SEMI-DEFINITE PROGRAMMING TECHNIQUES FOR STRUCTURED QUADRATIC INVERSE EIGENVALUE PROBLEMS

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Abstract. In the past decade or so, semi-definite programming (SDP) has emerged as a powerful tool capable of handling a remarkably wide range of problems. This article describes an innovative application of SDP techniques to quadratic inverse eigenvalue problems (QIEPs). The notion of QIEPs is of fundamental importance because its ultimate goal of constructing or updating a vibration system from some observed or desirable dynamical behaviors while respecting some inherent feasibility constraints well suits many engineering applications. Thus far, however, QIEPs have remained challenging both theoretically and computationally due to the great variations of structural constraints that must be addressed. Of notable interest and significance are the uniformity and the simplicity in the SDP formulation that solves effectively many otherwise very difficult QIEPs.

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1. Introduction.

The notion of quadratic pencils

$$\mathfrak{Q}(\lambda) := \mathfrak{Q}(\lambda; M, C, K) = \lambda^2 M + \lambda C + K, \tag{1.1}$$

where M, C and K are real-valued $n \times n$ coefficient matrices, plays a significant role in many important applications, including applied mechanics, circuit analysis, electrical oscillation, vibro-acoustics, or finite element models of some PDEs. The physical parameters usually are embedded in the coefficient matrices (M, C, K). In the forward setting, known as the quadratic eigenvalue problem (QEP) [26, 40], the dynamical behavior of the underlying physical system often can be interpreted via the eigenvalues and eigenvectors of the quadratic pencil (1.1) whose coefficient matrices M, C and K have already been determined from the specified physical parameters. In contrast, in the inverse setting, the quadratic inverse eigenvalue problem (QIEP) intends to validate, determine, or estimate the parameters of the system according to its observed or expected behavior.

By a *structured* QIEP, we mean the construction of real coefficient matrices M, C and K so that the resulting quadratic pencil (1.1) has k prescribed eigenpairs $\{(\lambda_j, \mathbf{x}_j)\}_{j=1}^k$ while the matrices M, C and K meet certain distinctive conditions imposed upon their respective structures. Without loss of generality, the prescribed eigenvalues and eigenvectors can be denoted respectively in real-valued form,

$$\Lambda_1 := \mathrm{diag}\left\{ \begin{bmatrix} \alpha_1 & \beta_1 \\ -\beta_1 & \alpha_1 \end{bmatrix}, \ldots, \begin{bmatrix} \alpha_{k_c} & \beta_{k_c} \\ -\beta_{k_c} & \alpha_{k_c} \end{bmatrix}, \lambda_{2k_c+1}, \ldots, \lambda_k \right\} \in \mathbb{R}^{k\times k}, \tag{1.2}$$

$$X_1 := [\mathbf{x}_{1R}, \mathbf{x}_{1I}, \ldots, \mathbf{x}_{k_cR}, \mathbf{x}_{k_cI}, \mathbf{x}_{2k_c+1}, \ldots, \mathbf{x}_k] \in \mathbb{R}^{n\times n}, \tag{1.3}$$

as was characterized in [18]. Then the coefficient matrices (M, C, K) should necessarily satisfy the linear algebraic system

$$MX_1\Lambda_1^2 + CX_1\Lambda_1 + KX_1 = 0_{n\times k}. \tag{1.4}$$

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Each prescribed eigenpair gives rise to n scalar algebraic equations. Thus with reasonable k the homogeneous system (1.4) should have a nontrivial solution space. Difficulties arise when our goal is to seek within this solution space three matrices (M, C, K) that satisfy additional constraints as we shall see below.

Considerable efforts have been taken to solve various special cases of QIEPs, both theoretically and computationally. A partial list of existent works includes those discussed in articles [16, 17, 19, 20, 29, 30, 32, 35, 37, 38] and books [15, 24, 25]. Still, research results advanced thus far for the QIEPs are incomplete and indeed quite limited. In particular, the most commonly assumed structure imposed upon (M, C, K) is symmetry. At most, the condition of positive definiteness is imposed upon one or two of the coefficient matrices, which is far from being sufficient to address, for example, the finer grain of inner-connectivity among elements of the underlying physical system or the requisite nonnegativity of the physical parameters for the sake of feasibility. There are other types of structures, such as a mixture of both positive definite matrices and skew-symmetric matrices for gyroscopic systems and matrices with prescribed entries for local model updating problems, for which no theory of solvability is available at all. Effective numeral algorithms for QIEPs are few and those already developed are cumbersome in three aspects — that they can handle only some limited and non-refined structures; that the approaches often involve matrix factorization and hence are suitable only for small-scale problems; and worst of all, that the techniques are designed for a particular vibration system which are structure dependent and cannot be generalized to other types of quadratic systems.

This purpose of this paper is to propose a general numerical scheme for solving QIEPs by employing semi-definite programming (SDP) techniques [2, 3, 41]. Semi-definite programming concerns finding a symmetric matrix to optimize a linear functional subject to linear constraints and the additional condition that the matrix be positive semi-definite, that is,

$$\begin{array}{ll} \text{Minimize}_X & C \bullet X \\ \text{Subject to} & A_i \bullet X = b_i, \quad i = 1, \ldots, m \\ & X \succeq 0, \end{array} \tag{1.5}$$

where C, $A_1, \ldots, A_m$ are given symmetric matrices in $\mathbb{R}^{n\times n}$, $X \succeq 0$ means that X is positive semi-definite, and $C \bullet X$ denotes the Frobenius inner product between C and X. In recent years, SDP has emerged as an important tool in mathematical programming for two reasons. The first reason is its versatility to model problems arising in broad discipline areas ranging from mathematical studies in combinatorial optimization, Boolean and non-convex quadratic programming, min-max eigenvalue problems, and matrix completion problems to engineering applications in nonlinear and time-varying system analysis, controller synthesis, computer-aided control system design, network queueing, optimal statistical model designs, and structural optimization. Examples of converting these problems into the standard primal problem (1.5) or its dual can be found in [41, 45, 9]. The second reason is that there is a considerable similarity between the notion of SDP and that of the well known linear programming (LP). In particular, much of the dual theory, interior point algorithms, convergence and polynomial time-complexity for LP can be extended to SDP [1, 34]. This generalization thus admits theoretically efficient solution procedures based on iterating interior points that either follow the central path or decrease a potential function. A profusion of research results are available in the literature. For example, the book on SDP [45] lists 877 references, while the online bibliography collected by Wolkowicz [44] lists 1299 as of this note being written. As casual users, we find that the comprehensive treatise in the two books [3, 9] and the two review articles [1, 41] offer quick and useful grasp of this interesting and intensely studied subject.

By exploiting the current development of SDP procedures, we are able to handle various types of structures for (M, C, K). We apply some state-of-the-art software packages [33, 39, 42] successfully to the otherwise very difficult QIEPs with structured coefficient matrices. Unlike other numerical

methods, the SDP approach presents a unified, efficient, and tractable scheme for solving QIEPs. Structural constraints such as positive definiteness, nonnegativity, mixture type, sparsity patterns, prescribed entries for QIEPs and the associated model updating problems can all be handled within the same framework. With the SDP tool in hand, we are at a vantage position to explore and propose new areas for further research about QIEPs.

This paper does not propose new algorithms. Rather, our point is to call attention to the applicability of SDP approaches to QIEPs. In the sequel, we briefly discuss how to formulate QIEPs under different structures for SDP applications. Our goal appears simply stated yet it involves vast undertakings: to convey the remarkable uniformity and simplicity of the SDP applications to many otherwise very diverse and very difficult QIEPs. This connection should be of great significance to QIEP practitioners. In the sense that "we have now come to a point where a problem with a solution implicitly described by an SDP can he considered solved, even though there is no analytic closed-form expression of the solution" [33], we may say that the structured QIEPs stated under the context of SDP have been solved relatively effectively. We shall demonstrate the great ability of the SDP approach by some interesting numerical experiments that could not have been accomplished so straightforwardly by currently available QIEP theory or algorithms.

2. When M, C and K are all symmetric and positive semi-definite. Suppose that M, C and K are all symmetric. It is known that so long as $k < k_{max}$, where

$$k_{max} = \begin{cases} 3\ell + 1, & \text{if } n = 2\ell, \\ 3\ell + 2, & \text{if } n = 2\ell + 1, \end{cases} \tag{2.1}$$

generically the solution space to (1.4) is not empty [13]. In many applications, however, one or all of M, C and K are required further to be positive semi-definite. Theoretical investigation of the solvability of the QIEP under this positive semi-definite condition is quite challenging. At present, only some partial results are available. The work by Lancaster and Prells [32], for instance, constitutes a construction of symmetric (M, C, K) with $M \succ 0$, $C \succeq 0$, and $K \succ 0$, but the establishment requires the complete information about eigenvalues and eigenvectors. With only partially prescribed eigenpairs available, Chu, Kuo and Lin [16] put forward a special solution for the symmetric QIEP, guaranteeing $M \succ 0$ and $K \succeq 0$. Shortly afterwards, a sufficient condition for the general solution parameterized in terms of the QR decomposition of the eigenvectors was developed by Kuo, Lin and Xu [29] for the case $k \leq n$ and by Cai, Kuo, Lin and Xu [11] for the case $k > n$. The feedback control approaches proposed in [19, 20, 35], on the other hand, generally can maintain symmetry only. In almost all developments thus far, the computation of a symmetric and positive semi-definite solution for the QIEP appears unwieldy and inefficient.

Taking advantage of the advances in SDP techniques, however, the difficult task ensuring that any of the three coefficient matrices are positive semi-definite can be accomplished via just a few lines of YALMIP commands such as the code demonstrated in Table 2.1. (In this particular instance, all three coefficient matrices are required to be positive semi-definite while only partial eigeninformation (Λ_1, X_1) is given. This is already a case that has not been studied in the literature before.) YALMIP is a free MATLAB-based toolbox that serves as a convenient interface for multiple external optimization solvers [33]. The YALMIP commands unify and facilitate the different formats in SDP software. When applied to QIEPs, it makes the description of various structural constraints extremely simple and offers via the well established SDP theory and algorithms a reliable and conclusive answer within the specified numerical tolerance.

By default, the YALMIP command `sdpvar(n,n)` defines a symmetric matrix as a variable. The algebraic condition (1.4) and the positive semi-definiteness are defined in the code as constraints `F` through the command `set`. We may also use this `set` command to take care of other types of structural constraints, which will be demonstrated in subsequent discussion. The simple code

```
% Define symmetric variables

sM = sdpvar(n,n);
sC = sdpvar(n,n);
sK = sdpvar(n,n);

% Specify equality and positive semi-definite constraints

F = set(sM*X_1*Lambda_1^2 + sC*X_1*Lambda_1 + sK*X_1 == zeros(n,k));
F = F + set(sM >= 0) + set(sC >= 0) + set(sK >= 0);

% Select SDPT3 as the solver

ops = sdpsettings('solver','sdpt3');

% Invoke SDP solver

solvesdp(F,[],ops);

% Retrieve numerical solution

M = double(sM)
C = double(sC)
K = double(sK)

% Check relative residual

norm(M*X_1*Lambda_1^2 + C*X_1*Lambda_1 + K*X_1,'fro')/norm([M,C,K],'fro')
```

TABLE 2.1
An SDP code calling routines from YALMIP to solve QIEPs for positive semi-definite (M, C, K).

illustrated in Table 2.1 defines the package SDPT3 [42] as the SDP solver and constructs a numerical solution, if exists, to the QIEP with positive semi-definite coefficient matrices. For a complete list of other commands and options to fine-tune the computation such as selecting a different SDP solver or stopping criterion, we refer to the reference [33] and the associated website.

We remark that in this particular YALMIP application, no objective function is given (denoted by `[]` in the command `solvesdp`), as the purpose for now is only to seek a feasible solution. Note that the set of feasible solutions, containing at least the trivial solution, is a convex cone. Once a numerical solution (M, C, K) is found (retrievable through the YALMIP command `double`), any positive scalar multiplication is also a solution. An appropriate normalization might be needed to assess the final reconstructed (M, C, K). A small relative residual, for example, might be a good indicator that the QIEP has been solved satisfactorily. Note that the unspecified eigenvectors of the reconstructed system are determined inherently and cannot be influenced from outside.

The code is so simple and the computation is so efficient that we are able investigate numerically many interesting questions. One such instance is whether symmetric QIEPs with positive semi-definite coefficient matrices can have k arbitrarily prescribed eigenpairs, provided all prescribed eigenvalues have negative real part (a necessary condition). In other words, can any arbitrarily pre-

k_c	6	5	5	5	5	5	5	4	4	3	3	4	4	3	2
k_p	0	2	2	1	0	0	1	4	3	4	3	2	2	3	4
k_n	0	1	0	2	3	2	1	1	2	3	3	3	2	4	4
k	12	13	12	13	13	12	12	13	13	13	12	13	12	13	12
☞	✓	❋	✓	❋	❋	✓	✓	✓	✓	✓	✓	✓	✓	❋	✓
t	8	7					6					5		4	3

TABLE 2.2
Existence of positive definite M, C and K to the symmetric QIEP with random eigenstructure.

scribed eigenstructure be completed so that the resulting quadratic pencil has symmetric and positive semi-definite coefficient matrices? To conduct a test, we randomly generate X_1 and Λ_1, subject to the condition that all prescribed eigenvalues have negative real part, and feed the information to the SDP code in Table 2.1. Our extensive experiment suggests a rather surprising discovery. We have observed in our repeated trials that if $k \le k_{max} - 4$ and regardless of how k is distributed among the numbers k_c of complex-conjugate eigenpairs, k_p of real eigenpairs with positive sign characteristic and k_n of real eigenpairs with negative sign characteristic [26, 32], the feasible set defined by F seems to always contain nontrivial solutions for random (X_1, Λ_1). Such an observation, if confirmed by theory, would be quite interesting.

To demonstrate our point, summarized in Table 2.2 are a few test results for the case $n = 10$ and hence $k_{max} = 16$. With the prescribed eigenpairs randomly generated according to the preselected (k_c, k_p, k_n), the symbols ✓ and ✗ indicate whether a nontrivial solution to the QIEP with positive semi-definite coefficient matrices can be found or not, whereas ❋ indicates that both cases are possible. The value t in the last row is the number of complex conjugate eigenvalues of the reconstructed pencil. We stress that these results are generic in the sense that each case has been run multiple times and no exception has been observed. As is seen, a positive semi-definite solution can always been found if $k \le 12$, whereas $k = 13$ serves as a borderline in which both possibilities can occur.

3. When M, C and K are a mixture of linear types. For vibrations of rotating machines or in moving coordinate frames, often a gyroscopic system appears. The quadratic pencil of the corresponding dynamical system generally assumes the form

$$\mathfrak{Q}(\lambda) := \mathfrak{Q}(\lambda; M, \mathscr{C}, \mathscr{G}, \mathscr{K}, \mathscr{N}) = \lambda^2 M + \lambda \underbrace{(\mathscr{C} + \mathscr{G})}_{C} + \underbrace{(\mathscr{K} + \mathscr{N})}_{K} \tag{3.1}$$

where M, $\mathscr{C}$, and $\mathscr{K}$, as the usual mass, damping and stiffness matrices, are symmetric and positive semi-definite, but $\mathscr{G}$ and $\mathscr{N}$, representing the gyroscopic and circulatory matrices, are skew-symmetric [47]. The combination such as $C = \mathscr{C} + \mathscr{G}$ or its like is referred to as a mixture of linear types, resulting so that C is neither symmetric, nor general, but is still of some special structure.

Numerical techniques for forward gyroscopic eigensystems have been discussed in [10, 22, 36], but the attention mostly is on the damping free case, i.e., $C = \mathscr{G}$ is skew-symmetric. A hybrid optimization method employing genetic algorithms and simulated annealing to identify bearing parameters of rotating machinery from bearing forces [4] is somewhat close to an inverse problem, but we have found no discussion on the gyroscopic inverse eigenvalue problem. Typical matrix analysis would have a hard time to differentiate the positive semi-definite matrix $\mathscr{C}$ from the skew-symmetric matrix $\mathscr{G}$ within the coefficient matrix C. We have to keep track of them separately. Toward that end, we can simply add the YALMIP commands

```
% Define skew-symmetric variables
```

k_c	8	7	7	7	6	6	7	7	6	6	6	6	5	5	5	5	3	4	3	0	3
k_r	0	1	0	2	3	2	4	3	5	6	5	4	7	6	5	4	6	6	6	6	8
k	16	15	15	16	15	14	18	17	17	18	17	16	17	16	15	14	12	14	12	6	14
☞	✓	✓	✓	✓	✓	✓	✗	✓	✓	✗	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
t	9			8			7						6					5			4

TABLE 3.1
Existence of positive definite M, $\mathscr{C}$, $\mathscr{K}$, and skew-symmetric $\mathscr{G}$ to the QIEP with random eigenstructure.

```
sG = sdpvar{n,n,'skew','real');
sN = sdpvar(n,n,'skew','real');
```

to define new variables for skew-symmetric matrices and modify the equality constraint to

```
F = set(sM*X_1*Lambda_1^2 + (sC+sG)*X_1*Lambda_1 + (sK+sN)*X_1 == zeros(n,k));
```

in the code in Table 2.1. The QIEP in the form (3.1) can now be handled without much trouble.

Each of the skew-symmetric matrices $\mathscr{G}$ and $\mathscr{N}$ adds $\frac{n(n-1)}{2}$ extra variables to the linear system (1.4). If merely the symmetry of M, $\mathscr{C}$ and $\mathscr{K}$ is required in (3.1), then it can be proved that there are always more variables than equations, implying that the equation (1.4) is solvable for any k. If the constraint of positive semi-definiteness for M, $\mathscr{C}$ and $\mathscr{K}$ is imposed, we wonder whether the inclusion of $\mathscr{G}$ or $\mathscr{N}$ would increase the solvability for a QIEP in the gyroscopic form (3.1) by allowing a larger number k of arbitrarily prescribed eigenpairs. Assume $\mathscr{N} = 0$, under the setting similar to the random tests in the preceding section except that $k_r = k - 2k_c$ stands for the total number of real eigenpairs (since real eigenvalues of a gyroscopic system have no sign characteristic, another significant difference between symmetric and non-symmetric pencils), the numerical evidence summarized in Table 3.1 seems to support affirmatively this added effect of the skew-symmetric matrix $\mathscr{G}$ — the maximal allowable k of arbitrarily prescribed eigenpairs increases by $\lfloor \frac{n-1}{2} \rfloor$.

Other types of coefficient matrices, such Toeplitz, Hankel, or even palindromic structures can easily be incorporated into the SDP formulation for QIEPs. In fact, the inverse eigenvalue problems being linear in the coefficient matrices, the entire procedure discussed in this paper can be carried over with little effort to matrix polynomials of general degrees. So far as we know, no theory is available for structured inverse eigenvalue problems when the underlying matrix polynomials are of degree higher than 2. In contrast, the SDP approach makes a numerical exploration or justification possible.

4. When M, C and K inherit sparsity patterns. Finite element models often result in M, C and K having specific sparsity patterns. The inter-connectivity among elements of a mass-spring system or an RLC circuit also gives rise to sparsity patterns. For the sake of physical feasibility, the construction of the coefficient matrices (M, C, K) should respect these inherent structures. Additionally, the physical parameters, such as mass, stiffness, voltage, resistance and so on, are embedded in the coefficient matrices (M, C, K) in a fixed but often mixed way. Merely solving the QIEP subject to the structural constraint is not enough. We must insist that the recovered parameters remain nonnegative for physical realization. In this context, we say that the QIEP must satisfy both the connectivity constraint and the nonnegative constraint.

Because the structure of the QIEP varies according to applications, even merely formulating the problem is a laborious task itself. Having in hand a tool that alleviates the burden of problem dependency and offers a universal approach to arbitrary physical systems should be highly desirable

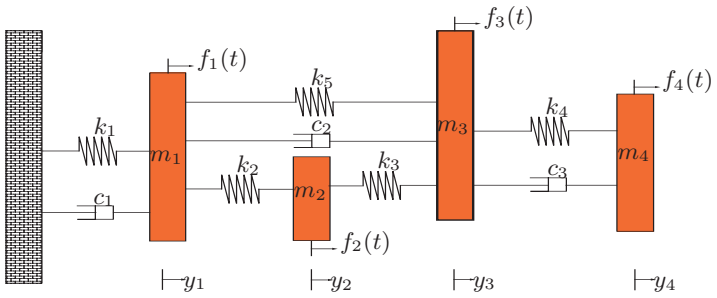

FIG. 4.1. *A four-degree-of-freedom mass-spring system.*

for practitioners. A general-purpose problem-independent software approach using the notion of truncated QR decomposition to create, if possible, a consistent linear inequality system has been proposed recently in [21]. In contrast, the SDP application offers a versatile alternative optimization approach to this type of QIEPs.

To demonstrate a possible sparsity patten, consider a mass-spring system where we assume that the restoring force follows Hooke's law and that the damping is negatively proportional to the velocity. For a mass-spring system with n degrees of freedom whose motion is limited to one dimension, let $\mathbf{m}$, $\mathbf{c}$, and $\mathbf{k}$ denote the vectors of masses, damping and stiffness coefficients, respectively. The following rules restricting the structure of (M, C, K) are well developed in the field of structural mechanics [27].

1. The mass matrix M is a diagonal matrix with masses $(m_1, \ldots, m_n)$ along its diagonal.
2. The damping matrix C is symmetric and positive semi-definite.
   (a) If there are ℓ dampers, identified by $i_1, \ldots, i_\ell$, between the p-th mass and the q-th mass, then the entries C_{pq} and C_{qp} of the damping matrix are given by $-\sum_{s=1}^{\ell} c_{i_s}$, where c_{i_s} is the damping coefficient of damper i_s. Otherwise, $C_{pq} = C_{qp} = 0$.
   (b) If the p-th mass is connected to the dampers $j_1, \ldots, j_\ell$, then $C_{pp} = \sum_{s=1}^{\ell} c_{j_s}$.
3. The stiffness matrix K is symmetric and positive semi-definite.
   (a) If there are ℓ springs, identified by $i_1, \ldots, i_\ell$, between the p-th mass and the q-th mass, then the entries K_{pq} and K_{qp} of the stiffness matrix are given by $-\sum_{s=1}^{\ell} k_{i_s}$, where k_{i_s} is the stiffness coefficient of spring i_s. Otherwise $K_{pq} = K_{qp} = 0$.
   (b) If the p-th mass is connected to the springs $j_1, \ldots, j_\ell$, then $K_{pp} = \sum_{s=1}^{\ell} k_{j_s}$.

Indeed, it can be proved that the symmetry and positive semi-definiteness of M, C and K are natural consequences of their respective structure specified above, provided that all physical parameters $\mathbf{m}$, $\mathbf{c}$, and $\mathbf{k}$ are nonnegative. Thus in solving the QIEPs for mass-spring systems, it is more important to maintain the connectivity and nonnegativity than to merely maintain symmetry and positive semi-definiteness. Such a demand can easily be met after formulating the QIEP as an SDP problem. It might be more informative to use the system depicted in Figure 4.1 to demonstrate the SDP formulation. Following the above rules, we define the parameters $\mathbf{m}$, $\mathbf{c}$, and $\mathbf{k}$, the equality and the nonnegative constraints by YALMIP commands as below.

```
% Define physical parameters

sm = sdpvar(4,1);
sc = sdpvar(3,1);
sk = sdpvar(5,1);
```

```
% Define coefficient matrices based on connectivity constraints

sM = diag(sm);
sC  = [sc(1)+sc(2)   0      -sc(2)        0;
            0        0        0           0;
          -sc(2)     0   sc(2)+sc(3)   -sc(3);
            0        0      -sc(3)      sc(3)];
sK  = [sk(1)+sk(2)+sk(5)       -sk(2)        -sk(5)           0;
            -sk(2)          sk(2)+sk(3)      -sk(3)           0;
            -sk(5)             -sk(3)   sk(3)+sk(4)+sk(5)  -sk(4);
              0                  0           -sk(4)         sk(4)];

% Define equality and nonnegativity constraints

F = set(sM*X_1*Lambda_1^2 + sC*X_1*Lambda_1 + sK*X_1 == zeros(n,k));
F = F + set(sm > 0) + set(sc > 0) + set(sk > 0);
```

Note that the connectivity constraint is built in the intermediate matrix variables `sM`, `sC`, and `sK` which are used to define the equality constraint (1.4). The true `sdpvar` variables in the calculation are the physical parameters `sm`, `sc`, and `sk` of which the nonnegative constraint is enforced through the `set` command that defines element-wise inequalities. In fact, given any configuration of inner-connectivity in a mass-spring system, the rules outlined earlier can be implemented to systematically generate the associated structure. Thus, the above YALMIP code can be fully automated for large-scale mass-spring systems. To avoid arbitrary scaling, we recommend the employment of an artificial objective function such as

```
solvesdp(F,(sm(1,1)-1)^2,ops);
```

with the hope that the first mass m_1 is normalized to unity, if a solution exists. It is important to note that this is a convex programming problem even with the added normalization, so the return from the SDP computation tells either a "completely successful reconstruction" with positive physical parameters or an "utterly disastrous failure" giving rise to either the trivial solution or a degenerated system.

Observe that the mass-spring system in Figure 4.1 has 12 parameters. Can the parameters be chosen so that the new system has k arbitrarily prescribed eigenpairs? Even more restrictively but curiously, given an original system with parameters $(\mathbf{m}_0, \mathbf{c}_0, \mathbf{k}_0)$, can the mass parameters $\mathbf{m}$ (or others) alone be altered so that the new system $(\mathbf{m}, \mathbf{c}_0, \mathbf{k}_0)$ has one specific but arbitrarily prescribed eigenpair? All these sorts of questions can easily be investigated numerically by slight modifications of the above SDP code, showing the potency of the SDP methods despite the lack of theoretical insight into the respective QIEPs. It turns out that the answers to the two questions just raised are negative. This could perhaps be explained by the notion that the eigenstructure of a structured pencil should be somehow structured and an arbitrary assignment is almost surely doomed.

5. When M, C and K have prescribed entries. Thus far we have been discussing the construction of the coefficient matrices (M, C, K) as a whole. In the preceding section, we have begun to see that the inner-connectivity among elements of a physical system may limit the structure of (M, C, K) to a certain sparsity pattern. Even there our concern has been about finding values for the entire parameters so that the reconstructed system has a prescribed eigenstructure. In practice, quite often the reconstruction or modification is limited to a local portion of the system. By a "local portion" we refer to either a specific part of the matrix or a subset of the parameters. In the latter case, we have already demonstrated in the preceding section the idea of adjusting masses with the

(unrecognized) hope of achieving one or more specific eigenpairs. For the former, one scenario is vibrating structures such as bridges, highways, buildings or automobiles where, modeled by finite element methods, the spatial representation for a portion of the substructures such as the abutments, the foundations, or chassis is fixed.

Some (linear) inverse eigenvalue problems with prescribed entries are related to other mathematical problems in somewhat different context [14]. Constructive proofs exist to a certain extent (and those proofs, such as the classical Schur-Horn theorem and the Hershkowitz theorem, are amazingly elegant in their own right) but beyond these, very few theories or numerical algorithms are available. Presumably analyzing and solving QIEPs with prescribed entries would be much harder. With the aid of the SDP techniques and the YALMIP interface, however, all we need to do is to assign the prescribed values to the relevant entries when defining the `sdpvar` variables. One such example has already been given in Section 4 where the sparsity pattern is interpreted and fixed by those zeros. Depending on the goals, we could assign any values to any positions in the matrices and proceed to call for an SDP solver which will give us a make-or-break decision on the solvability. We shall give more examples in the next section.

6. Model Updating Problems. Model updating problems (MUP) fall within the realm of QIEPs, but are more special in that a quadratic pencil

$$\mathfrak{Q}_0(\lambda) := M_0\lambda^2 + C_0\lambda + K_0 \tag{6.1}$$

is already given, where M_0, C_0 and K_0 are matrices in $\mathbb{R}^{n\times n}$ with specified structures. The MUP seeks to find a best least squares update

$$\mathfrak{Q}(\lambda) := M\lambda^2 + C\lambda + K$$

in the sense that $\|(M_0, C_0, K_0) - (M, C, K)\|_F$ (or a weighted variant) is minimized subject to the conditions that $\mathfrak{Q}(\lambda)$ has k prescribed eigenpairs, that is, the algebraic constraint (1.1), and that (M, C, K) satisfies a certain prescribed structure. Model updating problems have emerged as an important tool for the design, construction and maintenance of mechanical systems [6, 7]. A good exposition about general principles of model updating can be found in the book by Friswell and Mottershead [24]. The basic idea is to improve the "intricacy level" of the model through its eigenstructure. When natural frequencies and mode shapes of the current model (6.1) do not match with experimentally measured or desirable frequencies and mode shapes, refining, correcting, or updating the current dynamic model on the basis of actual test data becomes necessary, with many important consequences.

The structure imposed upon (M, C, K) can be quite general. Any of the previously discussed structures such as positive semi-definiteness or prescribed entries is allowable. In the literature, however, the most commonly discussed case is when $M = M_0 \succeq 0$ and C and K are symmetric [5, 23, 28, 31]. The other slightly easier case is when $C = C_0 = 0$ [6, 12, 19, 43, 46]. We believe that the absence in the literature of other possible structures is not due to their lack of interest, but rather because of the difficulties associated with these constraints. Currently, the three most prevailing numerical methods seem to be the Lagrange multiplier approach adopted in [24], the direct approach via dimension reduction proposed in [28], and the Newton-type iteration used in [5]. Since the underlying structure is application dependent, it seems that we can tackle structured quadratic model updating only problem by problem. Indeed, it was exclaimed in [24] that "Updating is a process fraught with numerical difficulties."

Updating, even subject to the more stringent structural constraints, is not necessarily so numerically difficult with the employment of the SDP techniques. First, taking into account the desirable structures, we use the `sdpvar` command to define the variables (M, C, K) as we have described earlier. In the event that the sparsity pattern in the original matrix, say, K_0, is to be maintained in the updated K, for example, we could issue commands such as

```
sK = sdpvar(n,n).*abs(sign(K_0));
```

where the element to element multiplication `.*` is to pick up the nonzero locations of K_0. Suppose next that the closeness of the updated model to the original model is measured by the objective function

$$J = \mu\|M_0^{-\frac{1}{2}}(M_0 - M)M_0^{-\frac{1}{2}}\|_F^2 + \nu\|M_0^{-\frac{1}{2}}(C_0 - C)M_0^{-\frac{1}{2}}\|_F^2 + \|M_0^{-\frac{1}{2}}(K_0 - K)M_0^{-\frac{1}{2}}\|_F^2, \qquad (6.2)$$

where μ and ν are some preselected weight factors [28]. Note that J is a convex but nonlinear function in (M, C, K). We can easily rewrite this objective function as a second-ordered Lorentz cone programming problem via the YALMIP commands

```
% Define variables, structured if necessary

sdpvar uM uC uK
sM = ...
   :

W = inv(M_0^(1/2));

% Define equality and structural constraints

F = ...

% Include rotated Lorentz cones

F = F + set(rcone(reshape(W*(M_0-sM)*W,n^2,1),uM,1/2));
F = F + set(rcone(reshape(W*(C_0-sC)*W,n^2,1),uC,1/2));
F = F + set(rcone(reshape(W*(K_0-sK)*W,n^2,1),uK,1/2));

% Define objective function and call for second order cone programming solvers

sdpsolve(F,mu*uM + nu*uC + uK,ops);
```

where the command `rcone(z,x,y)` with a column vector `z` and scalars $x, y > 0$ defines a rotated second order cone constraint $\|\mathbf{z}\|_2^2 < 2xy$. We stress that the structural constraints to be imposed upon the updated model (M, C, K) at the line `F = ...` can be quite general, including all or some of the following: (M, C, K) positive semi-definite, sparsity patterns, mixture of linear types, or fixed entries, as we have discussed earlier. Again, taking advantage of convex programming, the return from the SDP computation gives us a make-or-break decision about whether the update is achievable and, if it is, we always obtain a global by optimal solution.

It will be instructive to consider the problem of updating the (linear) pencil

$$(\lambda M_0 - K_0)\mathbf{x} = 0, \qquad (6.3)$$

where M_0 and K_0 are the matrices BCSSTM01 and BCSSTK01, respectively, from the set BCSSTRUC1 in the Harwell-Boeing Collection [8]. These are 48×48 matrices whose sparsity patterns are plotted in Figure 6.1. Specifically, M_0 is a nonnegative diagonal matrix with 24 zeros along its diagonal while K_0 is symmetric and positive definite, implying that (6.3) has 24 positive eigenvalues with the remaining 24 at positive infinity. Furthermore, the two coefficient matrices are not of comparable scales, $\|M_0\|_F \approx 7.7 \times 10^2$ and $\|K_0\|_F \approx 7.5 \times 10^9$. All of these aspects suggest that eigenvalue computation would be particularly challenging. Indeed, the residuals $\|(\lambda M_0 - K_0)\mathbf{x}\|_2$ of

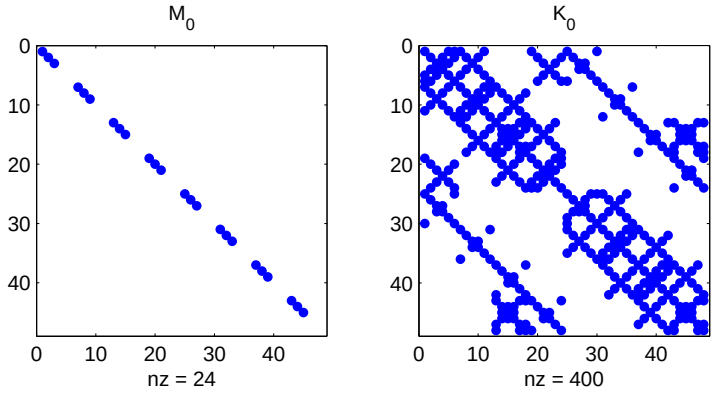

FIG. 6.1. *The 48 × 48 sparse matrices M_0 and K_0.*

the 24 real-valued eigenvalues and eigenvectors computed by MATLAB ranges from 5.5×10^{-9} to 1.6×10^{-6}. Because of this disparity, we have to use our discretion carefully in deciding whether the SDP schemes have returned a satisfactory updating in its computation, for which we now explain our experiment below.

Knowing that the magnitude of K_0 is several order higher that of M_0, it is perhaps more suitable in practice to weight the approximations for M_0 and K_0 separately. To demonstrate that the SDP calculation can endeavor to overcome the dissimilarity in the data, however, we shall use the objective value of

$$J(M, K) = \|M_0 - M\|_F^2 + \|K_0 - K\|_F^2 \tag{6.4}$$

to measure the nearness of updating. Given one eigenpair $(\lambda, \mathbf{x})$ of the quadratic pencil (6.3) with $\|(\lambda M_0 - K_0)\mathbf{x}\|_2 \approx 5.5 \times 10^{-9}$, we want to find out the nearest coefficient matrices M and K satisfying an updated eigenpair $(\mu, \mathbf{x})$ where, while keeping the eigenvector invariant, we let the new eigenvalue be a random number generated by

$$\mu = \lambda(1 + \sigma|\text{randn}(1)|)$$

with $\sigma = 10^{-p}$, $p = 1, \cdots, 10$ and randn(1) expresses a pseudo-random value derived from a normal distribution with mean zero and standard deviation one. For different p values, our idea is to let σ represent the "variance" of the "one-sided" perturbation while we intend to observe a general performance. Two steps are involved in the updating procedure:

$$(\mu, \mathbf{x}) \xrightarrow{\text{feasibility}} (M, K) \xrightarrow{\text{optimization}} (M_0, K_0).$$

First, we need to decide whether a linear pencil (M, K) with eigenpair $(\mu, \mathbf{x})$, the same sparsity pattern as (M_0, K_0), and positive semi-definiteness exists. We measure the feasibility by the residual

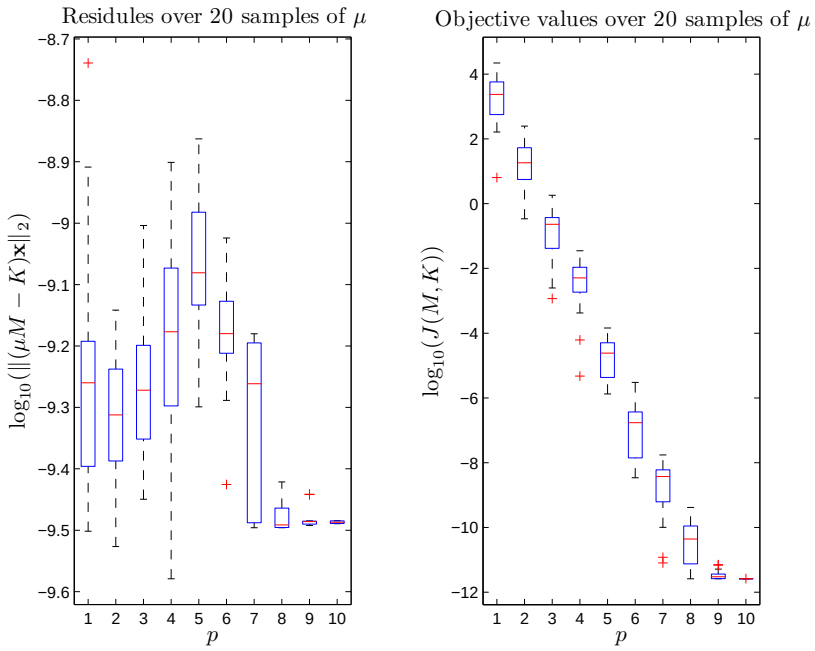

FIG. 6.2. *Performance of model updating for Harwell-Boeing test data BCSST*01 (n = 48).*

value $\|(\mu M-K)\mathbf{x}\|_2$. Ideally, the residual should be zero but the floating-point arithmetic can return at most the machine precision if we carefully fine tune the `sdpsettings` in YALMIP. In fact, we have already pointed out that the singularity and the imbalance in scaling of the original pencil (M_0, K_0) make it fairly difficult to verify feasibility even with exact data. The graph on the left side in Figure 6.2 indicates boxplots of residuals over 20 samples of μ for each p. Assuming reasonably small residuals as an indication of feasibility, the second step is to search for the unique pair (M, K) within this "feasible set" that is nearest to (M_0, K_0). The graph on the right side in Figure 6.2 represent boxplots of the corresponding objective values $J(M, K)$. It is interesting to note how the objective values deteriorate rapidly as the perturbation of eigenvalues for the structured pencil increases in the sense of its variance. Keep in mind, however, the large magnitude of entries in K_0. The relative discrepancy $\left(\frac{\|M_0-M\|_F}{\|M_0\|_F}, \frac{\|K_0-K\|_F}{\|K_0\|_F}\right)$ returned by the SDP approach is surprisingly reasonable.

The peculiarity in the BCSST*01 data makes the performance of the SDP procedure dubious. If the SDP solver `SeDuMi` is employed in YALMIP, for example, it returns the warning message "no sensible solution found". To avoid this confusion, we consider next the BCSST*02 pair which actually results from applying static condensation to the oil rig model. These are matrices of size 66×66 where M_0 is a diagonal matrix but K_0 is a dense matrix, both of which are symmetric and positive semi-definite matrices yet M_0 is nonsingular. There is still a considerable variation of magnitudes in the entries with $\|M_0\|_F \approx 8.2 \times 10^{-1}$ and $\|K_0\|_F \approx 5.3 \times 10^4$, but the eigenvalue computation by MATLAB is much more stable with $\|(\lambda M_0 - K_0)\mathbf{x}\|_2$ approximately of the order 10^{-11}. Given one computed eigenpair $(\lambda, \mathbf{x})$ with $\|(\lambda M_0 - K_0)\mathbf{x}\|_2 \approx 2.2 \times 10^{-11}$, we carry out a similar experiment using the SDP procedure and report the test results in Figure 6.3. Without the sparsity constraint of K_0 and the singularity of M_0 as for the BCSST*01 pair, the model updating for BCSST*02 seems more satisfactory.

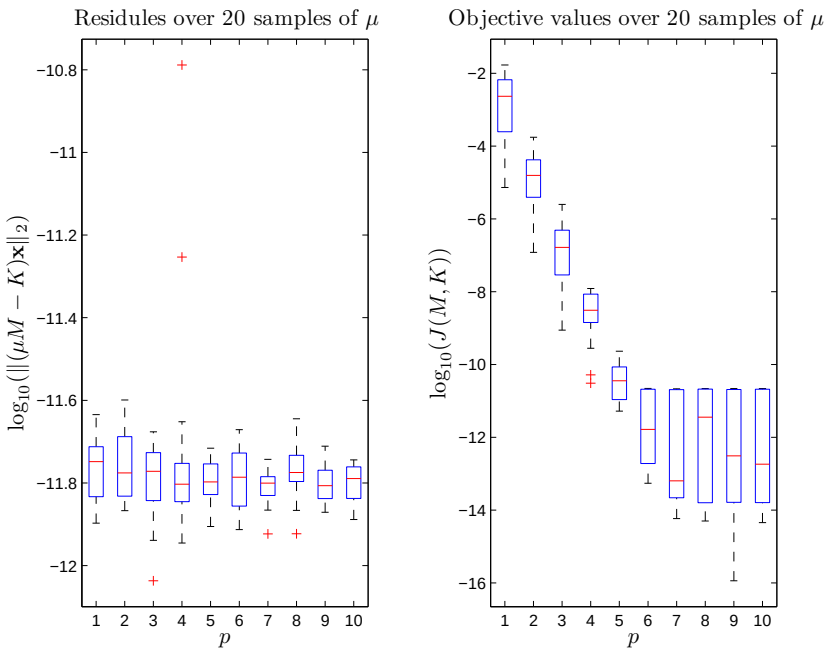

FIG. 6.3. *Performance of model updating for Harwell-Boeing test data BCSST*02 (n = 66).*

7. Conclusion. Without venturing into any mathematical details of SDP theory, but merely using existing SDP software packages, we have tried to convey, through this short communication, the applicability of SDP techniques to a variety of structured QIEPs. The QIEPs and the associated MUPs are of practical importance in sciences and engineering, but current developments in both theory and computation have been able to deal with only some rudimentary structures. We demonstrate how the otherwise very challenging issue of solvability and complicated task of computation for various structures, some of which have not even been considered in the literature, can be handled through just a few lines of coding in YALMIP. This SDP approach offers a unified and effectual avenue of attack on the QIEPs in general and the MUPs in particular, which we think deserves attention from practitioners in this field.

Finally, we have to point out one possible drawback of the SDP approach. Without special-purpose algorithms that better exploit the sparsity of the underlying problem, the interior point methods which are the main engine behind most SDP algorithms probably cannot handle large scale problems effectively. Todd [41] suggested that "The interior-point methods we have discussed can solve most problems with up to about a thousand linear constraints and matrices of order up to a thousand or so." Aside from this issue, we find that the uniformity and simplicity of SDP formulation for QIEPs are remarkably interesting and significant.

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