STRUCTURED QUADRATIC INVERSE EIGENVALUE PROBLEM, II. GENERALLY LINKED SYSTEMS

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Abstract. Quadratic pencils arise in many areas of important applications. The underlying physical systems often impose inherent structures, which include the predetermined inner-connectivity among elements within the physical system and the mandatory nonnegativity of physical parameters, on the pencils. In the inverse problem of reconstructing a quadratic pencil from prescribed eigeninformation, respecting the desirable structure becomes important but more challenging both theoretically and practically. The issue of whether a structured inverse eigenvalue problem is solvable or not is problem dependent and has to be addressed structure by structure. In an earlier work, physical systems that can be modeled under the paradigm of a serially linked mass-spring system have been considered via specifically formulated inequality systems. In this paper, the framework is generalized to arbitrary generally linked systems. In particular, given any configuration of inner-connectivity in a mass-spring system, this paper presents a mechanism that systematically and automatically generates a corresponding inequality system. A numerical approach is proposed to determine whether the inverse problem is solvable and, if yes, computes the coefficient matrices while providing an estimate of the residual error. The most important feature of this approach is that it is problem independent, that is, the approach is general and robust for any kind of physical configuration. The ideas discussed in this paper have been implemented into a software package by which some numerical experiments are reported.

Key words. quadratic pencil, inverse eigenvalue problem, connectivity, nonnegativity, linear inequality system, QR decomposition, low rank approximation, quadratic programming, maximin problems.

AMS subject classifications. 65F18, 15A22, 93B55

1. Introduction. Many physical phenomena are modeled by or can be reduced to a second-order ordinary differential system

$$M\ddot{\mathbf{y}} + C\dot{\mathbf{y}} + K\mathbf{y} = f(t), \tag{1.1}$$

where $\mathbf{y}(t) \in \mathbb{R}^d$ varies in time t and M, C, $K \in \mathbb{R}^{d\times d}$ are matrices constant in t. It is well known that if $\mathbf{y}(t) = \mathbf{v}e^{\lambda t}$ represents a fundamental solution of (1.1), then the scalar λ and the vector $\mathbf{v}$ must solve the quadratic eigenvalue problem (QEP)

$$(\lambda^2 M + \lambda C + K)\mathbf{v} = 0. \tag{1.2}$$

Given a physical application such as those arisen from applied mechanics, circuit analysis, electrical oscillation, vibro-acoustics, or finite element models of some PDEs, often the physical parameters are embedded in the coefficient matrices (M, C, K). In a forward setting, the bearing of the dynamical system (1.1) are to be interpreted via the eigenvalues and eigenvectors of the QEP whose coefficient matrices M, C and K have already been determined from the specified physical parameters. In contrast, the quadratic inverse eigenvalue problem (QIEP) intends to validate, determine, or estimate the parameters of the system according to its observed or expected behavior.

In recent years, there have been considerable research interests in the inverse eigenvalue problems. See, for example, the review article [5], the books by Chu and Golub [6], Friswell and Mottershead [14], Gladwell [15], and the extensive references collected therein. Among the various settings for inverse eigenvalue problems, the QIEPs are perhaps the most important in practice and challenging in theory. Many questions remain open for further research [7].

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Generally speaking, a QIEP can be formulated as follows [4]:

(**QIEP**). Construct a nontrivial quadratic pencil $Q(\lambda) = \lambda^2 M + \lambda C + K$ so that its matrix coefficients (M, C, K) are of a specified structure and $Q(\lambda)$ has a specified set $\{(\lambda_i, \mathbf{v}_i)\}_{i=1}^k$ as its eigenpairs.

The key words in the definition of a QIEP are the phrase "of a specified structure". What makes the QIEP hard are the following factors:

- *Often only partial eigeninformation is available.* In vibration industries, engineers have in hand techniques to retrieve eigeninformation. However, for large scale systems, not all eigeninformation is retrievable and, for high complexity systems, not all eigeninformation retrieved is reliable. In the field of model updating, sometimes certain eigenvectors are required to satisfy some specific conditions [2, 28]. In all, only a few measured eigenvectors and eigenvalues are available [1, 11, 14, 13, 18]. It is more sensible and practically implementable to employ only the available, reliable, yet limited eigenpair information in the reconstruction or updating process.
- *Often the coefficient matrices are structured.* Depending on the physical applications, the coefficient matrices (M, C, K) often inherit some common structures such as symmetry or positive definiteness. The most challenging structure, nonetheless, comes from the inner-connectivity of elements in the original physical configuration. The connectivity constraint mandates a certain zero patterns or algebraic relationships among the entries of (M, C, K). What is even more compounding is that these patterns and relationships vary from problem to problem. For feasibility, it is necessary that the QIEP takes this connectivity into account, which is the first main thrust of this paper.
- *Often the parameters to be recovered must be nonnegative.* The physical parameters, such as mass, stiffness, voltage, resistance and so on, are embedded in the coefficient matrices (M, C, K) in a fixed but often mixed way. Merely solving the QIEP subject to the structural constraint is not enough. We must insist that the recovered parameters remain nonnegative for physical realization, which is the second main thrust of this paper.

Research results advanced thus far for the QIEPs cannot address all the issues mentioned above. A partial list of existent works on this subject and their limitations are briefed below. In [23], Ram and Elhay studied the QIEP with symmetric tridiagonal coefficient matrices and two pairs of eigenvalues. The structure involved in the QIEP studied by Starek and Inman [24] is for nonproportional underdamped systems. Lancaster and Prells [21] proposed a way to construct symmetric coefficient matrices (M, C, K) subject to the constraint that C is positive semi-definite, but they required that the complete set of eigenvalues and eigenvectors be given and that the eigenvalues be all simple and nonreal. Later, Lancaster [20] considered the QIEP where only the spectrum is given and semisimple, in which case special eigenvectors are found first to construct symmetric coefficient matrices (M, C, K). In [7], Chu, Kuo and Lin put forward a parametric characterization for the QIEP with partially prescribed eigenvectors where the reconstructed M is guaranteed to be positive definite, C symmetric, and K positive semi-definite. The result was further generalized in [19]. More recently, a representation of symmetric (M, C, K) in terms of eigenvalues and eigenvectors is discussed in [8], which is real-valued version of the well known GLR theory [16] and provides a theoretical basis for the QIEP. See also [9, 10, 22] for the related work on applying the notions of feedback control to reassign the eigenstructure.

We must stress that in all the above-mentioned articles and in most others in the literature, very few have paid attention to the constraints of either *connectivity* or *nonnegativity*. Such a negligence in the literature certainly is not due to its lack of interest, but rather because of the difficulties associated with these constraints. Since the underlying structure is application dependent, it seems that we can tackle the structured QIEP only problem by problem. A general theory would be hard to come by.

Only recently, Chu, Del Buono and Yu [4] considered for the first time the special structure of a linearly linked system where, given at most two eigenpairs, the structured QIEP was solved numerically for symmetric and tridiagonal coefficient matrices that respect both the connectivity and the nonnegativity. Still, it appears very difficult to generalize the mathematics exerted in deriving the special inequality systems in [4] for numerical computation to other types of physical configurations. Our main contribution in this paper thus is to develop a package of software that, based on a user's description of the physical configuration and available eigenpair information, automates the generation of a linear inequality system which is necessary for respecting the connectivity and the nonnegativity. Since the measured eigeninformation is not always precise, we allow some leeway in the accuracy. Our algorithm can check the consistency of the inequality system within a user-supplied tolerance. If the inequality system is solvable within the tolerance, we compute a solution to the structured QIEP and return the residual error for the user's discernment of accepting or rejecting the constructed model.

Because the structure of the QIEP varies according to applications, even merely formulating the problem is a laborious task itself. Our software relieves the burden of problem dependency and offers a universal approach to arbitrary physical systems. We believe our approach is innovative and should be an important contribution to the field. Our package is still in beta test, but since our approach appears to have a broad range of applications, it might be appropriate to communicate two goals via this presentation: first, we explain our ideas of developing a general-purpose problem-independent formulation for the structured QIEP; and secondly, we provide the mathematical basis of our error control strategy.

This paper is organized as follows. In Section 2 we outline the motivation behind deriving a linear inequality system. The inequality system, arising from rank deficiency and varying from structure to structure, is a necessary condition for a nonnegative solution to a QIEP with prescribed connectivity. By this inequality system, we transform a structured QIEP into a maximin problem for the sake of efficiency in numerical computation. If the prescribed eigeninformation is inexact, which often is the case in practice, the expected rank deficiency might not occur and the corresponding inequality system would have only a trivial solution. We propose a strategy of truncation in Section 3 that generates an approximate but consistent subsystem. We offer a posterior estimate on the resulting residual error which could be used to assess the quality of the reconstructed model. Before the linear inequality system can be induced, we have to first convert a given configuration of connectivity into a linear equality system. A general idea about this formation, which depends on the physical law that governs how the elements interact, is discussed in Section 4. We focus on the mass-spring system governed by Hooke's law for the sake of exposition, but our setting is general enough to include many other physical systems. The point to make is that we are able to handle models with arbitrary generally-linked connectivity, effectively generalizing the notion proposed in [4]. Finally, equipped with our software, we are able to explore some interesting perturbation results numerically in Section 5.

2. Motivation. Since we are interested only in real matrices, it is natural to expect that the prescribed eigenpairs $\{(\lambda_i, \mathbf{v}_i)\}_{i=1}^k$ are closed under complex-conjugation. Without loss of generality, we shall denote the prescribed eigenpairs in the matrix form (Λ, X), where $\Lambda \in \mathbb{R}^{k\times k}$ is block diagonal with at most 2×2 blocks along the diagonal wherever a complex-conjugate pair of eigenvalues appears in the prescribed spectrum, and $X \in \mathbb{R}^{n\times k}$ represents the "eigenvector matrix" in the sense that each pair of column vectors associated with a 2×2 block in Λ retains the real and the imaginary parts, respectively, of the original complex eigenvector. In this way, we may identify the given eigenpairs (Λ, X) as an element in $\mathbb{R}^{k\times k} \times \mathbb{R}^{d\times k}$. The QIEP therefore amounts to solving the algebraic equation

$$MX\Lambda^2 + CX\Lambda + KX = 0, \tag{2.1}$$

for the unknowns (M, C, K).

Specifically speaking, the coefficient matrices that we intend to reconstruct are of the form $(M(\mathbf{m}), C(\mathbf{c}), K(\mathbf{k}))$, where $(\mathbf{m}, \mathbf{c}, \mathbf{k})$ are physical parameters which must be nonnegative. In a mass-

spring system, for example, $(\mathbf{m}, \mathbf{c}, \mathbf{k})$ would stand for the mass, damping, and stiffness, respectively. The functions of $(M(\mathbf{m}), C(\mathbf{c}), K(\mathbf{k}))$ are characterized by the inner connectivity among elements in the system and, in this way, constitute a structural constraint on how the QIEP should be solved. This constraint, appearing in specific zero patterns or algebraic relationships, defines a much refined texture of the coefficient matrices than the general requirements such as merely symmetry and positive semi-definiteness. On the other hand, the nonnegativity constraint due to the physical nature of the relevant parameters imposes another challenge to the structured QIEPs.

Assuming that the connectivity information has been taken into account to form the structure of coefficient matrices (M, C, K), whose mechanism will be explained subsequently, we first outline the idea of linear inequality systems which must be satisfied in order to meet the nonnegativity constraint. This necessary condition is fundamental to our algorithm.

Typically, the functions $(M(\mathbf{m}), C(\mathbf{c}), K(\mathbf{k}))$ are linear, following from a certain physical law. We thus can rewrite (2.1) as a linear equality system

$$f(\mathbf{m}, \mathbf{c}, \mathbf{k}) = 0 \tag{2.2}$$

in terms of the unknown parameters $(\mathbf{m}, \mathbf{c}, \mathbf{k})$. The structured QIEP is thereby transformed into finding the nonnegative solutions for the linear equality system (2.2). More specifically, denote $\mathbf{x} := (x_1, \ldots, x_n)^\top = (\mathbf{m}, \mathbf{c}, \mathbf{k})^\top$, where n is the total number of physical parameters. Since each of the k prescribed eigenpair gives rise to d equations, we need to find a nonnegative solution for the homogeneous system

$$f(\mathbf{m}, \mathbf{c}, \mathbf{k}) =: A\mathbf{x} = 0, \tag{2.3}$$

where A is a $dk \times n$ matrix whose construction will be characterized afterward.

Suppose the prescribed eigenpairs (Λ, X) are exact for some quadratic pencils. Then the system (2.3) must have a nontrivial solution $\mathbf{x} = (\mathbf{m}, \mathbf{c}, \mathbf{k})^\top$, implying that the matrix A must be rank deficient. Let s denote the rank of the matrix A and write the *compact* QR decomposition of the matrix A as

$$A = QRP, \tag{2.4}$$

where $Q \in \mathbb{R}^{dk \times s}$ satisfies $Q^\top Q = I_s$ and $P \in \mathbb{R}^{n \times n}$ is a permutation matrix such that $R \in \mathbb{R}^{s \times n}$ is an upper "trapezoidal" matrix with non-increasing absolute values along its diagonal. Partition the matrix R further as

$$R = [S, T], \tag{2.5}$$

where

$$S := \begin{pmatrix} r_{11} & r_{12} & \cdots & r_{1s} \\ & r_{22} & \cdots & r_{2s} \\ & & \ddots & \vdots \\ & & & r_{ss} \end{pmatrix}$$

is invertible by assumption. Define

$$\overline{\mathbf{x}} = (\overline{x}_1, \ldots, \overline{x}_n)^\top := P\mathbf{x}.$$

Trivially we see the following equivalence:

$$A\mathbf{x} = 0 \Longleftrightarrow R\overline{\mathbf{x}} = [S, T] \begin{pmatrix} \overline{x}_1 \\ \vdots \\ \overline{x}_s \\ \overline{x}_{s+1} \\ \vdots \\ \overline{x}_n \end{pmatrix} = 0$$

$$\Longleftrightarrow \begin{pmatrix} \overline{x}_1 \\ \vdots \\ \overline{x}_s \end{pmatrix} = \begin{pmatrix} g_1(\overline{x}_{s+1}, \ldots, \overline{x}_n) \\ \vdots \\ g_s(\overline{x}_{s+1}, \ldots, \overline{x}_n) \end{pmatrix} := -S^{-1}T \begin{pmatrix} \overline{x}_{s+1} \\ \vdots \\ \overline{x}_n \end{pmatrix}. \tag{2.6}$$

Since all the parameters are nonnegative, we have recast the structured QIEP as an inequality system.

THEOREM 2.1. *Given eigenpairs (Λ, X), define matrices S and T according to (2.5) from the corresponding matrix A in (2.3) which has a fixed structure. Then finding the nonnegative parameters $(x_1, \ldots, x_n)$ for the structured QIEP is equivalent to finding nonnegative solutions $(\overline{x}_{s+1}, \ldots, \overline{x}_n)$ to the inequality system*

$$S^{-1}T \begin{pmatrix} \overline{x}_{s+1} \\ \vdots \\ \overline{x}_n \end{pmatrix} \leq 0. \tag{2.7}$$

Solving inequality system of convex functions is a classical problem. Extensive discussions can be found in the literature, including the well known Farkas Lemma [3], the von Neumann theorem [27], and the Ky Fan theorem [12]. Many software packages are readily available. For our application, however, our favorite is the maximin formulation which we find particularly easy and effective to use.

Observe that, by the definition in (2.6), the functions $g_i(\overline{x}_{s+1}, \ldots, \overline{x}_n), 1 \leq i \leq s$, are homogeneous. Any solution, if exists, can be scaled by positive constants. For convenience, we restrict the solutions $\overline{x}_{s+1}, \ldots, \overline{x}_n$ to the interval $[0, 1]$. Certainly, the trivial solution $\overline{x}_{s+1} = \ldots = \overline{x}_n = 0$ is of no interest. If the minimum of the evaluations of $\{g_1, \ldots, g_s\}$ is nonnegative at a certain nontrivial feasible point $\overline{x}^*_{s+1}, \ldots, \overline{x}^*_n$, then obviously entries of the vector

$$P^\top (g_1^*, \ldots, g_s^*, \overline{x}^*_{s+1}, \ldots, \overline{x}^*_n)^\top$$

give rise to the parameters $(\mathbf{m}, \mathbf{c}, \mathbf{k})^\top$ for the physical system. Conversely, if the maximum of $\min\{g_1, \ldots, g_s\}$ over the entire feasible set is negative, then the inequality system (2.7) has no nonnegative solution and, thus, the structured QIEP is not solvable. By this reasoning, we have transformed a structured QIEP to a maximin problem:

$$\max_{0 \leq \overline{x}_{s+1}, \ldots, \overline{x}_n \leq 1} \min\{g_1, \ldots, g_s\}, \tag{2.8}$$

and the solvability is determined by a nonnegative objective value in (2.8).

As each member g_i, $i = 1, \ldots, s$, is linear in $\overline{x}_{s+1}, \ldots, \overline{x}_n$, the MIN function $\min\{g_1, \ldots, g_s\}$ is concave. The feasible set $\{(\overline{x}_{s+1}, \ldots, \overline{x}_n) \in \mathbb{R}^{n-s} | 0 \leq \overline{x}_i \leq 1,\ i = s+1, \ldots, n\}$ is convex. Therefore, we generally expect that the solution to (2.8) is unique and global. There are readily available routines to solve (2.8). For example, the MATLAB routine FMINIMAX implements a sequential quadratic programming method and seems capable of handling our problem reasonably well.

Take cautions in that the rank deficiency of the matrix A is due to our (unrealistic) assumption that the prescribed eigenpairs are exact for some (unknown) quadratic pencils. Without the rank deficiency,

we might not even be able to formulate the inequality systems. One of our main contributions in this paper is to take the inexactness into consideration when forming and truncating an inequality system. We are able to analyze the robustness of the resulting solution. We detail our approach in the next section.

3. Handling inexact eigenpairs. Due to the imperfection of available information gathering devices, it is literally impossible to measure all eigenpair information in a large and complicated physical system. In practice, even those obtainable eigenvalues and eigenvectors would often carry noises. It is more sensible to consider the solvability of the QIEP under the context that the prescribed eigenpairs (Λ, X) are partial and inexact.

Because of the noises in Λ and X, the matrix A formed in (2.3) may be of full rank. Even if it is rank deficient, the cutoff line in determining its numerical rank may be obscure. Without preprocessing A, the solutions to $A\mathbf{x} = 0$ would either be zero or far off the target. For engineering applications, the fact of the matter is that an approximate solution that makes more physical sense, such as satisfying the nonnegative constraint, is sometimes acceptable. We propose two methods for finding approximate nonnegative solutions to the system $A\mathbf{x} = 0$.

3.1. QP approach. Perhaps the most straightforward formulation for finding an approximate solution to the system (2.2) is to consider the quadratic programming (QP) problem:

$$\begin{aligned} \text{minimize} \quad & \|f(\mathbf{m}, \mathbf{c}, \mathbf{k})\|_2^2 \\ \text{subject to} \quad & \mathbf{m}, \mathbf{c}, \mathbf{k} \geq 1. \end{aligned} \tag{3.1}$$

Ideally, the structured QIEP has a positive solution if and only if the objective value in (3.1) is zero at a positive minimizer. The homogeneity of the objective function implies that any positive scaling of this positive minimizer remains to be a minimizer. To avoid the infinitesimal scaling, we require $(\mathbf{m}, \mathbf{c}, \mathbf{k}) \geq 1$ instead of $(\mathbf{m}, \mathbf{c}, \mathbf{k}) \geq 0$ in the constraint of (3.1). We caution that the presence of any zero physical parameters, which should have been ruled out in the first place, will cause the scaling to fail. This case, fortunately, is not generic. We caution again that the presence of noises in the measured eigeninformation (Λ, X) often causes the matrix A to be of full rank, resulting only the trivial solution for the system (2.3), which is of little interest in practice and is precisely the issue we want to deal with.

Our rumination about the QP approach is that since the function f itself is not exact, it makes no sense to find its roots exactly. We propose to approximately solve (2.2) via minimizing the objective function in (3.1) by tuning some properly preselected stopping criteria. In this way, when the optimization process is terminated after meeting the preset criteria, we know for sure that the computed parameters $(\mathbf{m}^*, \mathbf{c}^*, \mathbf{k}^*)$ are positive and return a residual of magnitude $\|f(\mathbf{m}^*, \mathbf{c}^*, \mathbf{k}^*)\|_2$. If the residual is small, then we may think *heuristically* that we have approximately constructed the quadratic pencil. If magnitudes of elements among the positive solution $(\mathbf{m}^*, \mathbf{c}^*, \mathbf{k}^*)$ are scattered over a relatively large scale, then it might be an indication that the smallest elements are (numerically) zero. It must be stressed, however, that a small residual does not necessarily imply that $(\mathbf{m}^*, \mathbf{c}^*, \mathbf{k}^*)$ is close to the exact solution of $f(\mathbf{m}, \mathbf{c}, \mathbf{k}) = 0$. On the other hand, since we are only interested in constructing an approximate model with not so perfectly accurate eigeninformation, sometimes it is sufficient (or is the only choice) for engineering applications to use the residual as a pertaining merit function.

Finally, we point out that the optimization (3.1) is a convex programming problem. Any local solution is necessarily a global solution. Solving the above problem as is given, nonetheless, may become expensive when many parameters are involved. The next approach exploits the possibility of dimension reduction.

3.2. Maximin approach. Were the prescribed eigenpairs (Λ, X) exact for some quadratic pencils, then the resulting linear system (2.2) would be consistent for some nonnegative $(\mathbf{m}, \mathbf{c}, \mathbf{k})$ as we

have seen in Section 2. The task of determining whether the given (Λ, X) is feasible before a model is constructed is an interesting question of theoretical importance in its own right, but is beyond the scope of this paper. Instead, inspired by the discussion in Section 2, we offer two numerical methods in this section that can *approximately* solve the system in (2.2) without knowing a prior whether the given (Λ, X) is feasible. We shall explain the meaning of an approximate solution in the sequel. Our main point is that, even if the given (Λ, X) is inexact, we can offer a posterior estimate for the residual of the constructed model which can be used at discretion to accept or reject the constructed model.

To begin with, we establish the matrix $A \in \mathbb{R}^{dk\times n}$ from the given k eigenpairs (whose construction can be automated as we shall see in Section 4.2). Perform the *full* QR decomposition with column pivoting of the matrix A to obtain the following relationship

$$A\mathbf{x} = 0 \Longleftrightarrow R\overline{\mathbf{x}} = 0,$$

where $R \in \mathbb{R}^{dk\times n}$ is an upper trapezoidal matrix whose diagonal elements are arranged in accordance with non-increasing absolute values by the permutation matrix P and $\overline{\mathbf{x}} := P\mathbf{x}$. In general, A is of full rank and thus $\mathbf{x} = 0$ is the only solution, which is of little value. To obtain some meaningful nontrivial solutions, we must replace R by some low rank approximations. There are two ways to determine the low rank.

3.2.1. Static rank. Let ϵ be a predetermined threshold. Define R_ϵ to be the submatrix of R consisting of rows of R whose diagonal elements are greater than ϵ. Suppose R_ϵ is made of t rows. We consider a nonnegative solution to $R_\epsilon\overline{\mathbf{x}} = 0$, if exists, as an approximate solution to $R\overline{\mathbf{x}} = 0$. Obviously, the threshold ϵ must be such that $t < n$ since, otherwise, $R_\epsilon\overline{\mathbf{x}} = 0$ still has only trivial solution. The choice of ϵ is often on a trial and error basis. The value of ϵ should be large enough to bring forth substantial truncation so that $t < n$, but also should be small enough to avoid losing too much information about A.

Assuming $t < n$, partition $R_\epsilon = [S_\epsilon, T_\epsilon]$, where $S_\epsilon \in \mathbb{R}^{t\times t}$ is invertible. From the equivalence

$$R_\epsilon\overline{\mathbf{x}} = 0 \Longleftrightarrow \begin{pmatrix} \overline{x}_1 \\ \vdots \\ \overline{x}_t \end{pmatrix} = \begin{pmatrix} g_1(\overline{x}_{t+1}, \ldots, \overline{x}_n) \\ \vdots \\ g_t(\overline{x}_{t+1}, \ldots, \overline{x}_n) \end{pmatrix} := -S_\epsilon^{-1}T_\epsilon \begin{pmatrix} \overline{x}_{t+1} \\ \vdots \\ \overline{x}_n \end{pmatrix},$$

we intend to solve the inequality system

$$S_\epsilon^{-1}T_\epsilon \begin{pmatrix} \overline{x}_{t+1} \\ \vdots \\ \overline{x}_n \end{pmatrix} \le 0, \tag{3.2}$$

for some $(\overline{x}_{t+1}, \ldots, \overline{x}_n)^\top \ge 0$. Note that the system (3.2) involves only $n-t$ variables in t inequalities, which could mean a sizable dimension reduction from the original system.

Similar to the ideas discussed in Section 2, we recast the problem in (3.2) as a maximin problem:

$$\max_{0\le\overline{x}_{t+1},\ldots,\overline{x}_n\le 1} \min\{g_1, \ldots, g_t\}, \tag{3.3}$$

The consistency of the inequality system (3.2) is equivalent to the nonnegativity of the objective value of the maximin problem (3.3). If a feasible solution exists, the vector $P^\top(g_1^*, \ldots, g_t^*, \overline{x}_{t+1}^*, \ldots, \overline{x}_n^*)^\top$ is considered as the physical parameters $(\mathbf{m}^*, \mathbf{c}^*, \mathbf{k}^*)^\top$. The effect of ϵ on the residual $\|M^*X\Lambda^2 + C^*X\Lambda + K^*X\|_F$, where (M^*, C^*, K^*) are the structured coefficient matrices associated with $(\mathbf{m}^*, \mathbf{c}^*, \mathbf{k}^*)^\top$, will be analyzed later.

In most cases, we have no idea about how to truncate the matrix R. Too much truncation means less precise solution. Too little truncation means no feasible solution. The idea of dynamic rank in the next section offers an automatic, self-examining truncation strategy.

3.2.2. Dynamic rank. Suppose that the matrix R has numerical rank s. For $1 \le i \le s$, let $R^{[i]}$ denote the submatrix of the *first* i rows of R. Partition $R^{[i]} = [S_i, T_i]$ where, by construction, $S_i \in \mathbb{R}^{i\times i}$ is invertible. For each i, consider the optimization problem,

$$\max_{0\le \overline{x}_{i+1},\ldots,\overline{x}_n \le 1} \min\{g_1^{[i]}, \ldots, g_i^{[i]}\}, \tag{3.4}$$

where

$$\begin{pmatrix} \overline{x}_1 \\ \vdots \\ \overline{x}_i \end{pmatrix} = \begin{pmatrix} g_1^{[i]}(\overline{x}_{i+1}, \ldots, \overline{x}_n) \\ \vdots \\ g_i^{[i]}(\overline{x}_{i+1}, \ldots, \overline{x}_n) \end{pmatrix} := -S_i^{-1} T_i \begin{pmatrix} \overline{x}_{i+1} \\ \vdots \\ \overline{x}_n \end{pmatrix}.$$

Starting with $i = s$ and gradually decreasing to $i = 1$, we solve (3.4) successively. The process is terminated at the first (largest) i for which we obtain a nonnegative objective value. Our motivation is to keep the integrity of R as much as possible while seeking consistency. By taking advantage of the fact that our maximin problem returns a global solution, we throw away a portion of R only when it absolutely cannot compromise with the consistency.

The dynamic rank approach differs from the static rank approach in that it avoids truncating the matrix R too late. Suppose the dynamic rank approach stops at i, then for any $\epsilon < r_{ii}$ the system (3.2) inconsistent. On the other hand, with $\epsilon = r_{ii}$, we have $R_\epsilon = R^{[i]}$. Obviously, for any $\epsilon > r_{ii}$, the "much", if not overly, truncated system $R_\epsilon \overline{\mathbf{x}} = 0$ still has a nonnegative solution, except that it results in a larger residual as we shall see below.

3.2.3. Posterior error estimate. In both static and dynamic rank approaches mentioned above, the essence is to approximate the original system $A\mathbf{x} = 0$ constructed from a prescribed eigenpair (Λ, X) by a lower rank matrix in exchange for nonnegative solutions. As our low rank approximation is obtained directly from the QR decomposition of A, the classical perturbation analysis about the QR decomposition such as those in [25, 26] are not immediately relevant to our error estimate. What is really needed is to understand the effect of the truncation of R on the solution $\mathbf{x}$, toward which we analyze below.

Recall that in both approaches, we obtain a nonnegative solution $\overline{\mathbf{x}}_\epsilon^*$ to the truncated linear system $R_\epsilon \overline{\mathbf{x}} = 0$ for a certain threshold ϵ. We consider $\mathbf{x}_\epsilon^* := P^\top \overline{\mathbf{x}}_\epsilon^*$ as an approximate nonnegative solution to the linear system $A\mathbf{x} = 0$ in (2.3). The following theorem quantifies the residual $\|A\mathbf{x}_\epsilon^*\|_2$ in terms of ϵ.

THEOREM 3.1. *Given a matrix $A \in \mathbb{R}^{dk\times n}$, let $A = QRP$ denote its full QR decomposition, where R is an upper trapezoidal matrix with its diagonal elements arranged in descending absolute value by the permutation matrix P. Given a positive number ϵ, let R_ϵ denote the truncated submatrix of R consisting of rows of R whose diagonal elements are greater than ϵ. Suppose $\overline{\mathbf{x}}_\epsilon$ is a solution to the system $R_\epsilon \overline{\mathbf{x}} = 0$ (For our application, we are interested in a nonnegative solution $\overline{\mathbf{x}}_\epsilon^*$). Define $\mathbf{x}_\epsilon := P^\top \overline{\mathbf{x}}_\epsilon$. Then*

$$\frac{\|A\mathbf{x}_\epsilon\|_2}{\|\mathbf{x}_\epsilon\|_2} = \mathcal{O}(\epsilon)$$

Proof. For any integer $1 \le j \le dk$, let $R_{[j]}$ denote the lower right submatrix of R by deleting its first $j-1$ rows and columns. Recall that the sorted QR decomposition has the property that $|r_{jj}|$ is greater than or equals to the 2-norm of any columns in $R_{[j]}$. Consequently, it is true that

$$\|R_{[j]}\|_2 \le \sqrt{n-j+1}\|R_{[j]}\|_1 \le \sqrt{n-j+1}\sqrt{dk-j+1}|r_{jj}|.$$

Since $\overline{\mathbf{x}}_\epsilon$ is a solution of $R_\epsilon \overline{\mathbf{x}} = 0$, we have

$$\begin{aligned}\|A\mathbf{x}_\epsilon\|_2 = \|R\overline{\mathbf{x}}_\epsilon\|_2 &= \|R\overline{\mathbf{x}}_\epsilon - R_\epsilon\overline{\mathbf{x}}_\epsilon + R_\epsilon\overline{\mathbf{x}}_\epsilon\|_2 \\ &\le \|R - R_\epsilon\|_2 \cdot \|\overline{\mathbf{x}}_\epsilon\|_2 + \|R_\epsilon\overline{\mathbf{x}}_\epsilon\|_2 \\ &< \sqrt{n-i}\sqrt{dk-i}\|\mathbf{x}_\epsilon\|_2\epsilon.\end{aligned}$$

The last inequality follows from the fact that, for a given ϵ, the only possible nonzero elements in $R - R_\epsilon$ would be those in $R_{[i+1]}$ where i is the smallest integer such that $|r_{i+1,i+1}| < \epsilon$ and the fact that $\|\overline{\mathbf{x}}_\epsilon\|_2 = \|\mathbf{x}_\epsilon\|_2$. □

A few comments are worth mentioning in the applications of Theorem 3.1 to the structured QIEPs. First, our algorithm checks to determine R_ϵ so that $R_\epsilon\overline{\mathbf{x}} = 0$ has a nonnegative solution. In order to achieve consistency, the rank of R_ϵ might need to be fairly low, that is, the value of ϵ might be relatively large. In fact, it is possible that the consistency cannot be satisfied for any ϵ. However, once such a nonnegative solution is found to exist, then the ratio of norms of the residual vector relative to the nonnegative solution is independent of scaling. The scaling used in either the QP approach or the maximin approach is only for computational convenience. It has no effect on the relative error. Secondly, if (M^*, C^*, K^*) stands for the reconstructed pencil from the nonnegative approximate parameters $\mathbf{x}^* = (\mathbf{m}^*, \mathbf{c}^*, \mathbf{k}^*)^\top$ after taking the connectivity constraint into account, then with respect to the prescribed eigenpair (Λ, X) we have the estimate $\|M^*X\Lambda^2 + C^*X\Lambda + K^*X\|_F = \|A\mathbf{x}^*\|_2$, which is of order $\mathcal{O}(\epsilon)$ relative to $\|\mathbf{x}^*\|_2$. Users can determine from this estimate whether the reconstruction is acceptable or not.

4. Algorithm for QIEP. Before we give further details about its implementation, we highlight the design of a numerical way for solving a structured QIEP in the following prototype algorithm:

ALGORITHM 4.1. (QIEP Algorithm)

Input values:

- *The desirable connectivity configuration.*
- *Partial eigeninformation in the form of eigenpairs closed under complex conjugation.*

Algorithm description:

- *Step 1. Identify the inherent structure of the coefficient matrices M, C, K.*
- *Step 2. Prepare $X\Lambda^2$ and $X\Lambda$ for forming A.*
- *Step 3. Transform the equation (2.1) into $A\mathbf{x} = 0$ with variable $\mathbf{x} = (\mathbf{m}, \mathbf{c}, \mathbf{k})^\top$ of size $n = \dim(\mathbf{m}) + \dim(\mathbf{c}) + \dim(\mathbf{k})$.*
- *Step 4. Apply either the QP or the maximin techniques to search for an approximate nonnegative solution.*

Output:

- *If the QP approach has a feasible solution, or the maximin approach returns a nonnegative objective value, the structured QIEP is deemed solvable within a specified tolerance on its residual.*
- *Otherwise, the structured QIEP has no solution.*

We point out that Algorithm 4.1 is general and robust in that there is no restriction on how much eigeninformation should be given and that, within a prescribed tolerance, the numerical approach has both high "specificity" in determining that a QIEP is not solvable and high "sensitivity" in predicting a problem is solvable. Furthermore, if the transformation required by Step 3 can be carried out systematically and automatically, as we shall see in the next section, then the algorithm can handle almost any kind of physical configurations. A general-purpose computer software can be developed.

4.1. Identify inherent structure. We have already described some basic ideas of finding a nonnegative solution for an approximate system of $A\mathbf{x} = 0$. It remains to discuss how the matrix A can be generated from a prescribed connectivity configuration. Our thrust in this section is to present a scheme that can automatically identify the structure of the coefficient matrices (M, C, K)

according to the underlying physical principles, e.g., Hooke's law, Ohm's law, Kirchhoff's laws, and so on. The length restriction prevents us from giving a comprehensive presentation over the entire scope of applications, but the following discussion can be modified if the underlying physical principle is changed.

Following [4], our focus at the moment is on any physical systems that can be modeled via the vibration of a mass-spring system. Assuming that the restoring force follows Hooke's law and that the damping is negatively proportional to the velocity, we demonstrate the construction of the coefficient matrices (M, C, K) which then will be converted to the matrix A. In the literature, a typical way of characterizing the structure of a general vibrating system is that the mass matrix M is diagonal, both the damping matrix C and the stiffness matrix K are symmetric, M is positive definite, and K is positive semi-definite. Such a characterization is correct in general but certainly not enough in its specifics. The matter of the fact is that M, C, K are further structured, which are related to the internal connectivity of the masses, dampers, and springs. A different configuration of the connectivity leads to a different matrix structure. That said, we can explore *any* specific structure according to the following rules which are well developed in the field of structural mechanics [17].

THEOREM 4.1. *For a mass-spring system with d degrees of freedom where the motion is limited to one dimension, let* $\mathbf{m}$, $\mathbf{c}$, *and* $\mathbf{k}$ *denote the vectors of masses, damping and stiffness coefficients, respectively. The matrices (M, C, K) inherit the following structure:*

1. *The mass matrix M is a diagonal matrix with masses $(m_1, \ldots, m_d)$ along its diagonal.*
2. *The damping matrix C is symmetric and positive semi-definite. If there are ℓ dampers, identified by $i_1, \ldots, i_\ell$, between the p-th mass and the q-th mass, then the entries C_{pq} and C_{qp} of the damping matrix are $-\sum_{s=1}^{\ell} c_{i_s}$, where c_{i_s} is the damping coefficient of damper i_s. Otherwise, $C_{pq} = C_{qp} = 0$. If the p-th mass is connected to the dampers $j_1, \ldots, j_\ell$, then $C_{pp} = \sum_{s=1}^{\ell} c_{j_s}$.*
3. *The stiffness matrix K is symmetric and positive semi-definite. If there are ℓ springs, identified by $i_1, \ldots, i_\ell$, between the p-th mass and the q-th mass, then the entries K_{pq} and K_{qp} of the stiffness matrix are $-\sum_{s=1}^{\ell} k_{i_s}$, where k_{i_s} is the stiffness coefficient of spring i_s. Otherwise $K_{pq} = K_{qp} = 0$. If the p-th mass is connected to the springs $j_1, \ldots, j_\ell$, then $K_{pp} = \sum_{s=1}^{\ell} k_{j_s}$.*

Example 1. Consider a four-degree-of-freedom vibrating system whose masses, dampers, and springs are connected as in Figure 4.1. The corresponding coefficient matrices (M, C, K) for the

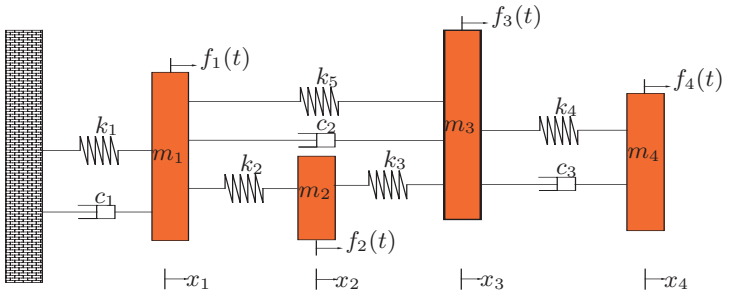

FIG. 4.1. *A four-degree-of-freedom mass-spring system*

dynamical system (1.1) should be structured as follows:

$$M = \begin{pmatrix} m_1 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \\ 0 & 0 & m_3 & 0 \\ 0 & 0 & 0 & m_4 \end{pmatrix}, C = \begin{pmatrix} c_1 + c_2 & 0 & -c_2 & 0 \\ 0 & 0 & 0 & 0 \\ -c_2 & 0 & c_2 + c_3 & -c_3 \\ 0 & 0 & -c_3 & c_3 \end{pmatrix}$$
$$K = \begin{pmatrix} k_1 + k_2 + k_5 & -k_2 & -k_5 & 0 \\ -k_2 & k_2 + k_3 & -k_3 & 0 \\ -k_5 & -k_3 & k_3 + k_4 + k_5 & -k_4 \\ 0 & 0 & -k_4 & k_4 \end{pmatrix} \tag{4.1}$$

It is readily provable that these matrices are symmetric and positive semi-definite. However, be aware of the extra texture of zero patterns and implicit algebraic relationships among rows in these matrices. Apparently, solving a QIEP merely for symmetric and positive definite matrices (M, C, K) is not enough. If the QIEP is to be solved for this specific mass-spring system sketched in Figure 4.1, then the (M, C, K) must respect the structure in (4.1). This is the kind of connectivity constraint we are referring to throughout our discussion.

If the underlying physical principle is changed, then the corresponding structure in (M, C, K) needs to be changed accordingly. Taking the circuit theory as an example, despite the known analogy between an RLC circuit and a mass-spring system, the resulting structures in (M, C, K) could be significantly different as we demonstrate in the next example.

Example 2. Consider a resonant circuit consisting of three inductors, four resistors, and three capacitors, connected in the way depicted in Figure 4.2. After applying Ohm's law and Kirchhoff's

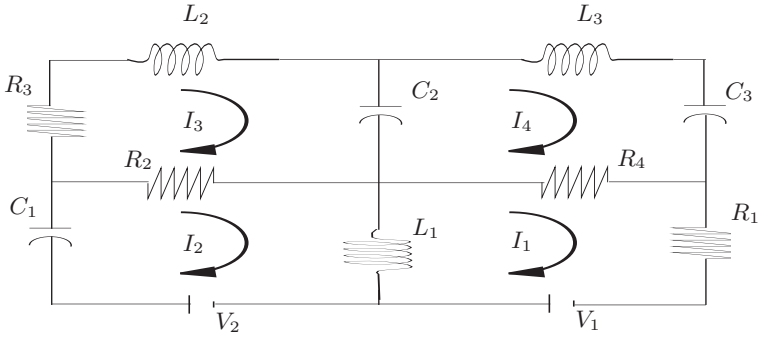

FIG. 4.2. *An RLC electronic network*

law to the circuit, we obtain the following coefficient matrices

$$M = \begin{pmatrix} -L_1 & L_1 & 0 & 0 \\ L_1 & -L_1 & 0 & 0 \\ 0 & 0 & L_2 & 0 \\ 0 & 0 & 0 & L_3 \end{pmatrix}, C = \begin{pmatrix} 0 & R_2 & -R_2 & 0 \\ R_1 + R_4 & 0 & 0 & -R_4 \\ 0 & -R_2 & R_2 + R_3 & 0 \\ -R_4 & 0 & 0 & R_4 \end{pmatrix},$$

$$K = \begin{pmatrix} 0 & \frac{1}{C_1} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{C_2} & -\frac{1}{C_2} \\ 0 & 0 & -\frac{1}{C_2} & \frac{1}{C_2} + \frac{1}{C_3} \end{pmatrix}$$

for the governing equation of the current in the circuit. In contrast to the previous structure associated with the mass-spring system, the structure associated with a general electric circuit may not be definite or even symmetric.

We need to establish a library that collects the different rules of constructing (M, C, K) such as that in Theorem 4.1 according to the different physical laws. Such a task will not be endeavored in this paper, but are gradually added to our software package module by module. Once the user classifies which physical law is to be used in the model, the library is able to automatically identify the corresponding structure for (M, C, K) based on the specified configuration of connectivity. Serving as the lowest level of input mechanism, we do allow users to enter their structure manually into our software package through a text interface, once the matrix structure is known by some other means. By resorting to the classical notion of identifying parameters $m_i = L_i, c_i = R_i, k_i = \frac{1}{C_i}$ between a mass-spring system and an RLC system, for example, Figure 4.3 is the text interface for the circuit in Figure 4.2.

FIG. 4.3. *Text interface for RLC electronic network*

Thus far, we are using MATLAB as the main engine in our computation. The above ASCII file which can be edited and saved as a detached text file is somewhat fashioned for MATLAB symbolic and numeric computation. Numbers in the first row represent respectively the degree of freedom of the system, the number of "masses", the number of "dampers", and the number of "springs". For easy identification, the symbols in the text file must be "m", "c" and "k" respectively. We find it convenient to always attach a "+" sign in front of every positive expression and allow only one space between two consecutive entries. There should be no space at the end of every line. (This minuscule abridgement of users' guide is for the MATLAB environment and are not critically important in the general ideas presented in this paper.) Once the file is uploaded, our algorithm can generate matrix A automatically from the prescribed eigenpair (Λ, X).

4.2. Construct matrix A. From the connectivity configuration and the underlying physical law, we should know precisely how each element in coefficient matrices (M, C, K) is related linearly to the physical parameters $(\mathbf{m}, \mathbf{c}, \mathbf{k})$. It remains to rewrite the linear equation (2.2) in terms of $(\mathbf{m}, \mathbf{c}, \mathbf{k})$, that is, a system of the form $A\mathbf{x} = 0$. It seems that the equivalence between (2.2) and (2.3) is mathematically trivial since we only need to collect like terms together. Our point, however, is to avoid the tedious task of bookkeeping case by case, especially when the involvement is complicated. We prefer to have an automated way of generating A from any specified structure of (M, C, K) and eigenpairs (Λ, X).

Partition $A\mathbf{x} = 0$ as

$$(A_{\mathbf{m}}, A_{\mathbf{c}}, A_{\mathbf{k}}) \begin{pmatrix} \mathbf{m} \\ \mathbf{c} \\ \mathbf{k} \end{pmatrix} = 0.$$

It suffices to demonstrate how the first term $MX\Lambda^2$ in (2.1) can be converted into the form $A_{\mathbf{m}}\mathbf{m}$ from a given structure of M. The matrix $A_{\mathbf{m}}$ is constructed in the order that the block of its first k rows is equivalent to the first row of M times $X\Lambda^2$, the next block of k rows to the second row of M times $X\Lambda^2$, and so on. The size of $A_{\mathbf{m}}$ should be $dk \times \dim(\mathbf{m})$. The other two matrices $A_{\mathbf{c}}$ and $A_{\mathbf{k}}$ can be constructed in a similar way with the respective structure of C and K in mind.

The matrix $A_{\mathbf{m}}$ is built row by row. It is illustrative to use the example in Figure 4.3 to set forth our idea of construction. Denote columns of $X\Lambda^2$ by $\mathbf{y}_1, \ldots, \mathbf{y}_k$, each of which is a vector in $\mathbb{R}^4$. The first row $[-m_1, m_1, 0, 0]$ of M is converted to $[\langle -1, 0, 0\rangle, \langle 1, 0, 0\rangle, \langle 0, 0, 0\rangle, \langle 0, 0, 0\rangle]$, where $\langle \alpha_1, \alpha_2, \alpha_3\rangle$ stands for the coefficients in the expression $\alpha_1 m_1 + \alpha_2 m_2 + \alpha_3 m_3$. The block of the first k rows of $A_{\mathbf{m}}$ is given by

$$\begin{pmatrix} \mathbf{y}_1^\top \\ \vdots \\ \mathbf{y}_k^\top \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \left(\underbrace{\begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{\mathfrak{M}_1} X\Lambda^2 \right)^\top .$$

Continuing this process, the second row of M is converted to an operator matrix $\mathfrak{M}_2$ of size 3×4 and the next k rows of $A_{\mathbf{m}}$ is generated by $(\mathfrak{M}_2 X\Lambda^2)^\top$.

In general, given any linearly structured matrices M, C, and K in terms of the physical parameters $(\mathbf{m}, \mathbf{c}, \mathbf{k})$, each row gives rise to corresponding operator matrices $\mathfrak{M}_i$, $\mathfrak{C}_i$, and $\mathfrak{K}_i$ of sizes $\dim(\mathbf{m}) \times d$, $\dim(\mathbf{c}) \times d$, and $\dim(\mathbf{k}) \times d$, respectively. The matrix A then is given by

$$A = \begin{pmatrix} (\mathfrak{M}_1 X\Lambda^2)^\top & (\mathfrak{C}_1 X\Lambda)^\top & (\mathfrak{K}_1 X)^\top \\ & \vdots & \\ (\mathfrak{M}_d X\Lambda^2)^\top & (\mathfrak{C}_d X\Lambda)^\top & (\mathfrak{K}_d X)^\top \end{pmatrix}. \tag{4.2}$$

We need to point out that the above expression is for explanatory purpose only. The operator matrices need not be generated explicitly. All we need are a few indices pointing to the locations where the connectivity is active. Consequently, the matrix multiplications involved in (4.2) are not as expensive as the expression seems to have suggested. The software we are developing has two input modes — either it picks out these indices from reading the text interface or, better yet, it generates the indices directly from the connectivity configuration and the underlying physical law.

To demonstrate the last point of input mode, we summarize in the following theorem without proof the rules by which the operator matrices $\mathfrak{M}_i, \mathfrak{C}_i, \mathfrak{K}_i$ for a mass-spring system can be generated directly from the connectivity configuration of the system, bypassing any reference to (M, C, K).

THEOREM 4.2. *Given a mass-spring system with d degrees of freedom where the motion is limited to one dimension, let* $\dim(\mathbf{m})$, $\dim(\mathbf{c})$, *and* $\dim(k)$ *denote the number of masses, dampers, and springs, respectively. Then for $1 \le p, q, \le d$, the operator matrices are given as follows:*

1. *Each $\mathfrak{M}_p$ is of size* $\dim(\mathbf{m}) \times d$. *Its entries are all zero except 1 at the (p, p) position.*
2. *Each $\mathfrak{C}_p$ is of size* $\dim(\mathbf{c}) \times d$. *Its entries are all zero with the following exceptions:*
   - *If there are ℓ dampers, identified by $i_1, \ldots, i_\ell$, linking the p-th mass to the q-th mass, then the q-th column of the matrix $\mathfrak{C}_p$ has -1 at its $i_1, \ldots, i_\ell$ entries. Likewise, the p-th column of $\mathfrak{C}_q$ has -1 at its $i_1, \ldots, i_\ell$ entries.*

- *If the p-th mass is connected to the dampers $j_1, \ldots, j_\ell$, then the p-th column of $\mathfrak{C}_p$ has 1 at its $j_1, \ldots, j_\ell$ entries.*

3. *Each $\mathfrak{K}_p$ is of size $\dim(\mathbf{k}) \times d$. Its entries are all zero with the following exceptions:*
   - *If there are ℓ springs, identified by $i_1, \ldots, i_\ell$, linking the p-th mass to the q-th mass, then the q-th column of $\mathfrak{K}_p$ has -1 at its $i_1, \ldots, i_\ell$ entries. Likewise, the p-th column of $\mathfrak{K}_q$ has -1 at its $i_1, \ldots, i_\ell$ entries.*
   - *If the p-th mass is connected to the springs $j_1, \ldots, j_\ell$, then the p-th column of $\mathfrak{K}_p$ has 1 at its $j_1, \ldots, j_\ell$ entries.*

5. Numerical experiment. Based on the ideas discussed thus far, we have developed a software package named Opt4QIEP for solving the structured quadratic inverse eigenvalue problems. The package is available upon request. In this section, we report some results from our numerical experiment with this package to evidence the working of our algorithm.

Our software can handle much more complicated systems, but for demonstration purpose, we shall restrict ourselves to the connectivity configured in Figure 4.1. Even so, we have observed some interesting points that are worth reporting.

To generate test data, we randomly generate positive values as the "real" physical parameters,

$$\begin{aligned}\mathbf{m} &= [0.42052, 0.95581, 0.94875, 0.65968]^\top,\\ \mathbf{c} &= [0.55187, 1.00000, 0.91316]^\top,\\ \mathbf{k} &= [0.71675, 0.90909, 0.73377, 0.38006, 0.32200]^\top,\end{aligned}$$

where we have scaled the numbers so that the largest parameter is normalized to unity. The coefficient matrices in the corresponding quadratic pencil $\lambda^2 M + \lambda C + K$ should follow the structure specified in (4.1). The resulting QEP has four pairs of complex conjugate eigenvalues and eigenvectors. We use only the following two eigenpairs as the *exact* eigeninformation (Λ, X) in our test:

$$\lambda_1 = -0.2560344023 + 1.5586653651i, \quad \lambda_2 = -0.0775078020 + 0.3777316241i,$$

$$\mathbf{v}_1 = \begin{pmatrix} -0.4849049878 + 1.2935348932i \\ -1.5045182718 - 1.2912238078i \\ 0.5225204188 + 1.0465626189i \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 0.4687637938 + 0.1026944256i \\ 0.6690611955 + 0.1297948996i \\ 0.8080014019 + 0.0892317746i \\ 1 \end{pmatrix},$$

where the last entry of each eigenvector has been normalized to unity. It is important to note that we have purposefully set exactly five digits for the physical parameters. On the other hand, we have displayed only the first ten digits for the exact eigenpairs which really should be accurate to the machine precision. Our intentions are to employ our software to explore the effect of various scenarios of inexact eigeninformation on the reconstruction.

Scenario 1. Suppose we are given only $(\lambda_1, \mathbf{v}_1)$ as the prescribed eigenpair. Note that this eigenpair agrees with the "truly exact" eigenpair only up to the tenth digit. Recall that our algorithm endeavors to construct internally a linear inequality system with nonnegative solutions. With this slightly perturbed eigeninformation, our Opt4QIEP does return a nonnegative solution which, after rounded to five digits, has computed parameters,

$$\begin{aligned}\mathbf{m}^* &= [0.42052, 0.95581, 0.94875, 0.65968]^\top,\\ \mathbf{c}^* &= [0.55187, 1.00000, 0.91316]^\top,\\ \mathbf{k}^* &= [0.71675, 0.90909, 0.73377, 0.38006, 0.32200]^\top.\end{aligned}$$

In other words, these computed parameters agree with the original physical parameters up to the fifth digit. The residual $\|M^* X\Lambda^2 + C^* X\Lambda + K^* X\|_F$ with the computed physical parameters is around 10^{-11}, justifying a successful calculation.

The coincidence of this computed physical parameters to the original physical parameters is somewhat surprising in that inverse problems generally are ill-posed, that is, the solutions typically are not unique. In our calculation, however, we have required the additional constraint of nonnegativity. Our inequality setting also enjoys the feature that a minimizer, if exists, is necessarily a global minimizer. Whether these conditions attribute to the coincidence as has been observed is a theoretical question of interest in its own right and deserves further investigation.

Scenario 2. We have mentioned earlier that any number of specified eigenpairs can be used in the reconstruction. Suppose now two eigenpairs $(\lambda_1, \mathbf{v}_1)$ and $(\lambda_2, \mathbf{v}_2)$ are used simultaneously. Given what has happened in Scenario 1, we are not surprised to obtain similar agreement between the computed parameters and the original parameters with residual $\|M^*X\Lambda^2 + C^*X\Lambda + K^*X\|_F \approx 10^{-11}$.

The difference between Scenario 1 and Scenario 2 is that this time Λ and X are of sizes 2×2 and 4×2, respectively. The "extra" eigeninformation with approximately the same accuracy as the previous eigeninformation increases the computational overhead (for computing A), but does not seem to have an impact on the final answer.

Scenario 3. In this test, we perturb the "exact" eigenpairs to the extent that the "prescribed" eigenpairs are given by

$$\widetilde{\lambda}_1 = -0.25603 + 1.55867i, \quad \widetilde{\lambda}_2 = -0.07751 + 0.37773i,$$

$$\widetilde{\mathbf{v}}_1 = \begin{pmatrix} -0.48490 + 1.29353i \\ -1.50452 - 1.29122i \\ 0.52252 + 1.04656i \\ 1 \end{pmatrix}, \quad \widetilde{\mathbf{v}}_2 = \begin{pmatrix} 0.46876 + 0.10269i \\ 0.66906 + 0.12979i \\ 0.80800 + 0.08923i \\ 1 \end{pmatrix}.$$

In other words, the "accuracy" of the prescribed eigenpairs is only at the fifth decimal. Note that we typically would have no assessment about the accuracy of a prescribed eigeninformation. Thus it is not clear at all that any quadratic pencil structured as in (4.1) can have $(\widetilde{\lambda}_1, \widetilde{\mathbf{v}}_1)$ and $(\widetilde{\lambda}_2, \widetilde{\mathbf{v}}_2)$ as its eigenpairs. This is precisely the difficulty of the structured QIEP, not to mention that we are seeking further nonnegative solutions.

Upon applying our Opt4QIEP, however, we are delighted to find a nonnegative solution

$$\begin{aligned} \mathbf{m}^* &= [0.42053, 0.95585, 0.94878, 0.65971]^\top, \\ \mathbf{c}^* &= [0.55188, 1.00000, 0.91319]^\top, \\ \mathbf{k}^* &= [0.71677, 0.90911, 0.73383, 0.38009, 0.32200]^\top, \end{aligned}$$

by which the residual $\|M^*\widetilde{X}\widetilde{\Lambda}^2 + C^*\widetilde{X}\widetilde{\Lambda} + K^*\widetilde{X}\|_F$ of the corresponding reconstructed quadratic pencil is of the order 10^{-6}. This example is a numerical justification of our theory in Theorem 3.1.

Scenario 4. To further dramatize the scenario and to demonstrate the situation where our code is able to indicate that a QIEP is *not* solvable when the prescribed eigenpairs are infeasible, consider the randomly generated eigenpairs

$$\widehat{\lambda}_1 = 0.6068 + 0.8913i, \quad \widehat{\lambda}_2 = 0.4860 + 0.7621i,$$

$$\widehat{\mathbf{v}}_2 = \begin{pmatrix} 0.0185 + 0.4057i \\ 0.8214 + 0.9355i \\ 0.4447 + 0.9169i \\ 1 \end{pmatrix}, \quad \widehat{\mathbf{v}}_2 = \begin{pmatrix} 0.7919 + 0.8936i \\ 0.9218 + 0.0579i \\ 0.7382 + 0.3529i \\ 1 \end{pmatrix}.$$

We use only four digits this time for the reason that, in contrast to the above scenarios, we generally do not have any clue in practice about the feasibility of prescribed eigenpairs at all. Thus, not only

that we are not sure whether a structured QIEP is solvable with $(\widehat{\lambda}_1, \widehat{\mathbf{v}}_1)$ and $(\widehat{\lambda}_2, \widehat{\mathbf{v}}_2)$ as its eigenpairs, neither can we foretell whether an approximate structured pencil exists. What we are certain about is that, if a solution or a nearby solution ever exists, our method will find it (high sensitivity) and that, if the QIEP is not solvable, our method will indicate either a large residual or no solution (high specificity).

Applying our software to the above two randomly generated eigenpairs, we do find a nonnegative "approximate" solution, but the residuals of the corresponding quadratic pencils are 1.2885 and 0.7667, respectively, at each eigenpair. It should be an obvious indication that the reconstructed model is unacceptable.

We conclude the discussion with this final remark. It seems reasonable to expect that the eigenstructure of structured problems should be inherently "structured". Given a specified structure for the coefficient matrices (M, C, K), it seems rational to expect that the corresponding eigenvalues and eigenvectors cannot be arbitrary. Scenario 4 exemplifies such a case. It remains an open question to determine the eigenstructure of a structured pencil. Our algorithm might serve as a numerical tool to help investigation in this direction.

6. Conclusion. Just as the quadratic eigenvalue problems are critical to scores of important applications, the inverse quadratic eigenvalue problems are equally vital in many different fields of disciplines. The structured quadratic inverse eigenvalue problems are particularly challenging in that the inner connectivity of the system and the nonnegativity of the parameters must be respected. Thus far, no theory can address these difficulties.

The main contribution of this paper is to offer a general purpose and robust numerical approach that, via a returned residual estimate, has high specificity in determining that a QIEP is not solvable and high sensitivity in predicting a problem is solvable. The method imposes no restriction on how much eigeninformation should be given and has the ability of handling almost all kinds of connectivity configurations.

The crux of our computation is the mechanism of establishing a proper inequality system and solving the system via optimization techniques. The setting allows us to find the nearest possible global solution up to scaling and thus provides us with the desirable robustness.

We have built a package of software that can automatically handle the above mentioned task from any input of connectivity configuration and eigeninformation by users. The beta release of this package is available upon request. We are continuing the expansion of its library to include various settings according to the underlying physical laws.

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