \\ (\Delta K_2)^\top & \Delta K_3 \end{array}\right] &:=& \left[\begin{array}{cc} Q_1R_1, & Q_2 \end{array}\right]^\top \Delta K \left[\begin{array}{cc} Q_1R_1, & Q_2 \end{array}\right]. \end{array} \tag{2.6}$$

By construction, it follows from (2.1) that

$$\left\{\begin{array}{lcl} \Delta M_1 &=& M_1\Phi M_1, \\ \Delta M_2 &=& M_1\Phi M_2, \\ \Delta K_1 &=& K_1\Lambda^{-1}\Phi\Lambda^{-\top}K_1. \end{array}\right. \tag{2.7}$$

We next turn our attention to the equation (1.5) which pertains to the updated eigeninformation. Upon substituting (1.7) into (1.5) and defining

$$\widetilde{\Sigma} = T\Sigma T^{-1},$$

we see that

$$(M_0 + \Delta M)X\widetilde{\Sigma}^2 + (C_0 + \Delta C)X\widetilde{\Sigma} + (K_0 + \Delta K)X = 0,$$

which is equivalent to

$$\left[\begin{array}{c} M_1 + M_1\Phi M_1 \\ (M_2 + M_1\Phi M_2)^\top \end{array}\right]\widetilde{\Sigma}^2 + \left[\begin{array}{c} C_1 + \Delta C_1 \\ (C_2 + \Delta C_2)^\top \end{array}\right]\widetilde{\Sigma} + \left[\begin{array}{c} K_1 + \Delta K_1 \\ (K_2 + \Delta K_2)^\top \end{array}\right] = 0.$$

By Lemma 2.2 again, there exists a symmetric matrix Ξ satisfying

$$\widetilde{\Sigma}^\top \Xi = \Xi\widetilde{\Sigma}, \tag{2.8}$$

and such that

$$\left\{\begin{array}{lcl} C_1 + \Delta C_1 &=& \Xi - (M_1 + M_1\Phi M_1)\widetilde{\Sigma} - \widetilde{\Sigma}^\top(M_1 + M_1\Phi M_1), \\ K_1 + \Delta K_1 &=& \widetilde{\Sigma}^\top(M_1 + M_1\Phi M_1)\widetilde{\Sigma} - \Xi\widetilde{\Sigma} = \widetilde{\Sigma}^\top(M_1 + M_1\Phi M_1)\widetilde{\Sigma} - \widetilde{\Sigma}^\top\Xi, \\ K_2 + \Delta K_2 &=& -(\widetilde{\Sigma}^\top)^2(M_2 + M_1\Phi M_2) - \widetilde{\Sigma}^\top(C_2 + \Delta C_2). \end{array}\right. \tag{2.9}$$

Combining (2.5) and (2.9), we conclude that

$$\left\{\begin{array}{lcl} \Delta C_1 &=& \Xi - M_1\widetilde{\Sigma} - \widetilde{\Sigma}^\top M_1 - M_1\Phi M_1\widetilde{\Sigma} - \widetilde{\Sigma}^\top M_1\Phi M_1 - \Gamma + M_1\Lambda + \Lambda^\top M_1, \\ \Delta K_1 &=& \widetilde{\Sigma}^\top M_1\widetilde{\Sigma} + \widetilde{\Sigma}^\top M_1\Phi M_1\widetilde{\Sigma} - \widetilde{\Sigma}^\top\Xi - \Lambda^\top M_1\Lambda + \Lambda^\top\Gamma, \\ \Delta K_2 &=& -(\widetilde{\Sigma}^\top)^2 M_2 - \widetilde{\Sigma}^\top C_2 - (\widetilde{\Sigma}^\top)^2 M_1\Phi M_2 - \widetilde{\Sigma}^\top\Delta C_2 + (\Lambda^\top)^2 M_2 + \Lambda^\top C_2. \end{array}\right. \tag{2.10}$$

We summarize the needed updates in the following way for later reference,

$$\begin{array}{rcl}
\left[\begin{array}{cc} \Delta M_1, & \Delta M_2 \end{array} \right] & = & M_1\Phi \left[\begin{array}{cc} M_1, & M_2 \end{array} \right], \\
\left[\begin{array}{cc} \Delta C_1, & \Delta C_2 \end{array} \right] & = & M_1\Phi \left[\begin{array}{cc} \Gamma - M_1\Lambda, & \Lambda^\top M_2 + C_2 \end{array} \right] \\
& & +(\Lambda^\top - \widetilde{\Sigma}^\top - \widetilde{\Sigma}^\top M_1\Phi) \left[\begin{array}{cc} M_1, & M_2 \end{array} \right] \\
& & + \left[\begin{array}{cc} (M_1 + M_1\Phi M_1)(\Lambda - \widetilde{\Sigma}) + \Xi - \Gamma - M_1\Phi\Gamma, & \Delta W \end{array} \right], \\
\left[\begin{array}{cc} \Delta K_1, & \Delta K_2 \end{array} \right] & = & (\Lambda^\top - \widetilde{\Sigma}^\top - \widetilde{\Sigma}^\top M_1\Phi) \left[\begin{array}{cc} \Gamma - M_1\Lambda, & \Lambda^\top M_2 + C_2 \end{array} \right] \\
& & -\widetilde{\Sigma}^\top \left[\begin{array}{cc} (M_1 + M_1\Phi M_1)(\Lambda - \widetilde{\Sigma}) + \Xi - \Gamma - M_1\Phi\Gamma, & \Delta W \end{array} \right],
\end{array} \tag{2.11}$$

with

$$\Delta W := \Delta C_2 - [M_1\Phi(\Lambda^\top M_2 + C_2) + (\Lambda^\top - \tilde{\Sigma}^\top - \tilde{\Sigma}^\top M_1\Phi)M_2].$$

Finally, we look into the equation (1.6) which is needed for the no spill-over to the unmeasured or unknown eigeninformation. By defining

$$\mathcal{Z} = \left[\begin{array}{cc} Q_1R_1 & Q_2 \end{array} \right]^{-1} Z,$$

then the equation

$$M_0 \left[\begin{array}{cc} X & Z \end{array} \right] \left[\begin{array}{cc} \Lambda & \\ & \Upsilon \end{array} \right]^2 + C_0 \left[\begin{array}{cc} X & Z \end{array} \right] \left[\begin{array}{cc} \Lambda & \\ & \Upsilon \end{array} \right] + K_0 \left[\begin{array}{cc} X & Z \end{array} \right] = 0$$

is reduced to

$$\begin{array}{r}
\left[\begin{array}{cc} M_1 & M_2 \\ M_2^\top & M_3 \end{array} \right] \left[\left[\begin{array}{c} I \\ 0 \end{array} \right], \ \mathcal{Z} \right] \left[\begin{array}{cc} \Lambda & \\ & \Upsilon \end{array} \right]^2 + \left[\begin{array}{cc} C_1 & C_2 \\ C_2^\top & C_3 \end{array} \right] \left[\left[\begin{array}{c} I \\ 0 \end{array} \right], \ \mathcal{Z} \right] \left[\begin{array}{cc} \Lambda & \\ & \Upsilon \end{array} \right] \\
+ \left[\begin{array}{cc} K_1 & K_2 \\ K_2^\top & K_3 \end{array} \right] \left[\left[\begin{array}{c} I \\ 0 \end{array} \right], \ \mathcal{Z} \right] = 0.
\end{array}$$

By Lemma 2.3 and (2.5), we obtain the relationship

$$\Lambda^\top \left\{ \left[\begin{array}{cc} M_1, & M_2 \end{array} \right] \mathcal{Z}\Upsilon + \left[\begin{array}{cc} \Gamma - M_1\Lambda, & \Lambda^\top M_2 + C_2 \end{array} \right] \mathcal{Z} \right\} = 0,$$

which implies that

$$\left[\begin{array}{cc} M_1, & M_2 \end{array} \right] \mathcal{Z}\Upsilon + \left[\begin{array}{cc} \Gamma - M_1\Lambda, & \Lambda^\top M_2 + C_2 \end{array} \right] \mathcal{Z} = 0. \tag{2.12}$$

As far as the equation (1.6) is concerned, an algebraic manipulation by using (2.11) and (2.12) yields the following facts,

$$\begin{array}{l}
(M_0 + \Delta M)Z\Upsilon^2 + (C_0 + \Delta D)Z\Upsilon + (K_0 + \Delta K)Z = 0 \\
\Rightarrow \left[\begin{array}{cc} M_1 + \Delta M_1, & M_2 + \Delta M_2 \end{array} \right] \mathcal{Z}\Upsilon^2 + \left[\begin{array}{cc} C_1 + \Delta C_1, & C_2 + \Delta C_2 \end{array} \right] \mathcal{Z}\Upsilon \\
\qquad + \left[\begin{array}{cc} K_1 + \Delta K_1, & K_2 + \Delta K_2 \end{array} \right] \mathcal{Z} = 0 \\
\Leftrightarrow \left[\begin{array}{cc} (M_1 + M_1\Phi M_1)(\Lambda - \widetilde{\Sigma}) + \Xi - \Gamma - M_1\Phi\Gamma, & \Delta W \end{array} \right] \mathcal{Z}\Upsilon \\
\qquad -\widetilde{\Sigma}^\top \left[\begin{array}{cc} (M_1 + M_1\Phi M_1)(\Lambda - \widetilde{\Sigma}) + \Xi - \Gamma - M_1\Phi\Gamma, & \Delta W \end{array} \right] \mathcal{Z} = 0.
\end{array} \tag{2.13}$$

We now have all the tools needed to characterize the parameter matrix Φ. As (Λ, X) is being replaced by (Σ, Y) and $Y = XT$, we claim that the two matrices Γ and Ξ employed in (2.5) and (2.9) are also related, indeed, in a special way.

LEMMA 2.4. *Assume that the unmeasured eigenvectors Z is of full row rank and that the unmeasured eigenvalues Υ and the newly measured eigenvalues Σ are disjoint. Then it is true that $\Xi = \Gamma$.*

Proof. By assumption, Υ and $\widetilde{\Sigma} = T\Sigma T^{-1}$ have no common eigenvalues. By Lemma 2.1 we see from (2.13)

$$\left[\begin{array}{cc} (M_1 + M_1\Phi M_1)(\Lambda - \widetilde{\Sigma}) + \Xi - \Gamma - M_1\Phi\Gamma, & \Delta W \end{array} \right] \mathcal{Z} = 0.$$

Since $\mathcal{Z}$ is of full row rank, we see that

$$\Delta C_2 = M_1\Phi(\Lambda^\top M_2 + C_2) + (\Lambda^\top - \widetilde{\Sigma}^\top - \widetilde{\Sigma}^\top M_1\Phi)M_2, \tag{2.14}$$

$$\Xi = (I + M_1\Phi)[M_1(\widetilde{\Sigma} - \Lambda) + \Gamma]. \tag{2.15}$$

Recall that the matrix ΔK_1 can be expressed by two different ways (2.7) and (2.11), where the latter can further be simplified by (2.15) so that

$$\Delta K_1 = K_1\Lambda^{-1}\Phi\Lambda^{-\top}K_1 = (\Lambda^\top - \widetilde{\Sigma}^\top - \widetilde{\Sigma}^\top M_1\Phi)(\Gamma - M_1\Lambda).$$

On the other hand, we can write the middle equation in (2.5) as $\Gamma = M_1\Lambda - \Lambda^{-\top}K_1$, therefore

$$K_1\Lambda^{-1}\Phi\Lambda^{-\top}K_1 = (\Lambda^\top - \widetilde{\Sigma}^\top - \widetilde{\Sigma}^\top M_1\Phi)(-\Lambda^{-T}K_1).$$

It follows that

$$\widetilde{\Sigma} - \Lambda = \Phi\Lambda^{-\top}(K_1 - \Lambda^\top M_1\widetilde{\Sigma}) = \Phi[M_1(\Lambda - \widetilde{\Sigma}) - \Gamma]. \tag{2.16}$$

Simplifying (2.15) by using (2.16) in the following way,

$$\Xi = (I + M_1\Phi)[M_1(\widetilde{\Sigma} - \Lambda) + \Gamma] = M_1(\widetilde{\Sigma} - \Lambda) + \Gamma + M_1(\Lambda - \widetilde{\Sigma}) = \Gamma,$$

we see that the two matrices Ξ and Γ are identical. □

LEMMA 2.5. *Assume that $\widetilde{\Sigma} - \Lambda = T\Sigma T^{-1} - \Lambda$ is nonsingular. Then the parameter matrix Φ is uniquely determined by the formula*

$$\Phi = [\Gamma T(\Lambda T - T\Sigma)^{-1} - M_1]^{-1}. \tag{2.17}$$

Proof. By (2.16), $K_1 - \Lambda^\top M_1\widetilde{\Sigma}$ is invertible and

$$\Phi = (\widetilde{\Sigma} - \Lambda)(K_1 - \Lambda^\top M_1\widetilde{\Sigma})^{-1}\Lambda^\top = (T\Sigma - \Lambda T)(K_1T - \Lambda^\top M_1T\Sigma)^{-1}\Lambda^\top \tag{2.18}$$

Since $\Lambda^{-\top}K_1T - M_1T\Sigma = M_1(\Lambda T - T\Sigma) - \Gamma T$, the assertion is proved. □

Note that by Lemma 2.4, the equation (2.8) is equivalent to

$$\Sigma^\top T^\top \Gamma T = T^\top \Gamma T\Sigma. \tag{2.19}$$

Together with the property (2.4), it is clear that the Φ specified in (2.17) is symmetric. In this course of discussion, we have developed several necessary conditions which we now summarize in the following theorem.

THEOREM 2.6. *Suppose that the triplet $(\Delta M, \Delta C, \Delta K)$ in the form (2.1) for some symmetric matrix $\Phi \in \mathbb{R}^{k\times k}$ solves the MUP. Assume that the following generic conditions among the outgoing eigenpairs (Λ, X), the updated eigenpairs (Σ, Y), and the untouched eigenpairs (Υ, Z) are satisfied:*

a. $Y = XT$ *for some nonsingular matrix* $T \in \mathbb{R}^{k\times k}$.
b. Z *is of full row rank.*
c. The spectra of Υ *and* Σ *are disjoint.*
d. $T\Sigma T^{-1} - \Lambda$ *is nonsingular.*

Then it is necessary that

1. *The matrix* $X^\top K_0 XT - \Lambda^\top X^\top M_0 XT\Sigma$ *is nonsingular.*
2. *The matrix* $T^\top \left(X^\top C_0 X + \Lambda^\top X^\top M_0 X + X^\top M_0 X\Lambda\right) T\Sigma$ *is symmetric.*
3. $\Phi = (T\Sigma - \Lambda T)(X^\top K_0 XT - \Lambda^\top X^\top M_0 XT\Sigma)^{-1}\Lambda^\top$.
4. $M + \Delta M$ *and* $K + \Delta K$ *are positive definite.*

Proof. It is clear from the definition in (2.2) that $M_1 = X^\top M_0 X$, $C_1 = X^\top C_0 X$, and $K_1 = X^\top K_0 X$. The second condition above follows from (2.19) if Γ is replaced by (2.3). The first and the third conditions are established in the proof of Lemma 2.5. The fourth condition is inherent in the assumption. □

The matter of fact is that the converse of Theorem 2.6 is also true. Specifically, we make the following claim that the above necessary conditions are also sufficient.

THEOREM 2.7. *Suppose that the outgoing eigenpairs* (Λ, X)*, the updated eigenpairs* (Σ, Y)*, and the untouched eigenpairs* (Υ, Z) *satisfy the generic conditions (a)-(d) in Theorem 2.6. Assume that Conditions 1-2 are satisfied. Then the triplet* $(\Delta M, \Delta C, \Delta K)$ *in the form (2.1) with* Φ *being uniquely given as in Condition 3 solves the two equations (1.5) and (1.6). If the resulting* $M_0 + \Delta M$ *and* $K_0 + \Delta K$ *are positive definite, then the MUP is solvable; otherwise, the original model* (M_0, C_0, K_0) *cannot be updated by* (Σ, Y) *without losing positive definiteness.*

Proof. We need to check three things to warrant the assertion. First, Condition 1 implies that the specific Φ given by Condition 3 is well defined. Condition 2 is equivalent to the equation (2.19) where Γ is defined by (2.3), which shows that Φ is symmetric. The triplet $(\Delta M, \Delta C, \Delta K)$ in the form (2.1) with any symmetric matrix Φ already satisfies (1.6). Secondly, defining $\Xi = \Gamma$ in (2.9), by Lemma 2.2, we then can trace the proof of Theorem 2.6 backward to prove that the equation (1.5) is satisfied. Thirdly, the uniqueness of Φ in Lemma 2.5 implies that the Conditions 1-3 either make or break the solvability of the MUP, depending upon whether the resulting $M + \Delta M$ and $K + \Delta K$ are positive definite or not. □

Example 1. Consider the statically condensed oil rig model (M_0, C_0, K_0) represented by the triplet BCSSTRUC1 in the Harwell-Boeing collection [36]. In this model, M_0 and $K_0 \in \mathbb{R}^{66\times 66}$ are symmetric and positive definite and $C_0 = 1.55 I_{66}$. There are 132 eigenpairs. Suppose we want to replace the eight eigenvalues

$$\lambda_1 = -5.358410088235457, \quad \lambda_2 = -3.462830582716281,$$
$$\lambda_3 = -3.570946054521908, \quad \lambda_4 = -9.276066378899415,$$

$$\lambda_5 = -7.802118288361733 + 164.3321224340448i, \quad \lambda_6 = -7.802118288361733 - 164.3321224340448i,$$
$$\lambda_7 = -7.755809434339588 + 164.0571880852085i, \quad \lambda_8 = -7.755809434339588 - 164.0571880852085i,$$

by newly measured eigenvalues

$$\sigma_1 = -5.05, \quad \sigma_2 = -3.32,$$
$$\sigma_3 = -3.75, \quad \sigma_4 = -9.07,$$

$$\sigma_5 = -7 + 160i, \quad \sigma_6 = -7 - 160i,$$
$$\sigma_7 = -8 + 170i, \quad \sigma_8 = -8 - 170i,$$

while keeping the corresponding eigenvectors invariant. That is, $Y = X$ and $T = I_{66}$. We check to see that $X^\top K_0 X - \Lambda^\top X^\top M_0 X\Sigma$ is nonsingular, so Condition 1 in Theorem 2.6 is satisfied.

Condition 2 is reduced to the symmetry of $\Gamma\Sigma$ which is automatically satisfied. We compute the uniquely described Φ according to Condition 3 in Theorem 2.6 and the resulting $(\Delta M, \Delta C, \Delta K)$. We can verify numerically that the computed $M + \Delta M$ and $K + \Delta K$ are positive definite. We think that the model has been updated satisfactorily because the residuals of the updated model

$$\|(M + \Delta M)X\Sigma^2 + (C + \Delta C)X\Sigma + (K + \Delta K)X\|_2 = 2.830849890712934 \times 10^{-9},$$
$$\|(M + \Delta M)Z\Upsilon^2 + (C + \Delta C)Z\Upsilon + (K + \Delta K)Z\|_2 = 7.620711447892196 \times 10^{-9}.$$

is compatible with the residual of the original model

$$\|MX\Lambda^2 + CX\Lambda + KX\|_2 = 2.873651863994003 \times 10^{-9},$$
$$\|MZ\Upsilon^2 + CZ\Upsilon + KZ\|_2 = 7.617801878019591 \times 10^{-9}.$$

Example 2. Consider the same data set as above except that $Y = XT$ with

$$T = \operatorname{diag}\left\{0.9169, 0.8132, 0.6038, 0.4451, \begin{bmatrix} 0.9318 & 0.4186 \\ 0.4186 & -0.9318 \end{bmatrix}, \begin{bmatrix} 0.5252 & 0.6721 \\ 0.6721 & -0.5252 \end{bmatrix}\right\}.$$

It can be checked that Conditions 1 and 2 are satisfied. We compute the unique Φ and the resulting $(\Delta M, \Delta C, \Delta K)$ based on Theorem 2.6. Surely, we find that the updated residuals are reasonably small,

$$\|(M + \Delta M)Y\Sigma^2 + (C + \Delta C)Y\Sigma + (K + \Delta K)Y\|_2 = 2.822913902500785 \times 10^{-9},$$
$$\|(M + \Delta M)Z\Upsilon^2 + (C + \Delta C)Z\Upsilon + (K + \Delta K)Z\|_2 = 7.612797317470159 \times 10^{-9}.$$

However, it turns out that both $M + \Delta M$ and $K + \Delta K$ are not positive definite. This example demonstrates that the MUP may not be solvable even if the triplet $(\Delta M,\ \Delta C, \Delta K)$ satisfy (1.5) and (1.6).

Example 3. Consider the same data set again with $Y = XT$ where

$$T = \operatorname{diag}\left\{-0.4326, -1.6656, 0.1253, 0.2877, \begin{bmatrix} -1.1465 & 1.1909 \\ -1.1909 & -1.1465 \end{bmatrix}, \begin{bmatrix} 1.1892 & -0.0376 \\ 0.0376 & 1.1892 \end{bmatrix}\right\}.$$

Again, Conditions 1 and 2 are satisfied. We compute Φ and $(\Delta M, \Delta C, \Delta K)$. It is found that

$$\|(M + \Delta M)Y\Sigma^2 + (C + \Delta C)Y\Sigma + (K + \Delta K)Y\|_2 = 4.535516693254191 \times 10^{-9},$$
$$\|(M + \Delta M)Z\Upsilon^2 + (C + \Delta C)Z\Upsilon + (K + \Delta K)Z\|_2 = 7.620710821638826 \times 10^{-9},$$

while $M + \Delta M$ and $K + \Delta K$ are positive definite.

These three examples clearly illustrate the importance of a properly selected T.

3. Solvability under a specific structure.. Thus far, our theory of solvability has been developed under the assumption that the incremental matrices ΔM, ΔC, and ΔK are of the structure in (2.1). It has been proved in [4, Theorem 3.5] that for almost all values of the untouched eigenpair (Υ, Z) these conditions are also necessary. Theorems 2.6 and 2.7 therefore are perhaps the most general results for model updating while preserving both positive definiteness and no spill-over.

Still, it is curious to ask whether there are other types of solutions $(\Delta M, \Delta C, \Delta K)$ to the MUP which do not assume the generic form. This section provides an affirmative answer to this question for the case $k = n$. It suffices to limit ourselves to the case $Y = X$.

Let the matrix $M_1 = X^\top M_0 X \in \mathbb{R}^{n\times n}$ be partitioned into $t \times t$ blocks,

$$M_1 = \begin{array}{c} \begin{array}{cccccc} \mathit{1} & \cdots & \mathit{1} & \mathit{2} & \cdots & \mathit{2} \end{array} \\ \left[\begin{array}{cccccc} M_1^{(1,1)} & \cdots & M_1^{(1,s)} & M_1^{(1,s+1)} & \cdots & M_1^{(1,t)} \\ \vdots & \ddots & \vdots & & \ddots & \vdots \\ M_1^{(s,1)} & \cdots & M_1^{(s,s)} & M_1^{(s,s+1)} & \cdots & M_1^{(l,t)} \\ M_1^{(s+1,1)} & \cdots & M_1^{(s+1,s)} & M_1^{(s+1,s+1)} & \cdots & M_1^{(s+1,t)} \\ \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ M_1^{(t,1)} & \cdots & M_1^{(t,s)} & M_1^{(t,s+1)} & \cdots & M_1^{(t,t)} \end{array}\right] \end{array} \begin{array}{c} \mathit{1} \\ \vdots \\ \mathit{1} \\ \mathit{2} \\ \vdots \\ \mathit{2} \end{array}, \tag{3.1}$$

where each block is of the size as indicated along the border. With respect to this partitioning, there exists a permutation matrix $P \in \mathbb{R}^{n\times n}$ among blocks such that

$$P^\top M_1 P = \operatorname{diag}\{\mathcal{M}_{11}, \ldots, \mathcal{M}_{pp}\}, \tag{3.2}$$

where each $\mathcal{M}_{ii}$ is composed of one or more blocks $M_1^{(g,h)}$ and is block irreducible [20]. If $p = 1$, then M_1 is block irreducible, otherwise, if $p > 1$, then M_1 is block reducible.

Let Γ be the same as the one in the preceding section. Denote

$$\begin{aligned} P^\top \Lambda P &= \operatorname{diag}\{\Lambda_{11}, \ldots, \Lambda_{pp}\}, \\ P^\top \Sigma P &= \operatorname{diag}\{\Sigma_{11}, \ldots, \Sigma_{pp}\}, \\ P^\top \Gamma P &= \operatorname{diag}\{\Gamma_{11}, \ldots, \Gamma_{pp}\}, \end{aligned}$$

which is always possible because Λ defined in (1.3) and the corresponding Σ and Γ are block diagonal and have the same partitioning as M_1.

LEMMA 3.1. *There exists one permutation matrix $P \in \mathbb{R}^{n\times n}$ of blocks specified in (3.1) such that*

1. *$\mathcal{M}_{11}$ is either empty or is diagonal with $\mathcal{M}_{11}(\Sigma_{11} - \Lambda_{11}) + \Gamma_{11} = 0$;*
2. *For $i = 2, \cdots, p$, if $\mathcal{M}_{ii} \in \mathbb{R}$, then $\mathcal{M}_{ii}(\Sigma_{ii} - \Lambda_{ii}) + \Gamma_{ii} \neq 0$; otherwise, either $\mathcal{M}_{ii}$ has only one 2-by-2 block, or $\mathcal{M}_{ii}$ has at least two blocks and is block irreducible.*

Proof. Let P denote temporarily any permutation matrix that does (3.2). For each $i = 1, \ldots, p$, the submatrices $\mathcal{M}_{ii}$, Λ_{ii}, Σ_{ii} and Γ_{ii} are of the same sizes. For each fixed i, there are three possible cases for these four blocks $\mathcal{M}_{ii}$, Λ_{ii}, Σ_{ii}, Γ_{ii}: they are all scalars, or they contain only one 2-by-2 block from M_1, Λ, Σ and Γ, respectively, or they have at least two blocks from M_1, Λ, Σ and Γ, respectively, and $\mathcal{M}_{ii}$ is block irreducible. In the first case, we may introduce one more round of permutation on the same partition (3.1) to group those with $\mathcal{M}_{ii}(\Sigma_{ii} - \Lambda_{ii}) + \Gamma_{ii} = 0$ together and rename the resulting block as $\mathcal{M}_{11}$. ◻

To set forth the exploration of a "new" solution not in the generic form, we first develop some useful identities that any solution $(\Delta M, \Delta C, \Delta K)$ to (1.5) and (1.6) must satisfy. As a necessary condition, it should not be surprising that many of the facts derived in the preceding sections remain valid so long as their cogency is independent of any reference to Φ. For instance, we shall continue using (2.5) with an appropriate symmetric block diagonal matrix $\Gamma \in \mathbb{R}^{n\times n}$.

Since M_0 and X are nonsingular, we also can write

$$\Delta M = M_0 X \underbrace{(M_0 X)^{-1} \Delta M (M_0 X)^{-\top}}_{\Phi} X^\top M_0,$$

where Φ is symmetric. Without causing ambiguity, we use the same notation as before,

$$\left\{\begin{array}{rcl} \Delta M_1 &:=& X^\top \Delta M X = M_1 \Phi M_1, \\ \Delta C_1 &:=& X^\top \Delta C X, \\ \Delta K_1 &:=& X^\top \Delta K X, \\ \mathcal{Z} &:=& X^{-1} Z, \end{array}\right. \tag{3.3}$$

except that ΔC and ΔK are not necessarily in the parametric form in (2.1).

To satisfy the equation (1.5), observe that

$$\begin{aligned} &(M+\Delta M)X\Sigma^2 + (C+\Delta C)X\Sigma + (K+\Delta K)X = 0 \\ \Leftrightarrow\ &(M_1 + M_1\Phi M_1)\Sigma^2 + (C_1+\Delta C_1)\Sigma + (K_1+\Delta K_1) = 0. \end{aligned}$$

The second equation above enables us to carry through the steps taken in the preceding section with only a few modifications. In particular, by Lemma 2.2, there exists a symmetric matrix Ξ which is block diagonal with similar structure as that of Σ, satisfies

$$\Sigma^\top \Xi = \Xi\Sigma, \tag{3.4}$$

and is such that

$$\left\{ \begin{array}{lcl} C_1 + \Delta C_1 &=& \Xi - (M_1 + M_1\Phi M_1)\Sigma - \Sigma^\top(M_1 + M_1\Phi M_1), \\ K_1 + \Delta K_1 &=& \Sigma^\top(M_1 + M_1\Phi M_1)\Sigma - \Xi\Sigma = \Sigma^\top(M_1 + M_1\Phi M_1)\Sigma - \Sigma^\top\Xi, \end{array} \right. \tag{3.5}$$

It follows that (See (2.11))

$$\left\{ \begin{array}{lcl} \Delta M_1 &=& M_1\Phi M_1 \\ \Delta C_1 &=& M_1\Phi(\Gamma - M_1\Lambda) + (\Lambda^\top - \Sigma^\top - \Sigma^\top M_1\Phi)M_1 \\ && +(M_1 + M_1\Phi M_1)(\Lambda - \Sigma) + \Xi - \Gamma - M_1\Phi\Gamma, \\ \Delta K_1 &=& (\Lambda^\top - \Sigma^\top - \Sigma^\top M_1\Phi)(\Gamma - M_1\Lambda) \\ && -\Sigma^\top\left[(M_1 + M_1\Phi M_1)(\Lambda - \Sigma) + \Xi - \Gamma - M_1\Phi\Gamma\right]. \end{array} \right. \tag{3.6}$$

Similarly, we can prove that (See (2.12))

$$M_1\mathcal{Z}\Upsilon + (\Gamma - M_1\Lambda)\mathcal{Z} = 0, \tag{3.7}$$

that (See (2.13))

$$(M_1 + M_1\Phi M_1)(\Lambda - \Sigma) + \Xi - \Gamma - M_1\Phi\Gamma = 0,$$

and hence (See (2.15))

$$(I + M_1\Phi)[M_1(\Sigma - \Lambda) + \Gamma] = \Xi. \tag{3.8}$$

Applying the permutation matrix P assumed in Lemma 3.1 to both sides of (3.8) and noting that $P^\top \Xi P$ is block diagonal due to the structure of Ξ, we see that

$$\begin{aligned} &\begin{bmatrix} I + \mathcal{M}_{11}\Phi_{11} & \cdots & \mathcal{M}_{11}\Phi_{1p} \\ \vdots & \ddots & \vdots \\ \mathcal{M}_{pp}\Phi_{1p}^\top & \cdots & I + \mathcal{M}_{pp}\Phi_{pp} \end{bmatrix} \operatorname{diag}\left\{\mathcal{M}_{11}(\Sigma_{11} - \Lambda_{11}) + \Gamma_{11}, \ldots, \mathcal{M}_{pp}(\Sigma_{pp} - \Lambda_{pp}) + \Gamma_{pp}\right\} \\ &\qquad = \operatorname{diag}\left\{\Xi_{11}, \ldots, \Xi_{pp}\right\}, \end{aligned} \tag{3.9}$$

where we have denoted the symmetric matrices $P^\top \Phi P$ and $P^\top \Xi P$ by

$$\begin{aligned} P^\top \Phi P &= \begin{bmatrix} \Phi_{11} & \cdots & \Phi_{1p} \\ \vdots & \ddots & \vdots \\ \Phi_{1p}^\top & \cdots & \Phi_{pp} \end{bmatrix}, \\ P^\top \Xi P &= \operatorname{diag}\left\{\Xi_{11}, \ldots, \Xi_{pp}\right].\end{aligned}$$

By Lemma 3.1, it must be either Ξ_{11} is empty or $\Xi_{11} = 0$. In the latter, Φ_{11} is arbitrary but symmetric. From (3.9) we also see that the following two equalities hold,

$$\mathcal{M}_{11} \begin{bmatrix} \Phi_{12} & \cdots & \Phi_{1p} \end{bmatrix} \operatorname{diag}\{\mathcal{M}_{22}(\Sigma_{22} - \Lambda_{22}) + \Gamma_{22}, \ldots, \mathcal{M}_{pp}(\Sigma_{pp} - \Lambda_{pp}) + \Gamma_{pp}\} = 0, \quad (3.10)$$

$$\begin{bmatrix} I + \mathcal{M}_{22}\Phi_{22} & \cdots & \mathcal{M}_{22}\Phi_{2p} \\ \vdots & \ddots & \vdots \\ \mathcal{M}_{pp}\Phi_{2p}^\top & \cdots & I + \mathcal{M}_{pp}\Phi_{pp} \end{bmatrix} \operatorname{diag}\{\mathcal{M}_{22}(\Sigma_{22} - \Lambda_{22}) + \Gamma_{22}, \ldots, \mathcal{M}_{pp}(\Sigma_{pp} - \Lambda_{pp}) + \Gamma_{pp}\}$$
$$= \operatorname{diag}\{\Xi_{22}, \ldots, \Xi_{pp}\}. \quad (3.11)$$

Now we are ready to characterize necessary conditions for the solvability of the MUP. The emphasis is that $(\Delta M, \Delta C, \Delta K)$ does not assume the parametric form described in (2.1). The first condition is clear — since $K_0 + \Delta K$ is positive definite, the matrix Σ of updated eigenvalues must be nonsingular, thus, $\Sigma_{i,i}$ $(i = 1, \cdots, p)$ are also nonsingular.

We claim the following intermediate result concerning Ξ.

Lemma 3.2. *The matrix* $\operatorname{diag}\{\Xi_{22}, \ldots, \Xi_{pp}\}$ *is nonsingular.*

Proof. We prove by contradiction. Suppose that Ξ_{ii} is singular for some $i = 2, \ldots, p$. Recall that Ξ_{ii} is of size either 1×1 or 2×2.

If $\mathcal{M}_{ii}$ is a scalar, then $\Xi_{ii} = 0$. Since $M_1 + M_1\Phi M_1 = X^\top(M_0 + \Delta M)X$ is positive definite, the matrix

$$\begin{bmatrix} I + \mathcal{M}_{22}\Phi_{22} & \cdots & \mathcal{M}_{22}\Phi_{2p} \\ \vdots & \ddots & \vdots \\ \mathcal{M}_{pp}\Phi_{2p}^\top & \cdots & I + \mathcal{M}_{pp}\Phi_{pp} \end{bmatrix}$$

is nonsingular. The corresponding entry in (3.11) would force $\mathcal{M}_{ii}(\Sigma_{ii} - \Lambda_{ii}) + \Gamma_{ii} = 0$, which is impossible by the way we choose P in Lemma 3.1.

If $\mathcal{M}_{ii}$ consists of only a 2×2 block, the corresponding Ξ_{ii} is of the form $\Xi_{ii} = \begin{bmatrix} \xi_i & \eta_i \\ \eta_i & -\xi_i \end{bmatrix}$. The singularity of Ξ_{ii} would imply $\Xi_{ii} = 0$. The corresponding block in (3.11) would imply $\mathcal{M}_{ii}(\Sigma_{ii} - \Lambda_{ii}) + \Gamma_{ii} = 0$. By the structures that inhere in 2×2 matrices Σ_{ii}, Λ_{ii} and Γ_{ii}, it follows that $\mathcal{M}_{ii} = -(\Sigma_{ii} - \Lambda_{ii})^{-1}\Gamma_{ii}$ would have trace zero, which is impossible because $\mathcal{M}_{ii}$ is positive definite.

Finally, if $\mathcal{M}_{ii}$ consists of at least two blocks and is block irreducible, we may assume without loss of generality that the first diagonal block of Ξ_{ii} is singular which, regardless of its sizes, by the argument above must be zero. The corresponding first block column of $\mathcal{M}_{ii}(\Sigma_{ii} - \Lambda_{i,i}) + \Gamma_{i,i}$ therefore would be zero. But this is again impossible, because $\mathcal{M}_{ii}$ is block irreducible, $\Sigma_{ii} - \Lambda_{ii}$ is nonsingular, and both $\Sigma_{ii} - \Lambda_{ii}$ and Γ_{ii} are block diagonal. □

From (3.11), we thus can write

$$\begin{bmatrix} I + \mathcal{M}_{22}\Phi_{22} & \cdots & \mathcal{M}_{22}\Phi_{2p} \\ \vdots & \ddots & \vdots \\ \mathcal{M}_{pp}\Phi_{2p}^\top & \cdots & I + \mathcal{M}_{pp}\Phi_{pp} \end{bmatrix} =$$
$$\operatorname{diag}\{\Xi_{22}, \ldots, \Xi_{pp}\} \operatorname{diag}\{\mathcal{M}_{22}(\Sigma_{22} - \Lambda_{22}) + \Gamma_{22}, \ldots, \mathcal{M}_{pp}(\Sigma_{pp} - \Lambda_{pp}) + \Gamma_{pp}\}^{-1},$$

which implies together with (3.10) that

$$\begin{cases} \Phi_{ij} = 0, & \text{if } i \neq j, \\ \Phi_{ii} = \mathcal{M}_{ii}^{-1}\left\{\Xi_{ii}[\mathcal{M}_{ii}(\Sigma_{ii} - \Lambda_{ii}) + \Gamma_{ii}]^{-1} - I\right\}, & i = 2, \ldots, p. \end{cases} \quad (3.12)$$

In other words, we have just proved the following lemma.

LEMMA 3.3. *Suppose ΔM is part of a solution to the MUP. Let $\Phi := (M_0X)^{-1}\Delta M(M_0X)^{-\top}$ and P be the permutation matrix specified in Lemma 3.1. Then it must be such that*

$$P^\top \Phi P = \operatorname{diag}\{\Phi_{11},\ldots,\Phi_{pp}\}$$

where Φ_{11} is arbitrary but $\mathcal{M}_{11}+\mathcal{M}_{11}\Phi_{11}\mathcal{M}_{11}$ must be positive definite and Φ_{ii}, $i=2,\ldots,p$, is given by (3.12).

From the fact that $K_1 = \Lambda^\top M_1\Lambda - \Gamma\Lambda$, we write

$$P^\top K_1 P = \operatorname{diag}\{\mathcal{K}_{11},\ldots,\mathcal{K}_{pp}\},$$

with

$$\mathcal{K}_{ii} := \Lambda_{ii}^\top \mathcal{M}_{ii}\Lambda_{ii} - \Gamma_{ii}\Lambda_{ii} = \Lambda_{ii}^\top\mathcal{M}_{ii}\Lambda_{ii} - \Lambda_{ii}^\top\Gamma_{ii}, \quad i=1,\cdots,p. \tag{3.13}$$

It follows that for $i=2,\ldots,p$, the matrix

$$\Xi_{ii}(\Lambda_{ii}^\top\mathcal{M}_{ii}\Sigma_{ii}-\mathcal{K}_{ii})^{-1}\Lambda_{ii}^\top\mathcal{M}_{ii} = \Xi_{ii}[(\Sigma_{ii}-\Lambda_{ii})+\mathcal{M}_{ii}^{-1}\Gamma_{ii}]^{-1} = \mathcal{M}_{ii}+\mathcal{M}_{ii}\Phi_{ii}\mathcal{M}_{ii} \tag{3.14}$$

is positive definite because $M_1 + M_1\Phi M_1$ is positive definite.

Exploiting the structure of Ξ_{ii} and Γ_{ii} and rearranging the diagonal blocks if necessary, we may assume without loss of generality that

$$\Xi_{ii}^{-1}\Gamma_{ii} = \operatorname{diag}\{\varsigma_1,\ldots,\varsigma_{\ell_1},\Psi_{\ell_1+1},\ldots,\Psi_{\ell_1+\ell_2}\},$$

where $\varsigma_i\in\mathbb{R}$ for $i=1,\cdots,\ell_1$, and $\Psi_{\ell_1+j} = \begin{bmatrix}\varsigma_{\ell_1+j} & \vartheta_{\ell_1+j}\\ -\vartheta_{\ell_1+j} & \varsigma_{\ell_1+j}\end{bmatrix}$ for $j=1,\cdots,\ell_2$. Likewise, let $\mathcal{M}_{ii}$ be partitioned accordingly as

$$\mathcal{M}_{ii} = \begin{bmatrix} \Theta_{1,1} & \cdots & \Theta_{1,\ell_1} & \Theta_{1,\ell_1+1} & \cdots & \Theta_{1,\ell_1+\ell_2}\\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ \Theta_{1\ell_1}^\top & \cdots & \Theta_{\ell_1,\ell_1} & \Theta_{\ell_1,\ell_1+1} & \cdots & \Theta_{\ell_1,\ell_1+\ell_2}\\ \Theta_{1,\ell_1+1}^\top & \cdots & \Theta_{\ell_1,\ell_1+1}^\top & \Theta_{\ell_1+1,\ell_1+1} & \cdots & \Theta_{\ell_1+1,\ell_1+\ell_2}\\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ \Theta_{1,\ell_1+\ell_2}^\top & \cdots & \Theta_{\ell1,\ell_1+\ell_2}^\top & \Theta_{\ell_1+1,\ell_1+\ell_2}^\top & \cdots & \Theta_{\ell_1+\ell_2,\ell_1+\ell_2}\end{bmatrix}.$$

Note that by (3.14) and the symmetry of Ξ_{ii} and Γ_{ii}, it is easy to show that $i=2,\ldots,p$, the matrix $\left[\Gamma_{ii}\mathcal{M}_{ii}^{-1}+(\Sigma_{ii}-\Lambda_{ii})^\top\right]\Xi_{ii}$, is symmetric. Since $(\Sigma_{ii}-\Lambda_{ii})^T\Xi_{ii}$ is symmetric, so is the matrix $\Gamma_{ii}\mathcal{M}_{ii}^{-1}\Xi_{ii}$ and, hence, $\mathcal{M}_{ii}\Xi_{ii}^{-1}\Gamma_{ii}$. It follows that for all $j=1,\cdots,\ell_2$, the diagonal block $\Theta_{\ell_1+j,\ell_1+j}\Psi_{\ell_1+j}$ is symmetric, which can be true only if $\vartheta_{\ell_1+j}=0$ and, hence, $\Xi_{ii}^{-1}\Gamma_{ii}$ must be a diagonal matrix

$$\Xi_{ii}^{-1}\Gamma_{ii} = \operatorname{diag}\{\varsigma_1,\ldots,\varsigma_{\ell_1},\varsigma_{\ell_1+1}I_2,\ldots,\varsigma_{\ell_1+\ell_2}I_2\}. \tag{3.15}$$

We draw the following conclusion which is part of the necessary condition.

LEMMA 3.4. *Let Γ be the matrix defined by (2.3). For $i=2,\ldots,p$,*

1. *If $\Gamma_{ii}\neq 0$, then either both $\Gamma_{ii}(\Lambda_{ii}^\top\mathcal{M}_{ii}\Sigma_{ii}-\mathcal{K}_{ii})^{-1}\Lambda_{ii}^\top\mathcal{M}_{ii}$ and $\Gamma_{i,i}\Sigma_{i,i}(\Lambda_{ii}^T\mathcal{M}_{ii}\Sigma_{ii}-\mathcal{K}_{ii})^{-1}\mathcal{K}_{ii}$ are positive definite or both $\Gamma_{ii}(\Lambda_{ii}^\top\mathcal{M}_{ii}\Sigma_{ii}-\mathcal{K}_{ii})^{-1}\Lambda_{ii}^\top\mathcal{M}_{ii}$ and $\Gamma_{i,i}\Sigma_{i,i}(\Lambda_{ii}^T\mathcal{M}_{ii}\Sigma_{ii}-\mathcal{K}_{ii})^{-1}\mathcal{K}_{ii}$ are negative definite.*
2. *If $\Gamma_{ii}=0$, then Λ_{ii} is diagonal and $\Lambda_{ii}\Sigma_{ii}$ is positive definite.*

Proof. The facts that $\mathcal{M}_{ii}$ is block irreducible (by Lemma 3.1), (3.15) holds, and $\mathcal{M}_{ii}\Xi_{ii}^{-1}\Gamma_{ii}$ is symmetric imply that $\Xi_{ii}^{-1}\Gamma_{ii} = \eta_i I$ for some scalar η_i. Thus it is always true that

$$\Gamma_{ii} = \eta_i \Xi_{ii}. \tag{3.16}$$

If $\Gamma_{ii} \neq 0$, then $\eta_i \neq 0$ and by (3.14) we see that $\frac{1}{\eta_i}\Gamma_{ii}(\Lambda_{ii}^\top \mathcal{M}_{ii}\Sigma_{ii} - \mathcal{K}_{ii})^{-1}\Lambda_{ii}^\top \mathcal{M}_{ii}$ is positive definite. If $\Gamma_{ii} = 0$, then by (3.13) we have $\mathcal{K}_{ii} = \Lambda_{ii}^\top \mathcal{M}_{ii}\Lambda_{ii}$. By (3.14) again, we see that $\Xi_{ii}(\Sigma_{ii} - \Lambda_{ii})^{-1}$ is positive definite.

Suppose that Λ_{ii} is not diagonal, then $\Sigma_{i,i} - \Lambda_{ii}$ has at least one nonsingular diagonal block of the form $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$. Correspondingly, Ξ_{ii} has a diagonal block of the form $\begin{bmatrix} \xi & \eta \\ \eta & -\xi \end{bmatrix}$. It follows that $\Xi_{ii}(\Sigma_{ii} - \Lambda_{ii})^{-1}$ has a diagonal block with trace zero, which contradicts with the fact $\Xi_{ii}(\Sigma_{ii} - \Lambda_{ii})^{-1}$ is positive definite.

We have one more item to check, namely, the positive definiteness of $K_0 + \Delta K$. From (3.5), it is clear that $P^\top(K_1 + \Delta K_1)P$ is block diagonal and that its i-th diagonal block is given by $\Sigma_{ii}^\top(\mathcal{M}_{ii} + \mathcal{M}_{ii}\Phi_{ii}\mathcal{M}_{ii})\Sigma_{ii} - \Sigma_{ii}^\top \Xi_{ii}$ which is positive definite. Obviously, by (2.8), $\Sigma_{ii}^\top \Xi_{ii} = \Xi_{i,i}\Sigma_{i,i}$ for all i. Using (3.14), we can write

$$\begin{aligned}
\Sigma_{ii}^\top(\mathcal{M}_{ii} + \mathcal{M}_{ii}\Phi_{ii}\mathcal{M}_{ii})\Sigma_{ii} - \Sigma_{ii}^\top \Xi_{ii} &= \Sigma_{ii}^\top \Xi_{ii}(\Lambda_{ii}^\top \mathcal{M}_{ii}\Sigma_{ii} - \mathcal{K}_{ii})^{-1}\Lambda_{ii}^\top \mathcal{M}_{ii}\Sigma_{ii} - \Sigma_{ii}^\top \Xi_{ii} \\
&= \Xi_{ii}\Sigma_{ii}(\Lambda_{ii}^\top \mathcal{M}_{ii}\Sigma_{ii} - \mathcal{K}_{ii})^{-1}\Lambda_{ii}^\top \mathcal{M}_{ii}\Sigma_{ii} - \Xi_{ii}\Sigma_{ii} \\
&= \Xi_{ii}\Sigma_{ii}(\Lambda_{ii}^\top \mathcal{M}_{ii}\Sigma_{ii} - \mathcal{K}_{ii})^{-1}\mathcal{K}_{ii},
\end{aligned} \tag{3.17}$$

which is positive definite for $i = 2, \cdots, p$. Thus $\Gamma_{ii}\Sigma_{ii}(\Lambda_{ii}^\top \mathcal{M}_{ii}\Sigma_{ii} - \mathcal{K}_{ii})^{-1}\mathcal{K}_{ii}$ is positive definite for $i = 2, \cdots, p$. On the other hand, recall from (3.13) that if $\Gamma_{ii} = 0$, then $\mathcal{K}_{ii} = \Lambda_{ii}^\top \mathcal{M}_{ii}\Lambda_{ii}$. In this case, we can further deduce (3.17) to

$$\Sigma_{ii}^\top(\mathcal{M}_{ii} + \mathcal{M}_{ii}\Phi_{ii}\mathcal{M}_{ii})\Sigma_{ii} - \Sigma_{ii}^\top \Xi_{ii} = \Xi_{ii}\Sigma_{ii}(\Sigma_{ii} - \Lambda_{ii})^{-1}\Lambda_{ii} = \Xi_{ii}(\Sigma_{ii} - \Lambda_{ii})^{-1}\Lambda_{ii}\Sigma_{ii} \tag{3.18}$$

which is positive definite. By the fact that $\Xi_{ii}(\Sigma_{ii} - \Lambda_{ii})^{-1}$ is positive definite, we conclude that $\Lambda_{ii}\Sigma_{ii}$ is positive definite. ∎

We summarize the above necessary conditions in the following theorem. The most interesting development is that the necessary conditions are also sufficient.

THEOREM 3.5. *Suppose that $k = n$ and that $Y = X$. Let Γ be defined by (2.3) and P be defined by Lemma 3.1. Then the MUP is solvable if and only if the matrix Σ is nonsingular and for $i = 2, \cdots, p$, the following conditions among the corresponding blocks defined by P hold:*

1. *If $\Gamma_{ii} \neq 0$, then $\Gamma_{ii}(\Lambda_{ii}^\top \mathcal{M}_{ii}\Sigma_{ii} - \mathcal{K}_{ii})^{-1}\Lambda_{ii}^\top \mathcal{M}_{ii}$ and $\Gamma_{i,i}\Sigma_{i,i}(\Lambda_{ii}^T \mathcal{M}_{ii}\Sigma_{ii} - \mathcal{K}_{ii})^{-1}\mathcal{K}_{ii}$ are either both positive definite or both negative definite.*
2. *If $\Gamma_{ii} = 0$, then Λ_{ii} is diagonal and $\Lambda_{ii}\Sigma_{ii}$ is positive definite.*

Proof. Only the sufficiency needs to be proved. It will be most informative if we prove the sufficiency by constructing the solution $(\Delta M, \Delta C, \Delta K)$.

Clearly, Λ is nonsingular because K_0 is positive definite. We may therefore select a symmetric and positive definite matrix Φ_{11} such that

$$\Delta\mathcal{K}_{11} := \Sigma_{11}^\top(\mathcal{M}_{11} + \mathcal{M}_{11}\Phi_{11}\mathcal{M}_{11})\Sigma_{11} - \mathcal{K}_{11}$$

is positive definite. Set $\Xi_{11} = 0$. For $i = 2, \ldots, p$ and if $\Gamma_{ii} \neq 0$, we can choose by assumption a scalar $\omega_i \in \mathbb{R}$ such that both matrices Φ_{ii} and $\Delta\mathcal{K}_{ii}$ defined by

$$\begin{aligned}
\Phi_{ii} &:= \mathcal{M}_{ii}^{-1}\left[\omega_i\Gamma_{ii}(\Lambda_{ii}^\top \mathcal{M}_{ii}\Sigma_{ii} - \mathcal{K}_{ii})^{-1}\Lambda_{ii}^\top \mathcal{M}_{ii} - \mathcal{M}_{ii}\right]\mathcal{M}_{ii}^{-1}, \\
\Delta\mathcal{K}_{ii} &:= \omega_i\Gamma_{ii}\Sigma_{ii}(\Lambda_{ii}^\top \mathcal{M}_{ii}\Sigma_{ii} - \mathcal{K}_{ii})^{-1}\mathcal{K}_{ii} - \mathcal{K}_{ii},
\end{aligned}$$

are positive definite. Set $\Xi_{ii} = \omega_i \Gamma_{ii}$. Similarly, if $\Gamma_{ii} = 0$, we may choose a diagonal matrix Ξ_{ii} such that both matrices Φ_{ii} and $\Delta\mathcal{K}_{ii}$ defined by

$$\Phi_{ii} := \mathcal{M}_{ii}^{-1}[\Xi_{ii}(\Sigma_{ii} - \Lambda_{ii})^{-1} - \mathcal{M}_{ii}]\mathcal{M}_{ii}^{-1},$$
$$\Delta\mathcal{K}_{ii} := \Xi_{ii}(\Sigma_{ii} - \Lambda_{ii})^{-1}\Lambda_{ii}\Sigma_{ii} - \mathcal{K}_{ii},$$

are positive definite. The matrix

$$\Phi := P\mathrm{diag}\left\{\Phi_{11}, \ldots, \Phi_{pp}\right\} P^\top$$

is positive definite. By construction and the definition (3.13), the equation (3.8) is satisfied. Furthermore, defining

$$\begin{cases} \Delta M := M_0 X \Phi X^\top M_0, \\ \Delta C := X^{-\top}[-X^\top M_0 X \Phi \Lambda^{-T} X^\top K_0 X + (\Lambda^\top - \Lambda^\top - \Lambda^\top X^\top M_0 X \Phi) X^\top M_0 X] X^{-1}, \\ \Delta K := -X^{-\top}[(\Lambda^\top - \Lambda^\top - \Lambda^\top X^\top M_0 X \Phi)\Lambda^{-T} X^\top K_0 X] X^{-1}, \end{cases} \tag{3.19}$$

we find by using (2.5) and (3.8) that the corresponding $(\Delta M_1, \Delta C_1, \Delta K_1)$ defined in (3.19) can be expressed as

$$\begin{cases} \Delta M_1 &= M_1 \Phi M_1, \\ \Delta C_1 &= \Xi - (M_1 + M_1 \Phi M_1)\Sigma - \Sigma^\top (M_1 + M_1 \Phi M_1) - C_1, \\ \Delta K_1 &= \Sigma^\top (M_1 + M_1 \Phi M_1)\Sigma - \Sigma^\top \Xi - K_1, \end{cases}$$

showing that

$$(M_0 + \Delta M)\begin{bmatrix} X & Z \end{bmatrix}\begin{bmatrix} \Sigma & \\ & \Upsilon \end{bmatrix}^2 + (C_0 + \Delta C)\begin{bmatrix} X & Z \end{bmatrix}\begin{bmatrix} \Sigma & \\ & \Upsilon \end{bmatrix} + (K_0 + \Delta K)\begin{bmatrix} X & Z \end{bmatrix} = 0$$

Note that $M_0 + \Delta M = M + M\Phi M$ is positive definite. Since ΔK satisfies

$$P^\top \Delta K_1 P = \mathrm{diag}\left\{\Delta\mathcal{K}_{11}, \ldots, \Delta\mathcal{K}_{pp}\right\}$$

which by construction is positive definite, we conclude that $K + \Delta K$ is also positive definite and that the MUP is solved. □

Recall that our goal in this section is to explore a solution that is not in the parametric form assumed in the preceding section. Note that the "parameter matrix" Φ in the above proof does not enter the solution $(\Delta M, \Delta C, \Delta K)$ in (3.19) in the same way as that in the generic form characterized by (2.1).

4. Conclusions. Updating a real symmetric quadratic model while preserving positive definiteness and no spill-over remains a fundamental challenge in the field. In this paper, we have made some advances toward this challenge. Our main contributions of the present work are twofold:

1. Theorems 2.6 and 2.7 provide necessary and sufficient solvability conditions for the underlying problem when the triplet $(\Delta M, \Delta C, \Delta K)$ assumes the parametric form (2.1 which are known to be generic in the literature.
2. Theorem 3.5 characterizes another necessary and sufficient solvability conditions of the underlying problem for the case $k = n$ and $Y = X$ while not using the parametric form.

It is important to note that the techniques developed in Section 3 give complete answer to the MUP but only under the condition that precisely n eigenvalues are to be updated. Theorem 3.5 therefore includes and generalizes Theorems 2.6 and 2.7 under this special circumstance. The techniques cannot be carried through under other scenarios. Thus, for the case when less than

n eigenpairs are to be updated, the only result we know of is Theorem 2.6 and 2.7 under the assumption that $(\Delta M, \Delta D, \Delta K)$ is in the parametric form (2.1) which, fortunately, is generically true for almost all updated eigenpair (Σ, Y).

Our study represents some new steps toward the understanding of the MUP. Many questions remain open, including the preservation of semi-definiteness or skewness of the damping matrix C and the structured problem.

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