A MATHEMATICAL FRAMEWORK FOR THE LINEAR RECONSTRUCTOR PROBLEM IN ADAPTIVE OPTICS

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Abstract. The wave front field aberrations induced by atmospheric turbulence can severely degrade the performance of an optical imaging system. Adaptive optics refers to the process of removing unwanted wave front distortions in real time, i.e., before the image is formed, with the use of a phase corrector. The basic idea in adaptive optics is to control the position of the surface of a deformable mirror in such a way as to approximately cancel the atmospheric turbulence effects on the phase of the incoming light wave front. A phase computation system, referred to as a reconstructor, transforms the output of a wave front sensor into a set of drive signals that control the shape of a deformable mirror. The control of a deformable mirror is often based on a linear wave front reconstruction algorithm that is equivalent to a matrix-vector multiply. The matrix associated with the reconstruction algorithm is called the reconstructor matrix. Since the entire process, from the acquisition of wave front measurements to the positioning of the surface of the deformable mirror, must be performed at speeds commensurate with the atmospheric changes, the adaptive optics control imposes several challenging computational problems.

The goal of this paper is twofold: (i) to describe a simplified yet feasible mathematical framework that accounts for the interactions among main components involved in an adaptive optics imaging system, and (ii) to present several ways to estimate the reconstructor matrix based on this framework. The performances of these various reconstruction techniques are illustrated using some simple computer simulations.

1. Introduction.

Imaging through turbulence is a challenging task that has significant impacts on many important applications in defense, engineering and science. The presence of atmospheric turbulence, for example, has especially frustrated astronomers. In the absence of any correction to the turbulence, no design or optical quantity of a telescope can improve the degraded image. Atmospheric turbulence exists by simple factors such as mixing of warm and cold air layers that results in changes in air density. This non-uniformity of air density then slows the light wave forms by different degrees and hence distorts the image [18].

Researchers in science and engineering are actively seeking to overcome the degradation of astronomical image quality caused by the effects of atmospheric turbulence and other image degradation processes. Exciting technological breakthroughs are

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rapidly coming to the aid of scientists attempting to de-blur atmospheric images, see the survey articles [3, 10, 16]. New optical imaging methods are now being developed to give vital tools for ground-based, air-to-air, and air-to-ground imaging. Among these, one interesting approach is to perform turbulence compensation using mechanical means, now generally referred to as *adaptive optics* (AO). A quick briefing about the AO applications as well as current research activities in this area can be found in a recent SIAM News article [5]. Some more technical background can be found in [6, 7, 12, 19, 20] and the many references contained therein. The purpose of this paper is to recast some of the estimation methodology used in AO under a simple, unified mathematical framework.

In modern imaging facilities using AO technology, the improvement in optical image quality is most often attempted in two stages: The first stage involves real-time adaptive *deformable mirror* (DM) control. The idea is that at approximately the same time when the observed image is initially formed, optical systems also detect the distortions using either a natural guide star or a guide star artificially generated using range-gated laser backscatter. A *wave front sensor* (WFS) measures the optical phase distortions which can then be partially nullified by deforming a flexible mirror in the imaging system. Deformable mirrors operating in a closed-loop adaptive-optics system can partially compensate for the degradation effects of atmospheric turbulence. To be effective, these corrections have to be performed at real-time speed.

The second stage consists of off-line post-processing steps to restore the images. These steps involve the removal or minimization of noise or blur in an image using a priori knowledge about the degradation phenomena. This inverse problem, usually ill-posed and large-scaled, is generally solved by deconvolution techniques [2, 5, 18]. The study of post-processing image restoration needed for this second stage for reconstructing and restoring optical images is itself an area full of exciting and active mathematical research. The progress made is broad and significant. The power of these new techniques is very impressive. Some ongoing research subjects include regularization techniques, total variation techniques, phase diversity techniques, blind deconvolution methods, and many other articles presented in this conference.

The work in this paper concerns mainly the AO reconstructor problem involved in the first stage. The problem includes the determination of an optimal reconstructor matrix for phase reconstruction as well as a set of commands that control the surface of a DM. For ease of explanation, we sketch the main components of a closed-loop AO system in Figure 1.1. These include the deformable mirror, the wave front sensor, and the actuator command computer. Light in a narrow spectral band approaching the atmosphere from a distant light source, such as star, is usually modeled by a plane wave. When traveling through the atmosphere that does not have a uniform index of refraction, light waves are aberrated and no longer planar. In the closed-loop AO system depicted in Figure 1.1, this aberrated light is first reflected from the DM. Some of this light is focused to form an image, and some is diverted to the WFS that

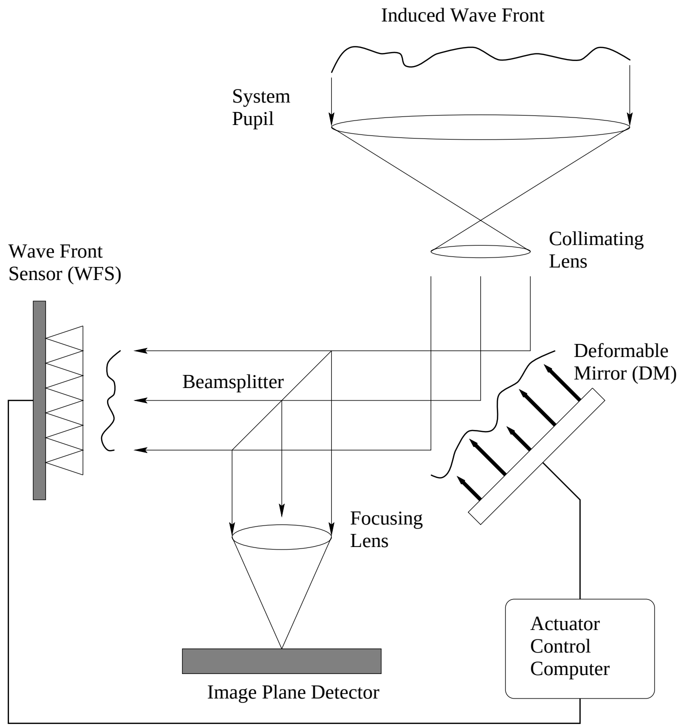

FIG. 1.1. *A simplified closed-loop AO system with main components.*

measures the wave front phase deformations. These WFS measurements are then fed to the actuator command computer that maps them into real time control commands for the DM. These control commands are used to adjust the DM actuators so as to compensate the wave front distortions. The underlying task is to somehow translate the atmospheric measurements to the actuator controls of the deformable mirrors. How this translation is done depends on the criterion selected, see [6, 11]. In this paper, several different types of linear estimators and their performances are discussed.

The basic problem in adaptive optics is thus to control the position of the surface of a deformable mirror in such a way as to approximately cancel the atmospheric turbulence effects on the phase of the incoming light wave front. A phase computation system, referred to as a *reconstructor*, transforms the output of a wave front sensor into a set of drive signals that control the shape of a deformable mirror. The control of a deformable mirror is often based on a linear wave front reconstruction algorithm that is equivalent to a matrix-vector multiply. The matrix associated with the reconstruction algorithm is called the *reconstructor matrix*.

The paper is organized as follows: We begin in Section 2 with a description of the dynamics associated with an AO system. We briefly discuss how the wave front phase is measured in terms of wave front slope, how the actuator command corrects the wave front phase and hence affects the wave front slope measurements, and how the measurements are gathered in a closed-loop AO system. The mathematics connecting these quantities sets forth several basic relationships that play fundamental roles in the reconstructor problem. Since wave front phase profiles normally are not available for real time processing, we discuss in Section 3 different approaches to estimate the wave front profile from observed wave front slope information. In Section 4, we discuss various algorithms to update the actuator control system using currently measured atmospheric conditions. These algorithms differ on the type of objective criterion selected. An important issue to be considered is the availability of information required by a particular objective criterion. If such required information is not available then the algorithm may incur higher computational cost. Finally, in Section 5 we illustrate and compare the performance of these various approaches via some simple computer simulations.

Although classical image restoration has been extensively studied [1, 2, 4, 13], aero-optics imaging through the atmosphere is a very difficult task. Some of the challenges in imaging processing through atmospheric turbulence include [3, 10, 11, 16, 22]:

1. *The real-time measurement of the atmospheric turbulence.* The statistical information of the atmosphere varies on time scales of minutes and will never be known exactly. The associated parameters therefore have to be estimated adaptively for optimal performance.
2. *The use of the atmospheric measurements to actuate the control systems of the deformable mirrors.* Since these measurements are time-varying and are only an estimate, the degree of validity that this information can be used in the communication among actuator, sensor, and control computer needs to be carefully identified, managed, and evaluated.
3. *The fast performance of large-scale data processing and computations.* In the first stage, the statistical information of the atmospheric turbulence needs to be processed in real-time to alter the shape of the DM to counteract distortions. In the post-processing stage, gigabytes of data need to be analyzed in order to enhance the image partially corrected by the earlier AO procedure [2, 5, 18].

In this paper, we limit ourselves to mathematical and statistical models only. Readers are referred to the excellent books by Roggemann and Welsh [18] and Hardy [11] for a discussion on factors that contribute to the limitations that keep an AO system from achieving its ideal performance.

2. Basic Relationships. For convenience, we shall denote the turbulence-induced phase profile at position $\vec{x}$ in the telescope aperture plane, determined by the primary mirror, at time t by $\phi(\vec{x}, t)$. Likewise, the deformable mirror command issued at time t for the ith DM actuator is denoted by $a_i(t)$. The wave front slope sensor measurement obtained from the kth subaperture of the WFS with *no* correction at time t is denoted by $s_k(t)$. The goal in positioning the DM surface via commands $a_i(t)$ is to represent an approximate conjugate of the turbulence-induced field $\phi(\vec{x}, t)$ so that the field reflected from the DM will have the aberration somewhat canceled and more closely approximate the field when no atmosphere turbulence is present. In this section we discuss the mathematical models representing these quantities.

The mirror surface is controlled by a number of actuators that basically push and pull on the mirror surface to cause it to deform. Assuming that there are m actuators and that the actuators response linearly to the commands, the DM surface can be modeled by

$$\hat{\phi}(\vec{x}, t) = \sum_{i=1}^{m} a_i(t) r_i(\vec{x}), \tag{2.1}$$

where $r_i(\vec{x})$, called the influence function on the DM surface at position $\vec{x}$, denotes the response of the ith actuator to a unit adjustment. Suppose we sample the DM surface at n surface positions $\vec{x}_j$, $j = 1, \ldots n$, then the relationship between the surface position and the actuator command can be described aS

$$\hat{\phi}(t) = Ha(t). \tag{2.2}$$

In the above, the n dimensional vector $\hat{\phi}(t) = [\hat{\phi}(\vec{x}_1, t), \ldots, \hat{\phi}(\vec{x}_n, t)]^T$ represents the discrete corrected phase profile at time t. The $n \times m$ *DM configuration matrix* H, whose ith column is the vector $[r_i(\vec{x}_1), \ldots, r_i(\vec{x}_n)]^T$, is independent of time.

The wave front sensors usually do not measure the wave front phase $\phi(t)$ directly. Instead, the spatial gradient of $\phi(t)$, commonly referred to as the wave front slope, is estimated. Without given specific details, we shall use the Hartman WFS (H-WFS) in this discussion. Readers are referred to [18] for more details on the physical configuration a H-WFS. In brief, the H-WFS spatially segments the incident wave front with an array of ℓ small regions in the telescope pupil. Each array element, referred to as a subaperture, focuses a spot onto an array of detectors in the focal plane. The average wave front slope associated with the kth subaperture given by

$$\begin{aligned} s_k(t) &= \int d\vec{x} W_k(\vec{x}) \nabla \phi(\vec{x}, t) \\ &= -\int d\vec{x} \nabla W_k(\vec{x}) \phi(\vec{x}, t), \end{aligned} \tag{2.3}$$

where $W_k(\vec{x})$ is the kth subaperture weighting function, accounts for the H-WFS slope measurement. Upon approximating the integral in (2.3) by some quadrature rules at designated positions $\vec{x}_j$, $j = 1, \ldots n$, together with possible measurement noises, a linear relationship between wave front phase and the H-WFS slope measurement can be described as

$$s(t) = W\phi(t) + \epsilon(t). \tag{2.4}$$

In the above, $\phi(t) = [\phi(\vec{x}_1, t), \ldots, \phi(\vec{x}_n, t)]^T$ represents the discrete phase profile at time t, the matrix $W = [w_{kj}] \in R^{\ell \times n}$ where w_{kj} denotes the jth quadrature weight of the integral (2.3) at abscissa $\vec{x}_j$, and $\epsilon(t)$ accounts for any measurement error or noise.

The corresponding H-WFS slope measurement of the corrected wave front phase $\hat{\phi}(t)$ can be measured as follows:

$$\hat{s}_k(t) = \sum_{i=1}^{m} \underbrace{\left(-\int d\vec{x}(\nabla W_k(\vec{x}) r_i(\vec{x})\right)}_{G_{ki}} a_i(t).$$

Once again, upon discretization, we can write

$$\hat{s}(t) = Ga(t) \tag{2.5}$$

where the matrix $G = [G_{ki}] \in R^{\ell \times m}$ must satisfy the relationship

$$WH = G. \tag{2.6}$$

It should be noted that the DM actuators are *not* capable of producing the exact wave front phase $\phi(\vec{x}, t)$ due to their finiteness of degrees of freedom. So $\hat{s} = Ga$ is never an exact measurement in practice.

In a closed-loop AO system such as the one demonstrated in Figure 1.1, the wave front that arrives at either the H-WFS or the image plane detector is the one that has been reflected from the DM. Thus the information obtained at the image plane detector is actually the residual phase error

$$\Delta\phi(t) := \phi(t) - Ha(t). \tag{2.7}$$

That is, after the AO correction, $\Delta\phi(t)$ is the observable instantaneous wave front distortion at time t. Likewise, the information available at the H-WFS is the feedback applied to $s(t)$ by DM actuator adjustment

$$\Delta s(t) := s(t) - Ga(t). \tag{2.8}$$

This is the observable H-WFS slope measurement at time t. Given the relationship (2.4), it is easy to see that an identical linear relationship

$$\Delta s(t) = W\Delta\phi(t) + \epsilon(t) \tag{2.9}$$

holds between the residual phase error $\Delta\phi(t)$ and and the feedback H-WFS slope measurement $\Delta s(t)$.

3. Minimum Variance Estimator. The ultimate goal in an AO system is to retrieve sufficient information about the turbulence-induced wave front phase $\phi(t)$ in order to determine the actuator command vector $a(t)$ to control the shape of the DM. In this section, we present four possible ways to estimate the wave front phase based on the mathematical framework outlined in §2.

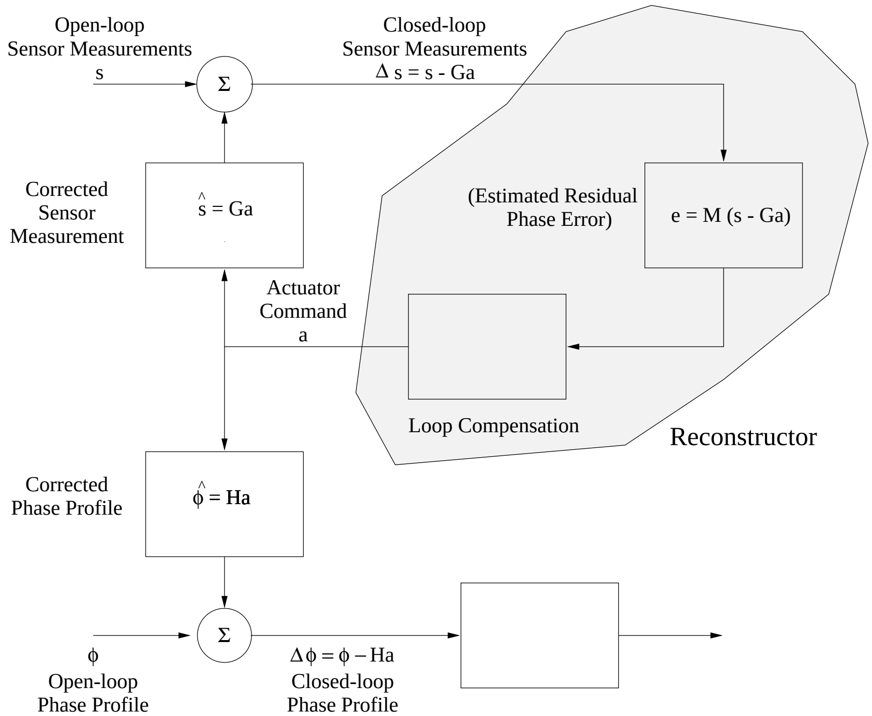

FIG. 3.1. *A close-loop AO control model*

We are dealing with atmospheric turbulence that is usually random in nature. It is convenient to adopt the conventional notation and terminology from statistics for the discussion in the sequel. That is, let $\mathcal{E}[x]$ denote the mathematical expectation from the underlying distribution function $p(x)$ of a random variable x. Define the mean $\mu_x := \mathcal{E}[x]$, the covariance $V_x := \mathcal{E}[(x - \mu_x)(x - \mu_x)^T]$, and so on [15].

The problem at hand is equivalent to finding a mapping from H-WFS measurements to the actuator control so as to result in the desired system performance. A schematic control-loop diagram corresponding to the AO system discussed in §1 is sketched in Figure 3.1. We have already described how a given actuator command $a(t)$ affects the residual phase error $\Delta\phi(t)$ and the feedback H-WFS slope measurement $\Delta s(t)$. The gray region in Figure 3.1 is meant to highlight the reconstructor problem of interest. Extensive research efforts are still ongoing in this direction [11, 20].

The spatial variations induced by the atmosphere in the optical path length between the object and telescope are generally recognized as the physical origin of the aberration. The primary function of deformable mirrors is to mechanically adjust the optical path length so as to partially compensate for this aberration before light is focused into an image. Therefore, it seems intuitive that the change $a(t)$ should be "proportional" (via the transformation H) to the residual phase error $\Delta\phi(t)$. However, in most AO systems the residual phase error $\Delta\phi(t)$ is not available directly, either because that such a quantity is not measurable from an image plane detector or because that its relationship to the image is highly nonlinear and cannot presently be inverted in real time. Indeed, the only actual measurement that is available in real time is $\Delta s(t)$. In fact, because of the delay needed to read out the H-WFS detectors, even the value of $\Delta s(t)$ itself is an integrated measurement of the average of the turbulence over the time interval $[t - \frac{\Delta t}{2}, t + \frac{\Delta t}{2}]$ where Δt is the time between successive H-WFS measurements. For the purpose of deriving the deformable mirror actuator command from $\Delta\phi(t)$ in real time, it becomes necessary to effectively estimate $\Delta\phi(t)$ from $\Delta s(t)$ based on the model (2.9).

We first present a classical result that plays a fundamental role throughout the discussion. The proof can be found in [14, 15]

THEOREM 3.1. *Suppose that the observation z is related to two uncorrelated random vectors β and η by the linear form*

$$z = B\beta + \eta. \tag{3.1}$$

Assume that $\mu_\eta = 0$. Then the vector

$$\hat{\beta} = \mu_\beta + A(z - B\mu_\beta), \tag{3.2}$$

where

$$A := \left[B^T V_\eta^{-1} B + V_\beta^{-1}\right]^{-1} B^T V_\eta^{-1}, \tag{3.3}$$

is the best unbiased, linear minimum variance estimate of β in the sense that $\mathcal{E}[\|\hat{\beta} - \beta\|_2^2]$ is minimized.

The obvious advantage of the above estimator $\hat{\beta}$ is that a full probabilistic description such as the distribution function for β or η is not required. Only the first and second statistical moments of β and η are needed. Knowledge about these moments often can be gathered from enough experimental samples. The disadvantage of this approach is that, in practice, it is fairly difficult to evaluate the inverses involved in the closed form (3.3) of the reconstructor matrix A.

The following result is an alternative way to evaluate the reconstructor A. In particular, it avoids information of moments about η and allows us to estimate the matrix A adaptively, with the arrival of any additional measurements.

COROLLARY 3.2. *Under the same assumption as in Theorem 3.1, the unbiased, linear minimum variance estimator $\hat{\beta}$ given in (3.2) can be computed equivalently from*

$$\hat{\beta} = \mu_\beta + \mathcal{E}[(\beta - \mu_\beta)(z - \mu_z)^T] \left(\mathcal{E}[(z - \mu_z)(z - \mu_z)^T]\right)^{-1} (z - \mu_z). \tag{3.4}$$

Proof. It is easy to see that

$$\begin{aligned} \mathcal{E}[(\beta - \mu_\beta)(z - \mu_z)^T] &= \mathcal{E}[(\beta - \mu_\beta)((B(\beta - \mu_\beta) + \eta)^T] \\ &= V_\beta B^T \end{aligned}$$

and that

$$\mathcal{E}[(z-\mu_z)(z-\mu_z)^T] = \mathcal{E}[(B(\beta-\mu_\beta)+\eta)(B(\beta-\mu_\beta)+\eta)^T]$$
$$= BV_\beta V_\beta^T B^T + V_\eta.$$

The matrix identity

$$V_\beta B^T \left(BV_\beta V_\beta^T B^T + V_\eta\right)^{-1} = \left[B^T V_\eta^{-1} B + V_\beta^{-1}\right]^{-1} B^T V_\eta^{-1}$$

is easily established by postmultiplying by $BV_\beta V_\beta^T B^T + V_\eta$ and premultiplying by $B^T V_\eta^{-1} B + V_\beta^{-1}$. □

If it is known a priori that the variables involved are Gaussian, then the density functions are completely determined by the information of the first two moments. In this case, the following theorem offers another interesting interpretation [15].

THEOREM 3.3. *Suppose the variables β and η in (3.1) are Gaussian. Then the linear minimum variance estimator $\hat{\beta}$ defined in (3.2) is the optimum when compared with any other linear or nonlinear estimators of β. Furthermore,*

$$\mu_{\beta|z} = \hat{\beta}, \tag{3.5}$$

$$V_{\beta|z} = \left[B^T V_\eta^{-1} B + V_\beta^{-1}\right]^{-1}. \tag{3.6}$$

Thus the estimator $\hat{\beta}$ maximizes the likelihood function $p(\beta|z)$, i.e., the conditional density function of β for observation z.

In the context of optical imaging, the random quantities involved often are Gaussian variables. The minimum variance estimator given by Theorem 3.1 therefore offers considerable tractability for wave front reconstructor problems. We now describe several possible approaches to the reconstructor problem:

Estimator 1. In the open-loop AO system, shown in Figure 3.2, where the received $s(t)$ is first measured in its uncorrected state, an estimate of $\phi(t)$ can be computed from wave front slope information $s(t)$ based on the model (2.4). From Theorem 3.1, the minimum variance estimator of ϕ at a fixed time t is given by

$$\hat{\phi} = \mu_\phi + A(s - W\mu_\phi) \tag{3.7}$$

with

$$A := \left[W^T V_\epsilon^{-1} W + V_\phi^{-1}\right]^{-1} W^T V_\epsilon^{-1}. \tag{3.8}$$

This is the ideal situation when the open-loop measurement s is available and when the stochastic information on the first two moments of ϕ and ϵ is known. Once the turbulence-induced wave front is estimated, the required corrections can then be computed and fed to the wave front compensation device. Obviously, the time required for the measurement and correction process must be less than the time change of the

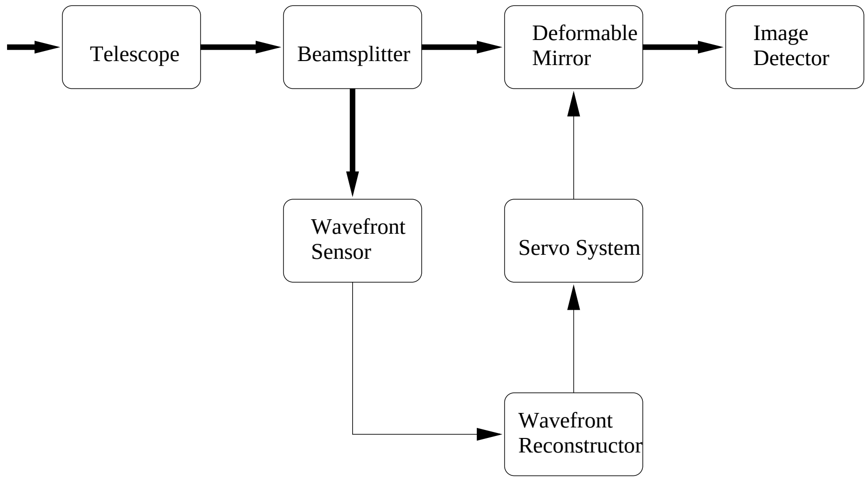

FIG. 3.2. *An open-loop AO configuration*

wave front. Also, this "one-shot" signal process requires that the correction loop be calibrated accurately; otherwise, any error in these components will show up directly in the optical output [11].

In a closed-loop environment, one possible scenario is that if simultaneous measurements Δs, $\Delta\phi$, and a are all available, then the open-loop information might be computed in principle from, for example, relationships such as (2.7) and (2.8). We may then proceed using (3.7) to retrieve information about ϕ. However, one should be cautious that, since G, H, and $a(t)$ are only known approximately, there might exist some long-term drifts and biases in the system. Some schemes more sophisticated than merely using (2.7) or (2.8) should be used instead, as we shall begin to describe below.

Estimator 2. As mentioned earlier, the residual phase error $\Delta\phi(t)$ is generally not available in real time. In a closed-loop environment using the H-WFS model, the relationship (2.9) can be used in a similar way as Estimator 1 to estimate $\Delta\phi(t)$ from $\Delta s(t)$, i.e.,

$$\Delta\hat{\phi} = \mu_{\Delta\phi} + A(\Delta s - W\mu_{\Delta\phi}), \tag{3.9}$$

where

$$A := \left[W^T V_\epsilon^{-1} W + V_{\Delta\phi}^{-1}\right]^{-1} W^T V_\epsilon^{-1}, \tag{3.10}$$

provided $\mu_{\Delta\phi}$ and $V_{\Delta\phi}$ are known.

In practice, the covariance matrices needed in (3.10) are not known a priori [6, 7, 8]. The dimensionality of observable data ranging from hundreds to thousands also makes it impractical to compute A using its closed form (3.10). Thus the challenge is to compute the reconstructor A in real time using measured data.

One possible approach is to employ recursive least squares techniques to estimate A. The following formulation was suggested to us by Ellerbroek [9]. A similar estimation can be established for the open-loop calculation required in (3.8). In order to afford the possibility of following the statistical variations of the observable data in a nonstationary environment such as the atmosphere, we define *ensemble averages* over the measurements by

$$\overline{\Delta\phi}_n = \overline{\Delta\phi}_{n-1} + \alpha(\Delta\phi_n - \overline{\Delta\phi}_{n-1}), \tag{3.11}$$

$$\overline{\Delta s}_n = \overline{\Delta s}_{n-1} + \alpha(\Delta s_n - \overline{\Delta s}_{n-1}), \tag{3.12}$$

where $\alpha \in (0,1)$ plays the role of the so called *forgetting factor* [12]. We also define *ensemble covariance* and *ensemble variance* matrices, respectively, by

$$B_n = (1-\alpha)\left[\alpha(\Delta\phi_n - \overline{\Delta\phi}_{n-1})(\Delta s_n - \overline{\Delta s}_{n-1})^T + B_{n-1}\right], \tag{3.13}$$

$$C_n = (1-\alpha)\left[\alpha(\Delta s_n - \overline{\Delta s}_{n-1})(\Delta s_n - \overline{\Delta s}_{n-1})^T + C_{n-1}\right]. \tag{3.14}$$

In the case of a stationary environment, we have the following results. The proofs are straightforward with tedious algebraic manipulation.

THEOREM 3.4. *Suppose $\{\Delta s_j\}$ and $\{\Delta\phi_j\}$ are random samples, respectively, from some stationary distributions for Δs and $\Delta\phi$. Then*

$$\begin{aligned} \mathcal{E}[\overline{\Delta\phi}_n] &= \mu_{\Delta\phi}, \\ \mathcal{E}[\overline{\Delta s}_n] &= \mu_{\Delta s}, \\ \lim_{n\to\infty} V_{\overline{\Delta\phi}_n} &= \frac{\alpha}{2-\alpha} V_{\Delta\phi}, \\ \lim_{n\to\infty} V_{\overline{\Delta s}_n} &= \frac{\alpha}{2-\alpha} V_{\Delta s}, \end{aligned}$$

i.e., the ensemble averages $\overline{\Delta\phi}_n$ and $\overline{\Delta s}_n$ are unbiased estimates of $\Delta\phi$ and Δs, but carry a considerable size of variances even as n goes to infinity.

THEOREM 3.5. *Under the same assumption as in Theorem 3.4, the expected value of ensemble covariance matrix of $\Delta\phi$ and Δs converges, i.e.,*

$$\lim_{n\to\infty} \mathcal{E}[B_n] = \frac{2(1-\alpha)}{2-\alpha} \mathcal{E}[(\Delta\phi - \mu_{\Delta\phi})(\Delta s - \mu_{\Delta s})^T]. \tag{3.15}$$

THEOREM 3.6. *Under the same assumption as in Theorem 3.4, the expected value of ensemble variance matrix of Δs converges, i.e.,*

$$\lim_{n\to\infty} \mathcal{E}[C_n] = \frac{4(1-\alpha)^2}{(2-\alpha)^2} \mathcal{E}[(\Delta s - \mu_{\Delta s})(\Delta s - \mu_{\Delta s})^T]. \tag{3.16}$$

Theorems 3.5, 3.6 together with Corollary 3.2 imply that, in a stationary environment, the reconstructor matrix A given in (3.10) is given by

$$A = \left(\frac{2-2\alpha}{2-\alpha}\right) \lim_{n\to\infty} \mathcal{E}[B_n](\mathcal{E}[C_n])^{-1}. \tag{3.17}$$

This relationship suggests that it might be reasonable to approximate the reconstructor matrix A from the product $B_n C_n^{-1}$. The point to make is that the rank-one update of C_n defined in (3.14) makes it possible to employ the Sherman-Morrison formula to facilitate the computation of its inverse.

Estimator 3. It is sometimes reasonable to assume that the elements in the noise vector ϵ are independent. Thus assuming $V_\epsilon = \sigma^2 I$, we reduce (3.8) to

$$A = [W^T W + \sigma^2 V_{\Delta\phi}^{-1}]^{-1} W^T. \tag{3.18}$$

This approach corresponds to some of the earliest techniques for the reconstructor problem, i.e., the reconstructor matrix A is essentially reduced to the Moore-Penrose generalized inverse $W^\dagger$ of W. Here, the residual phase error is estimated from

$$\Delta\hat{\phi} = (W^T W)^{-1} W \Delta s, \tag{3.19}$$

the solution to the least squares problem where $\|\Delta s - W\Delta\phi\|_2$ is minimized. From (3.18) we see that (3.19) is essentially valid when the noise variance $\sigma^2 I$ decreases to zero, i.e., no noise, or when no a priori information about the variable $\Delta\phi$ is known, i.e., $V_{\Delta\phi}^{-1} = 0$.

Estimator 4. Suppose we want to estimate the residual phase error using an equation of the form

$$\Delta\hat{\phi} = E\Delta s, \tag{3.20}$$

for some reconstructor matrix E. For the estimate to remain unbiased, it is necessary to require

$$EW = I. \tag{3.21}$$

Let $\mathcal{A}$ stand for the *vector space of actuator commands*, $\mathcal{S}$ stand for the *vector space of* H-WFS *slope measurements*, and Φ stand for the *vector space of phase profiles*. The diagram shown in Figure 3.3 clarifies the relationship between the various transformations involved. Recall that we already have the relationship $WH = G$. It follows that

$$EG = H. \tag{3.22}$$

It is known that $WW^\dagger$ is an orthogonal projection operator onto the range space of W. Define

$$\delta s = (I - WW^\dagger)\Delta s, \tag{3.23}$$

$$\delta\phi = \Delta\phi - W^\dagger \Delta s. \tag{3.24}$$

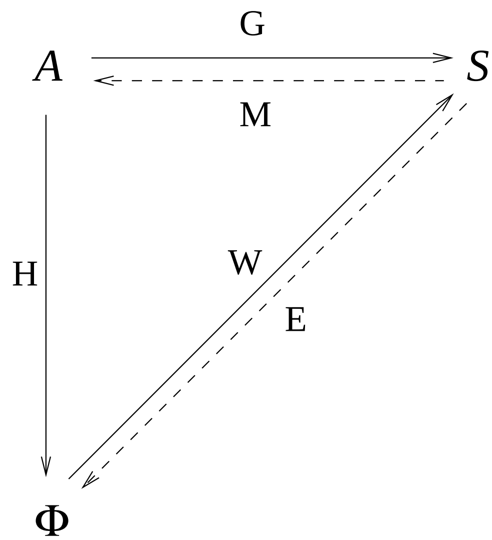

FIG. 3.3. *Diagram of mutual relationship*

Then δs may be interpreted as the orthogonal component of Δs in the direction perpendicular to the range space of W. Note also that $\delta\phi$ is precisely the difference between $\Delta\phi$ and its nominal least squares approximation $\Delta\hat{\phi}$ given by (3.19) in Estimator 3. It is easy to see from the model (2.9) that the relationship

$$\delta s = W\delta\phi + \epsilon \tag{3.25}$$

holds between δs and $\delta\phi$. Therefore, using a recursive least squares procedure similar to that described in Estimator 2, we can compute a reconstructor matrix $E_\perp$,

$$E_\perp = \mathcal{E}[(\delta\phi)(\delta s)^T]\left(\mathcal{E}[(\delta s)(\delta s)^T]\right)^{-1}. \tag{3.26}$$

The theory asserts that the quantity

$$\delta\hat{\phi} = E_\perp \delta s \tag{3.27}$$

is the best, *possibly biased*, linear minimum variance estimate of $\delta\phi$ in the sense that

$$\mathcal{E}[\|E_\perp\delta s - \delta\phi\|_2^2] \tag{3.28}$$

is minimized [14]. Now define

$$E := W^\dagger + E_\perp(I - WW^\dagger). \tag{3.29}$$

It is clear that condition (3.21) is satisfied. Furthermore,

$$\begin{aligned} \Delta\phi - E\Delta s &= \Delta\phi - \left(W^\dagger + E_\perp(I - WW^\dagger)\right)\Delta s \\ &= (\Delta\phi - W^\dagger\Delta s) - E_\perp(I - WW^\dagger)\Delta s \\ &= \delta\phi - E_\perp\delta s. \end{aligned} \tag{3.30}$$

From (3.28) we see that with the reconstructor E defined in (3.29), the quantity

$$\mathcal{E}[\|\Delta\phi - E\Delta s\|_2^2] \tag{3.31}$$

is minimized. This approach is significant in reducing the cost of computing $E_\perp$ and still maintaining an unbiased, linear minimum variance estimator for $\Delta\phi$. This is a slight modification of the approach due to Ellerbroek and Rhoadarmer [8].

4. Actuator Control. In the preceding section we discussed several ways to estimate $\Delta\phi(t)$ (or $\phi(t)$ in an open-loop environment.) Suppose an estimate to this quantity $\Delta\phi(t)$ is available. The question now is how this estimate should be utilized to obtain the *actuator command vector* $a(t)$. To measure the performance of the estimator, we assume that the vector space Φ of phase profiles is a Hilbert space with the weighted inner product

$$\langle f, g\rangle := f^T\Omega g \tag{4.1}$$

where Ω is a symmetric and positive definite weighting matrix. In this section, we address several possible strategies for this critical control issue.

Control 1. The ultimate goal of DM control is to minimize the residual phase error $\Delta\phi(t)$. Assume first that we know the wave front phase $\phi(t)$. Then the most intuitive control computation based on the model (2.4) would be

$$a(t) := (H^T\Omega H)^{-1}H^T\Omega\phi(t), \tag{4.2}$$

which minimizes $\langle\Delta\phi, \Delta\phi\rangle$ as an actuator command.

Control 2. In an open-loop environment where noise is present in the measurement $\phi(t)$, one can consider using either the estimator $\hat{\phi}(t)$ defined in (3.7) to configure the DM, or computing the control command $a(t)$ directly based on the WFS slope measurement $s(t)$. The former approach is similar to the closed-loop control which will be discussed later. We explore the latter idea as follows.

The problem is to compute a reconstructor matrix M to formulate an actuator command vector in the form

$$a(t) = Ms(t). \tag{4.3}$$

See Figure 3.3 for the role of M. In some simple models, it might be feasible to consider using the least square solution

$$a(t) := (G^TG)^{-1}G^Ts(t) \tag{4.4}$$

that minimizes $\|\Delta s\|_2^2$ as an actuator command. More generally, however, it is desired to issue a command $a(t)$ so that the residual phase error $\mathcal{E}[\langle\Delta\phi(t), \Delta\phi(t)\rangle]$ is minimized. Observe that

$$\begin{aligned}\sigma^2(M) &:= \mathcal{E}[\langle\phi - HMs, \phi - HMs\rangle] \\ &= \mathcal{E}[\langle\phi, \phi\rangle] - 2\langle H^T\Omega\mathcal{E}[\phi s^T], M\rangle_F + \langle M\mathcal{E}[ss^T]M^T, H^T\Omega H\rangle_F\end{aligned} \tag{4.5}$$

where we have used $\langle \cdot, \cdot \rangle_F$ to denote the Frobenius inner product for matrices. It is easy to see that the gradient of $\sigma^2(M)$ is given by

$$\nabla \sigma^2(M) = -2H^T \Omega \mathcal{E}[\phi s^T] + (H^T \Omega H) M \mathcal{E}[ss^T]. \tag{4.6}$$

It follows that the optimal command vector $a(t)$ is given by

$$a(t) = \left(H^T \Omega H\right)^{-1} \left(H^T \Omega \mathcal{E}[\phi s^T]\right) \left(\mathcal{E}[ss^T]\right)^{-1} s(t). \tag{4.7}$$

Obviously, to implement the above actuator control system requires detailed knowledge of statistics for both $\phi(t)$ and $s(t)$. This could prove to be a major drawback in practice.

Control 3. In a closed-loop environment, $\Delta\phi$ represents the residual error after the correction by the current DM command, which we denote by a_c. To improve the image, we expect that the new DM command a_+ should help to reduce the residual error. Suppose that the phase profile $\phi(t)$ has been stationary, then ideally we would like have $Ha_+ = \phi$ and hence

$$H\Delta a = \Delta\phi \tag{4.8}$$

where $\Delta a = a_+ - a_c$. This motivates us to consider the problem of minimizing the quantity $\langle H\Delta a - \Delta\phi, H\Delta a - \Delta\phi \rangle$. If we have perfect knowledge of $\Delta\phi$, then the new command is given by

$$a_+ = a_c + (H^T \Omega H)^{-1} H^T \Omega \Delta\phi. \tag{4.9}$$

This formulation is analogous to (4.2). One must realize, however, that in a closed-loop system the measured wave front slope is $\Delta s(t)$ and that $\Delta\phi$ is generally not available. Even so, we shall see below that (4.9) provides important insights into how the actuator command $a(t)$ should be updated.

Control 4. Thus far, we have assumed that the actuator control system would respond instantaneously to the the WFS slope measurements $s(t)$. In reality, there is a finite temporal delay due to the servo control loop, including, for example, time needed to read out the WFS information and to process the wave front data. Any time delay between the measurement and correction of a wave front disturbance results in a temporal error. Thus, when applying the next actuator command, one should also somehow try to compensate the effects of time delays. This is done in principle by predicting what the wave front structure will be at the time it is compensated. Since the correlation of atmospherically distorted wave fronts decays with time, one simple way of prediction is to use the current value of the wave front, with a decay factor, as the best estimate of the next sample. In AO systems, this idea is implemented using a temporal integrator or low-pass filter. We first explore the temporal integrator approach for a closed-loop system.

Recall that in a closed-loop system $\Delta\phi$ is generally not available. Realizing that the change of the DM command should still maintain a relationship similar to (4.9), we assume that the DM actuator command $a(t)$ follows the control law

$$\frac{da}{dt} = k\Delta\phi = kM(s - Ga) \tag{4.10}$$

where M is a constant matrix and k is a constant scalar. The differential equation is meant to serve as a servo-loop compensator because $a(t)$ is "filtered" before it is applied to the deformable mirror [6]. To be physically feasible, one has to assume that all eigenvalues of the matrix product MG have positive real part. See Figure 3.3 again for the role of M. It follows that the steady-state solution is given by

$$a(t) = \int_0^\infty e^{-kMG\tau} kMs(t-\tau)\, d\tau. \tag{4.11}$$

In [6], the linear constraint

$$MG = I \tag{4.12}$$

was imposed. In this case, we obtain

$$a(t) = My(t), \tag{4.13}$$

where

$$y(t) := \int_0^\infty e^{-k\tau} ks(t-\tau)\, d\tau \tag{4.14}$$

can be interpreted as the temporally filtered version of the slope measurements $s(t)$. The model (4.13) is exactly in the same format as (4.3) with $y(t)$ replacing $s(t)$, except that in (4.13) M must satisfy the constraint (4.12). Upon applying Lagrange multiplier techniques, it can be shown that, subject to the servo-loop compensator (4.10), the optimal command $a(t)$ that minimizes $\mathcal{E}[\langle \phi - Ha, \phi - Ha\rangle]$ is given by

$$a(t) = (R^{-1}BS^{-1} + (I - R^{-1}AS^{-1}G)(G^TSG)^{-1}G^TS^{-1})y(t) \tag{4.15}$$

where

$$R := H^T\Omega H, \tag{4.16}$$

$$B := H^T\Omega\mathcal{E}[\phi y^T], \tag{4.17}$$

$$S := \mathcal{E}[yy^T]. \tag{4.18}$$

Although the matrix

$$M = R^{-1}BS^{-1} + (I - R^{-1}AS^{-1}G)(G^TSG)^{-1}G^TS^{-1}$$

given in (4.15) is optimal, it is not necessarily practical for computation because of the inverses involved. One simple alternative, while still satisfying the constraint (4.12),

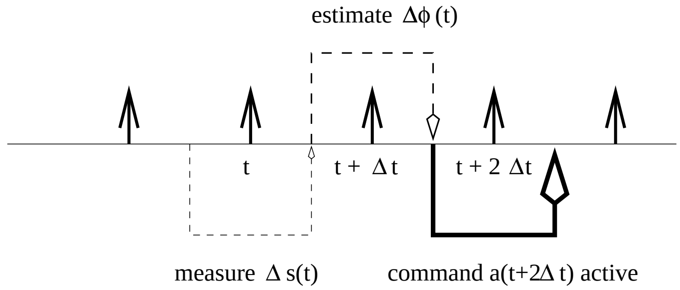

FIG. 4.1. *Time line for a 2-cycle delay AO system*

is to use model (4.4) where the slope measurement $s(t)$ is replaced by the temporally filtered slope measurement $y(t)$, i.e. [18],

$$a(t) = (G^T G)^{-1} G^T y(t). \tag{4.19}$$

The performance of this least squares solution that minimizes $\|y - Ga\|_2^2$ for a temporally filtered slope measurement $y(t)$ is yet to be evaluated.

Control 5. Because of the latency problem in AO systems, one must deal with the predicament of determining how to adjust the current actuator commands to correct the *future* phase profile ϕ which is not known at present. To illustrate the idea, we consider the case when there is a 2-cycle delay.

Let Δt denote the time between successive WFS measurements (and also the time between successive adjustments to the DM actuator commands.) Assume the AO system is operating in the following sequence of events: The photons which measure $\Delta s(t)$ are actually integrated on the WFS system over the interval $[t - 0.5\Delta t, t + 0.5\Delta t]$. The WFS information is then read out as slowly as possible to minimize the detector read noise over the interval $[t + 0.5\Delta t, t + 1.5\Delta t]$. The calculation of estimating $\Delta\phi(t)$ from $\Delta s(t)$ begins as soon as the first pixels are digitized shortly after $t + 0.5\Delta t$, but cannot be completed until the entire information has been read out just before $t + 1.5\Delta t$. At that point the command $a(t + 2\Delta t)$ is computed, sent to the DM, and remains in effect until before $t + 2.5\Delta t$. The time line for these series of events is depicted in Figure 4.1.

The new command $a(t + 2\Delta t)$ is usually computed by an autoregressive moving average (ARMA) process

$$a(t + 2\Delta t) := \sum_{k=0}^{p} c_k a(t + (1 - k)\Delta t) + \sum_{j=0}^{q} b_j M_j \Delta s(t - j\Delta t), \tag{4.20}$$

where the coefficients are derived to filter out some of the noise in the WFS measurements and to improve the stability of the control loop in the presence of latency and modeling errors. At this point, the computation of $a(t + 2\Delta t)$ becomes a classical single-input, single-out control problem.

surface positions n	$=$	5
number of actuators m	$=$	4
number of subapertures ℓ	$=$	3
size of random samples z	$=$	2500
H	$=$	$rand(n, m)$
W	$=$	$rand(\ell, n)$
G	$=$	WH
L_ϕ	$=$	$rand(n, n)$
L_ϵ	$=$	$diag(rand(\ell, 1))$
μ_ϕ	$=$	$zeros(n, 1)$
μ_ϵ	$=$	$zeros(\ell, 1)$

TABLE 5.1
Parameters used in simulation

It is interesting to note that the simplest filter, for example, is

$$a(t + 2\Delta t) = a(t + \Delta t) + b_0 M_0 \Delta s(t), \tag{4.21}$$

that uses the command $a(t+\Delta t)$, which was computed using information available just before $t + 0.5\Delta t$, as well as the most currently available $s(t)$. Note that (4.21) is quite similar to (4.9). Our speculation in (4.8) appears to indicate that this is a reasonable approach.

5. Numerical Simulation. Thus far we have presented a simple framework relating the main components in a general AO system. We have also described various models for estimating the wavefront profiles and for issuing the DM actuator commands. In this section, we want to demonstrate numerically how an AO system would behave using these models. We do not intend to give a comprehensive performance analysis. Our goal is simply to illustrate the formal mathematics behind these ideas. Therefore only Monte Carlo simulation of some small size problems is presented. In reality, the dimensions of authentic data are in hundreds.

For simulation purposes, we assume that both $\phi(t)$ and $\epsilon(t)$ are stochastically independent random variables with normal distributions. The parameters used for our simulation are listed in Table 5.1 where, for convenience, we use the MATLAB syntax $rand(n, m)$ to denote an $n \times m$ matrix with random entries chosen from a uniform distribution on the interval $(0, 1)$. Likewise, $randn$ denotes a normally distributed random variable with mean zero and variance one in the sequel. Define

$$\phi = \mu_\phi * ones(1, z) + L_\phi * randn(n, z),$$

where $*$ denotes standard matrix to matrix multiplication. Then we obtain z random samples for the multivariate normally distributed wavefront phase profile ϕ with mean

μ_ϕ and covariance matrix $V_\phi = L_\phi L_\phi^T$. In a similar way, the WFS measurement noises are simulated by

$$\epsilon = \mu_\epsilon * ones(1, z) + L_\epsilon * randn(\ell, z).$$

We remark that the covariance matrices in our simulation are derived from randomly generated L_ϕ and L_ϵ. In realistic AO evaluation, the covariacne matrices can be computed based on the turbulence models. See, for example, [6, 17, 21]. Define $s(t)$ according to model (2.4). It is easy to see that $s(t)$ enjoys a multivariate normal distribution with

$$\mu_s = W\mu_\phi + \mu_\epsilon,$$
$$V_s = WV_\phi W^T + V_\epsilon.$$

We first experiment with the estimators discussed in §3. To simulate the closed-loop environment, we assume that the configuration $a(t)$ of the DM actuator command at present time is $a(t) = randn(m, z)$, i.e., we assume each of the m actuators is stochastically independent and enjoys a normal distribution. The closed-loop simulation data $\Delta\phi(t)$ and $\Delta s(t)$ are then generated by using (2.7) and (2.8), respectively. The results of the various estimators discussed in Section 3 are plotted component by component in Figure 5.1. For example, we compare the distribution of the minimum variance estimator $\hat{\phi}$ of ϕ by using (3.7) with the original samples of ϕ at the upper-left corner of Figure 5.1. Likewise, the minimum variance estimator $\Delta\hat{\phi}$ of $\Delta\phi$ by using (3.9) is compared with the original samples of $\Delta\phi$ at the upper-right corner of Figure 5.1. The least squares estimator of $\Delta\phi$ using (3.19) and the estimator using (3.20) are compared with $\Delta\phi$, respectively, at the lower graphs in Figure 5.1. We see from repeated random simulations that all estimators proposed in this paper predict the original distribution of ϕ or $\Delta\phi$ reasonably well in the multivariate normally distributed case.

Next, we experiment with the actuator controls discussed in §4. Assume that we have perfect knowledge of ϕ. Then the control $a = H^\dagger\phi$ given by equation (4.2) should produce a minimum residual phase error. Using this ideal case as the basis, we compare in Figure 5.2 how other choices of control strategy will affect the resulting $\Delta\phi$. For each actuator command a, we plot the distributions of all n components of the resulting $\Delta\phi$ in the same frame. Note that these components are statistically correlated. One should notice from these graphs that while the control equation (4.4) minimizes $\|\Delta s\|^2$, it does not necessarily produce a good $\Delta\phi$. Notice also that the control equation (4.7) should produce the smallest $\mathcal{E}[\langle\Delta\phi(t), \Delta\phi(t)\rangle]$ among all possible controls in the form of (4.3). On the other hand, the control equation (4.9) corrects $\Delta\phi$ almost to the same effect as the ideal control equation (4.2).

Finally, to simulate the control with a 2-cycle delay, we assume that the distribution of ϕ stays stationary throughout the time interval when the iterations take place. We use estimator (3.19) as the reconstructor matrix for $\Delta\phi$. Together with (4.9), the matrix M_0 in (4.21) becomes $M_0 = H^\dagger W^\dagger$. For simulation purpose, we set $b_0 = 0.6$. Reported

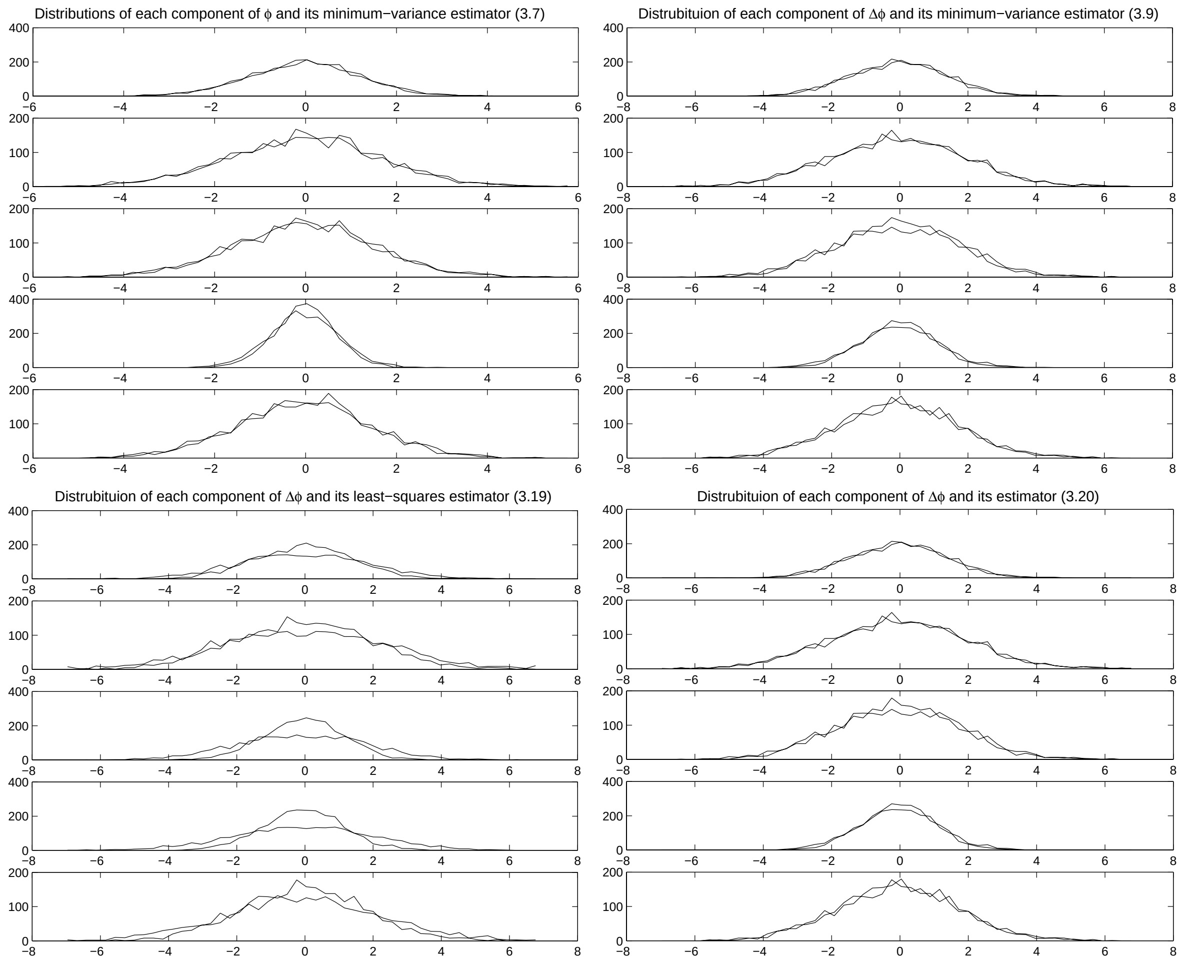

FIG. 5.1. *Estimating phase profiles from WFS slopes*

in Figure 5.3 is the dynamical behavior of means and variances of all n components of $\Delta\phi$ when the 2-cycle control (4.21) is used. To start out the iteration, two initial actuator commands must be given. We purposefully make bad initial guesses. It is encouraging to see that $\Delta\phi$ is eventually corrected to have mean approximately zero with constant (and smaller) variance. Note that because components of ϕ are internally correlated to begin with, one should not expect that all components of the corrected $\Delta\phi$ will have small variance simultaneously. The effect of the 20-th iteration of this delay control on $\Delta\phi$ is plotted at the bottom in Figure 5.2 for comparison with other types of controls.

6. Concluding Remarks. The randomness and time evolution of the atmospheric inhomogeneities make imaging through turbulence a difficult and challenging problem. Adaptive optics techniques afford a mechanical means of sensing and correct-

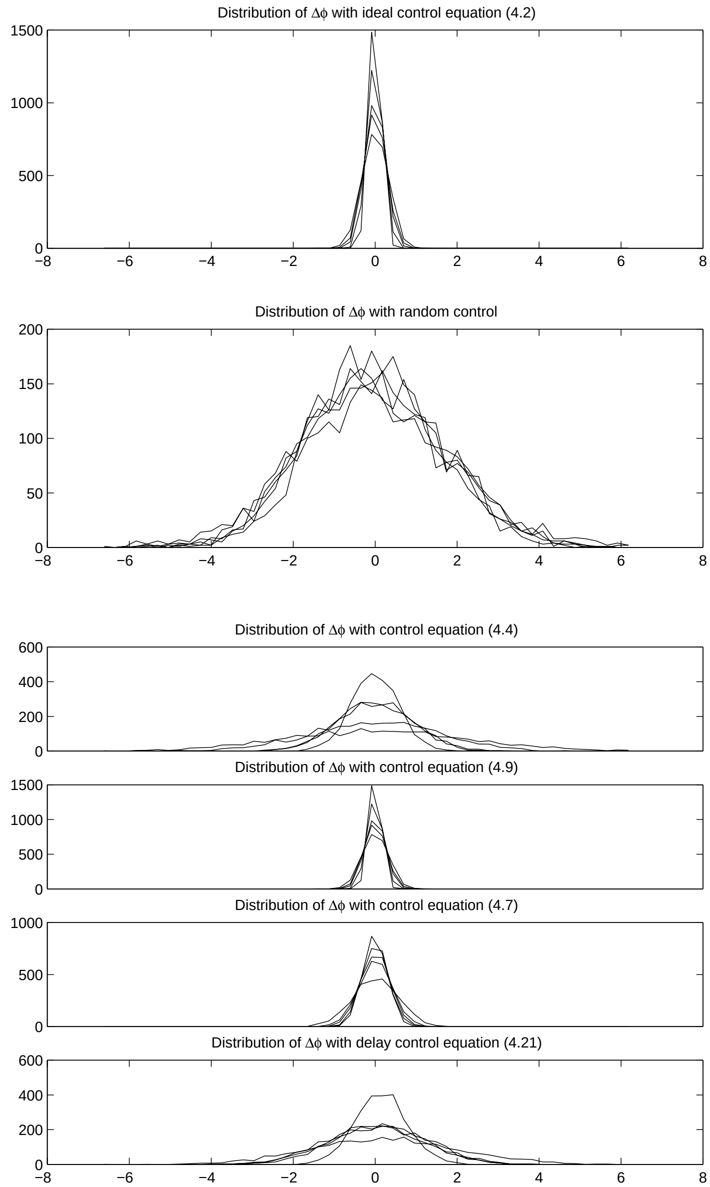

FIG. 5.2. *Effects on $\Delta\phi$ with different controls*

ing for turbulence effects as they occur. A simple mathematical framework connecting the major components of an AO system is outlined in this paper. From this framework, we set forth essential concepts of adaptive optics in terms of mathematical expressions. The discussion presented here integrates disparate viewpoints, notation, and analysis techniques. In particular, we describe the derivation of phase reconstruction matrices based on different types of objective criteria. It appears from repeated numerical simulations that all estimators proposed in this paper predict the original distribution of

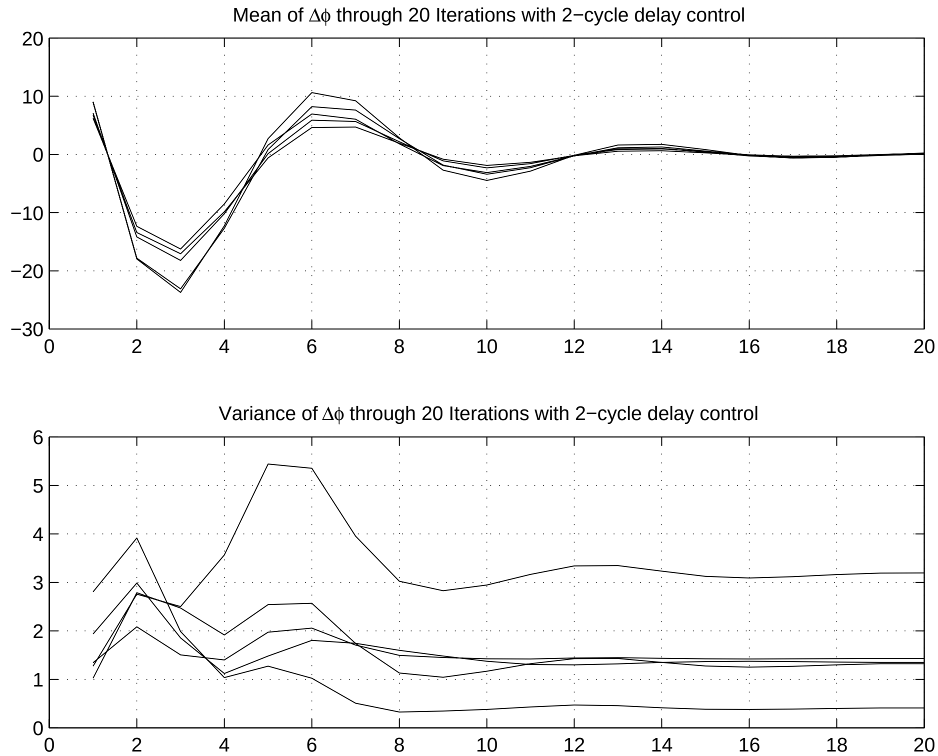

FIG. 5.3. *Convergence of* $\Delta\phi$ *with 2-cycle delay control (4.21)*

ϕ or $\Delta\phi$ reasonably well, at least in the multivariate normally distributed case. Cost effectiveness, not addressed in this paper, might become another important factor to be considered in real AO systems. On the other hand, under the situation where only closed-loop WFS information is available, the delay control scheme (4.20) appears to be able to correct the residual phase error $\Delta\phi$ competitively with any other controllers, provided coefficients in the scheme are properly selected.

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