ON INVERSE QUADRATIC EIGENVALUE PROBLEMS WITH PARTIALLY PRESCRIBED EIGENSTRUCTURE

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Abstract. The inverse eigenvalue problem of constructing real and symmetric square matrices M, C and K of size $n \times n$ for the quadratic pencil $Q(\lambda) = \lambda^2 M + \lambda C + K$ so that $Q(\lambda)$ has a prescribed subset of eigenvalues and eigenvectors is considered. This paper consists of two parts addressing two related but different problems.

The first part deals with the inverse problem where M and K are required to be positive definite and semi-definite, respectively. It is shown via construction that the inverse problem is solvable for any k given complex conjugately closed pairs of distinct eigenvalues and linearly independent eigenvectors, provided $k \leq n$. The construction also allows additional optimization conditions to be built into the solution so as to better refine the approximate pencil. The eigenstructure of the resulting $Q(\lambda)$ is completely analyzed.

The second part deals with the inverse problem where M is a fixed positive-definite matrix (and hence may be assumed to be the identity matrix I_n). It is shown via construction that the monic quadratic pencil $Q(\lambda) = \lambda^2 I_n + \lambda C + K$ with $n+1$ arbitrarily assigned complex conjugately closed pairs of distinct eigenvalues and column eigenvectors which span the space $\mathbb{C}^n$ always exists. Sufficient conditions under which this quadratic inverse eigenvalue problem is uniquely solvable are specified.

Key words. quadratic eigenvalue problem, inverse eigenvalue problem, partially prescribed spectrum, partial eigenstructure assignment

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1. Introduction.

Given $n \times n$ complex matrices M, C and K, the task of finding scalars λ and nonzero vectors $\mathbf{x}$ satisfying

$$Q(\lambda)\mathbf{x} = 0, \tag{1.1}$$

where

$$Q(\lambda) := Q(\lambda; M, C, K) = \lambda^2 M + \lambda C + K, \tag{1.2}$$

is known as the *quadratic eigenvalue problem* (QEP). The scalars λ and the corresponding vectors $\mathbf{x}$ are called, respectively, eigenvalues and eigenvectors of the quadratic pencil $Q(\lambda)$. Together, $(\lambda, \mathbf{x})$ is called an *eigenpair* of $Q(\lambda)$. The QEP has received much attention because its formation has repeatedly arisen in many different disciplines, including applied mechanics, electrical oscillation, vibro-acoustics, fluid mechanics, and signal processing. In a recent treatise, Tisseur and Meerbergen [17] surveyed a good many applications, mathematical properties, and a variety of numerical techniques for the QEP. It is known that the QEP has $2n$ finite eigenvalues over the complex field, provided that the leading matrix coefficient M is nonsingular. The QEP arising in practice often entails some additional conditions on the matrices. For example, if M, C and K represent the mass, damping and stiffness matrices, respectively, in a mass-spring system, then it is required that all matrices be real-valued and symmetric, and that M and K be positive definite and semi-definite, respectively. It is this class of constraints on the matrix coefficients in (1.2) that underlines our main contribution in this paper.

In mathematical modelling, we generally assume that there is a correspondence between the endogenous variables, that is, the internal parameters, and the exogenous variables, that is, the

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external behavior. In most of the applications involving (1.1), specifications of the underlying physical system are embedded in the matrix coefficients M, C and K while the resulting bearing of the system usually can be interpreted via its eigenvalues and eigenvectors. The process of analyzing and deriving the spectral information and, hence, inducing the dynamical behavior of a system from *a priori* known physical parameters such as mass, length, elasticity, inductance, capacitance, and so on is referred to as a *direct* problem. The *inverse* problem, in contrast, is to validate, determine, or estimate the parameters of the system according to its observed or expected behavior. The concern in the direct problem is to express the behavior in terms of the parameters whereas in the inverse problem the concern is to express the parameters in term of the behavior. The inverse problem is just as important as the direct problem in applications.

There has been a lot of interest in the inverse eigenvalue problem, including the notable pole assignment problem. Some general reviews and extensive bibliographies in this regard can be found, for example, in the first author's recent articles [3] and [4]. This paper concerns itself with the inverse problem of the QEP.

The term *inverse quadratic eigenvalue problem* (IQEP) adopted in the literature usually is for general matrix coefficients. In this paper we shall use it distinctively to stress the additional structure imposed upon the matrix coefficients. Two scenarios will be considered separately:

- Determine real, symmetric matrix coefficients M, C and K with M positive definite and K positive semi-definite so that the resulting QEP has a prescribed set of k eigenpairs.
- Assume that the symmetric and positive definite leading matrix coefficient M is known and fixed, determine real and symmetric matrix coefficients C and K so that the resulting QEP has a prescribed set of k eigenpairs.

Other types of IQEPs have been studied under modified conditions. For instance, the IQEP studied by Ram and Ehlay [13] is for symmetric tridiagonal coefficients and, instead of prescribed eigenpairs, two sets of eigenvalues are given. In a series of articles, Starek and Inman [16] studied the IQEPs associated with nonproportional underdamped systems. Settings for some other mechanical applications can be found at the web site [14]. Our study in this paper stems from the speculation that the notion of the IQEP has the potential of leading to an important modification tool for model updating [5], model tuning, and model correction [1, 10, 15, 18], when compared with an analytical model. We will discuss this specific application in a separate paper.

We note that in several recent works, including those by Chu and Datta [2], Nichols and Kautsky [12], as well as Datta, Elhay, Ram and Sarkissian [6, 7], studies are undertaken toward a feedback design problem for a second-order control system. That consideration eventually leads to either a full or a partial eigenstructure assignment problem for the QEP. The proportional and derivative state feedback controller designated in these studies is capable of assigning specific eigenvalues and making the resulting system insensitive to perturbations. Nonetheless, these results cannot meet the basic requirement that the quadratic pencil be symmetric.

In a large or complicated physical system, it is often impossible to obtain the entire spectral information. Furthermore, quantities related to high frequency terms generally are susceptible to measurement errors due to the finite bandwidth of measuring devices. Spectral information, therefore, should not be used at its full extent. For these reasons, it might be more sensible to consider an IQEP where only a *portion* of eigenvalues and eigenvectors is prescribed. A natural question to ask is how much eigeninformation is needed to ensure that an IQEP is solvable.

To facilitate the discussion, we shall describe the partial eigeninformation via the pair $(\Lambda, X) \in \mathbb{R}^{k\times k} \times \mathbb{R}^{n\times k}$ of matrices where

$$\Lambda = \operatorname{diag}\{\lambda_1^{[2]}, \ldots, \lambda_\ell^{[2]}, \lambda_{2\ell+1}, \ldots, \lambda_k\} \tag{1.3}$$

with

$$\lambda_j^{[2]} = \begin{bmatrix} \alpha_j & \beta_j \\ -\beta_j & \alpha_j \end{bmatrix} \in \mathbb{R}^{2\times 2}, \quad \beta_j \neq 0, \quad \text{for} \quad j = 1, \ldots, \ell, \tag{1.4}$$

and

$$X = [\mathbf{x}_{1R}, \mathbf{x}_{1I}, \ldots, \mathbf{x}_{\ell R}, \mathbf{x}_{\ell I}, \mathbf{x}_{2\ell+1}, \ldots, \mathbf{x}_k]. \tag{1.5}$$

The true eigenvalues and eigenvectors are readily identifiable via the transformation

$$R := \operatorname{diag}\left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}, \ldots, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}, I_{k-2\ell} \right\}, \tag{1.6}$$

with $i = \sqrt{-1}$. That is, by defining

$$\tilde{\Lambda} = R^H \Lambda R = \operatorname{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_{2\ell-1}, \lambda_{2\ell}, \lambda_{2\ell+1}, \ldots, \lambda_k\} \in \mathbb{C}^{k\times k}, \tag{1.7}$$

$$\tilde{X} = XR = [\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{2\ell-1}, \mathbf{x}_{2\ell}, \mathbf{x}_{2\ell+1}, \ldots, \mathbf{x}_k] \in \mathbb{C}^{n\times k}, \tag{1.8}$$

respectively, the IQEP is concerned about finding a *real-valued* quadratic pencil $Q(\lambda)$ (with its matrix coefficients possessing a certain specified structures) so that $Q(\lambda_j)\mathbf{x}_j = 0$ for all $j = 1, \ldots, k$. The true (complex-valued) eigenvalues and eigenvectors of the desired quadratic pencil $Q(\lambda)$ can be induced from the pair (Λ, X) of real matrices. In this case, note that $\mathbf{x}_{2j-1} = \mathbf{x}_{jR} + i\mathbf{x}_{jI}$, $\mathbf{x}_{2j} = \mathbf{x}_{jR} - i\mathbf{x}_{jI}$, $\lambda_{2j-1} = \alpha_j + i\beta_j$, and $\lambda_{2j} = \alpha_j - i\beta_j$ for $j = 1, \ldots, \ell$, whereas $\mathbf{x}_j$ and λ_j are all real-valued for $j = 2\ell + 1, \ldots, k$. For convenience, we shall denote henceforth the set of diagonal elements of $\tilde{\Lambda}$, which is precisely the spectrum of Λ, by $\sigma(\Lambda)$. We shall call (Λ, X) an *eigeninformation pair* of the quadratic pencil $Q(\lambda)$.

The two types of IQEP considered in this paper can be formulated as follows:

ISQEP (Inverse Standard Quadratic Eigenvalue Problem) Given an eigeninformation pair (Λ, X), find real and symmetric matrices M, C and K with M and K positive definite and semi-definite, respectively, so that the equation

$$MX\Lambda^2 + CX\Lambda + KX = 0, \tag{1.9}$$

is satisfied.

IMQEP (Inverse Monic Quadratic Eigenvalue Problem) Given an eigeninformation pair (Λ, X), find real and symmetric matrices C and K that satisfy the equation

$$X\Lambda^2 + CX\Lambda + KX = 0. \tag{1.10}$$

Before we move into further details, some remarks highlighting the fundamental differences between the two problems might help to capture the main points in the fairly involved mathematics later on.

1. In the IMQEP, it suffices to consider the monic quadratic pencil (1.10) for the more general case where the leading matrix coefficient M is positive definite and *fixed*. Since M is known, let $M = LL^\top$ denotes the Cholesky decomposition of M. Then

$$Q(\lambda)\mathbf{x} = 0 \quad \Leftrightarrow \quad \widetilde{Q}(\lambda)(L^\top \mathbf{x}) = 0, \tag{1.11}$$

where

$$\widetilde{Q}(\lambda) := \lambda^2 I_n + \lambda L^{-1} C L^{-\top} + L^{-1} K L^{-\top}. \tag{1.12}$$

Thus, without loss of generality, we may assume that the given matrix M in the IMQEP is the $n \times n$ identity matrix I_n to begin with. It is not the case with the ISQEP. The leading matrix coefficient M in the ISQEP is part of the unknowns to be determined.

2. Note that the IMQEP requires only symmetry and nothing else of the two matrix coefficients C and K. The symmetry of C and K implies that there are in total $n(n+1)$ unknowns to be determined in the inverse problem. Since each eigenpair $(\lambda, \mathbf{x})$ characterizes a system of n equations, it is natural to conjecture that a monic quadratic pencil could be determined from any given $n+1$ eigenpairs that are closed under complex conjugation. One of our main contributions in this paper is to substantiate this conjecture after a necessary condition is satisfied. We offer a constructive proof in this paper showing that the solution for the IMQEP is in fact unique.
3. In contrast, the positive definiteness imposed on the ISQEP is much more complicated than a mere count of the numbers of the unknowns and equations. It turns out that the amount of eigeninformation cannot contain more than n eigenpairs. We show that, given any $k \leq n$ distinct eigenvalues and linearly independent eigenvectors closed under complex conjugation, the ISQEP is always solvable but the solution often is not unique. Furthermore, the remaining unspecified eigenstructure of the reconstructed quadratic pencil is in fact quite limited. In particular, at the upper end when $k = n$, that is, when the number of prescribed eigenpairs is equal to the dimension of the ambient space, every prescribed eigenvalue is a double eigenvalue and the remaining eigenstructure is completely fixed.
4. Though both problems are solved by constructive proofs, the mathematical techniques employed to derive the main results for the two problems are indispensably different. It appears counter to the intuition that the IMQEP is much harder to analyze than the ISQEP.

It might be appropriate to attribute the first technique for solving the inverse problem of QEP to a short exposition in the book [9, p.173]. Unfortunately, the method derived from that discussion is not capable of producing symmetric C and K. Our contribution is innovative in four areas: First, we give a recipe for the construction of a solution to each of the two inverse problems. These recipes can be turned into numerical algorithms. Secondly, we specify necessary and sufficient conditions under which the IQEP is solvable. Thirdly, we completely characterize the eigenstructure of the reconstructed quadratic pencil. Finally, we propose a way to refine the construction process so that the best approximation subject to some additional optimal conditions can be established.

2. Solving ISQEP. In this section we present a general theory elucidating how the ISQEP could be solved with the prescribed spectral information (Λ, X). Our proof is constructive. As a by-product, numerical algorithms can also be developed thence. Examples of numerical schemes and applications will be discussed in Section 2.3. We shall assume henceforth, in the formulation of an ISQEP, that the given spectral information (Λ, X) is always in the form of (1.3) and (1.5).

2.1. Recipe of Construction. Starting with the given pair of matrices (Λ, X), consider the null space $\mathcal{N}(\Omega)$ of the augmented matrix

$$\Omega := \begin{bmatrix} X^\top & \Lambda^\top X^\top \end{bmatrix} \in \mathbb{R}^{k \times 2n}.$$

Denote the dimension of $\mathcal{N}(\Omega)$ by m. If X has linearly independently columns (as we will assume later), then $m = 2n - k$. Note that $m \geq n$, if we have assumed $k \leq n$ (for the reason to be seen later) in the formulation of the ISQEP. Let the columns of the matrix

$$\begin{bmatrix} U^\top \\ V^\top \end{bmatrix} \in \mathbb{R}^{2n \times m}$$

with $U^\top, V^\top \in \mathbb{R}^{n\times m}$ denote *any* basis of the subspace $\mathcal{N}(\Omega)$. The equation

$$\begin{bmatrix} X^\top & \Lambda^\top X^\top \end{bmatrix} \begin{bmatrix} U^\top \\ V^\top \end{bmatrix} = 0 \tag{2.1}$$

holds. Define the quadratic pencil $Q(\lambda)$ by the matrix coefficients

$$M = V^\top V, \tag{2.2}$$
$$C = V^\top U + U^\top V, \tag{2.3}$$
$$K = U^\top U. \tag{2.4}$$

We claim that the above definitions are sufficient for constructing a solution to the ISQEP. The theory will be established in several steps.

THEOREM 2.1. *Given any pair of matrices* (Λ, X) *in the form of (1.3) and (1.5), let* U *and* V *be an arbitrary solution to the equation (2.1). Then* (Λ, X) *is an eigenpair of the quadratic pencil* $Q(\lambda)$ *with matrix coefficients* M*,* C *and* K *defined according to (2.2), (2.3) and (2.4), respectively.*

Proof. Upon substitution, we see that

$$\begin{aligned} MX\Lambda^2 + CX\Lambda + KX &= V^\top VX\Lambda^2 + \left(V^\top U + U^\top V\right) X\Lambda + \left(U^\top U\right) X \\ &= V^\top \left(VX\Lambda + UX\right)\Lambda + U^\top \left(VX\Lambda + UX\right) = 0. \end{aligned}$$

The last equality is due to the properties of U and V in (2.1). ☐

By this construction, all matrix coefficients in $Q(\lambda)$ are obviously real and symmetric. Note also that both matrices M and K are positive semi-definite. However, it is not clear whether $Q(\lambda)$ is a trivial quadratic pencil. Toward that end, we claim that the assumption that X has full column rank is sufficient and necessary for the regularity of $Q(\lambda)$.

THEOREM 2.2. *The leading matrix coefficient* $M = V^\top V$ *is nonsingular, provided that* X *has full column rank. In this case, the resulting quadratic pencil* $Q(\lambda)$ *is regular, that is,* $\det(Q(\lambda))$ *is not identically zero.*

Proof. Suppose that $V^\top \in \mathbb{R}^{n\times m}$ is not of full row rank. There exists an orthogonal matrix $G \in \mathbb{R}^{m\times m}$ such that

$$V^\top G = \begin{bmatrix} V_1^\top & 0_{n\times m_2} \end{bmatrix},$$

where $V_1^\top \in \mathbb{R}^{n\times m_1}$ and $0_{n\times m_2}$ denotes the zero matrix of size $n \times m_2$. Note that $m_1 < n$ and $m_2 = m - m_1$. Postmultiply the same G to $U^\top$ and partition the product into

$$U^\top G = \begin{bmatrix} U_1^\top & U_2^\top \end{bmatrix},$$

where $U_1^\top \in \mathbb{R}^{n\times m_1}$ and $U_2^\top \in \mathbb{R}^{n\times m_2}$. Note that $m_2 > m - n$. On the other hand, we see from (2.1) that

$$X^\top U_2^\top = 0,$$

whereas the column of $U_2^\top$ are necessarily linearly independent by construction. It follows that

$$n - k \geq m_2 > m - n,$$

which contradicts with the fact that $m = 2n - k$. Thus, the matrix $V^\top$ must be of full row rank and then $M = V^\top V$ is nonsingular. ☐

THEOREM 2.3. *Suppose in a given pair of matrices* (Λ, X) *that all eigenvalues in* Λ *are distinct and that* X *is not of full column rank. Then the quadratic pencil* $Q(\lambda)$ *defined by (2.2), (2.3) and (2.4) is singular.*

Proof. It is easy to check that the equation (2.1) remains true if Λ and X are replaced by $\tilde{\Lambda}$ and $\tilde{X}$ defined in (1.7) and (1.8), respectively. Let μ be an arbitrary complex number not in $\sigma(\Lambda)$. Observe that

$$\left[\begin{array}{cc} \tilde{X}^\top & \tilde{\Lambda}^\top \tilde{X}^\top \end{array}\right] \left[\begin{array}{cc} I & -\mu I \\ 0 & I \end{array}\right] \left[\begin{array}{cc} I & \mu I \\ 0 & I \end{array}\right] \left[\begin{array}{c} U^\top \\ V^\top \end{array}\right] = 0.$$

It follows that

$$\left[\begin{array}{cc} \tilde{X}^\top & (\tilde{\Lambda}^\top - \mu I)\tilde{X}^\top \end{array}\right] \left[\begin{array}{c} \mu V^\top + U^\top \\ V^\top \end{array}\right] = 0.$$

By assumption, $\tilde{X}$ is not of full column rank. We may therefore assume that for some $2 \leq q \leq k$,

$$\tilde{\mathbf{x}}_q = \sum_{j=1}^{q-1} r_j \tilde{\mathbf{x}}_j,$$

where not all r_j, $j = 1, \ldots, q-1$, are zero. Define

$$\Gamma := \left[\begin{array}{cccccc} 1 & & & r_{1,q} & & 0 \\ & \ddots & & \vdots & & \\ & & \ddots & r_{q-1,q} & & \\ & & & 1 & & \\ & & & & \ddots & \\ & 0 & & & & 1 \end{array}\right] \in \mathbb{C}^{k \times k},$$

with $r_{j,q} = -\frac{\lambda_q - \mu}{\lambda_j - \mu} r_j$, $j = 1, \ldots, q-1$. Clearly,

$$\Gamma^\top \left[\begin{array}{cc} \tilde{X}^\top & (\tilde{\Lambda}^\top - \mu I)\tilde{X}^\top \end{array}\right] \left[\begin{array}{c} \mu V^\top + U^\top \\ V^\top \end{array}\right] = 0. \tag{2.5}$$

Notice that, by construction, the q-th row of $\Gamma^\top(\tilde{\Lambda}^\top - \mu I)\tilde{X}^\top$ is zero. Let $y(\mu)^\top$ denote the q-th row of $\Gamma^\top \tilde{X}^\top$, which cannot be identically zero because the spectrum of Λ are distinct. We thus see from (2.5) that

$$y(\mu)^\top \left(\mu V^\top + U^\top\right) = 0.$$

It follows that $y(\mu)^\top Q(\mu) = 0$. Since $\mu \in \mathbb{C}$ is arbitrary, $Q(\lambda)$ must be singular. ∎

We conclude this section with one important remark. The rank condition $k = n$ plays a pivotal role in ISQEP. It is the critical value for the regularity of the quadratic pencil $Q(\lambda)$ defined by the matrix coefficients (2.2), (2.3) and (2.4). In fact, it is clear now that corresponding to any given $\Lambda \in \mathbb{R}^{k \times k}$, $X \in \mathbb{R}^{n \times k}$ in the form of (1.3) and (1.5), the quadratic pencil $Q(\lambda)$ can always be factorized into the product

$$Q(\lambda) = \left(\lambda V^\top + U^\top\right)(\lambda V + U). \tag{2.6}$$

If $k > n$, then $\operatorname{rank}(\lambda V + U) \leq 2n - k < n$ and hence $\det(Q(\lambda)) \equiv 0$ for all λ. It is for this reason that we always assume that $k \leq n$ in the formulation of an ISQEP.

2.2. Eigenstructure of $Q(\lambda)$. We have shown in the preceding section how to define the matrix coefficients so that the corresponding quadratic pencil possesses a prescribed set of k eigenvalues and eigenvectors. The ISQEP thereby is solved via construction. An interesting question to ask is how the unspecified eigenpair in the constructed pencil should look like. In this section we examine the remaining eigenstructure of the quadratic pencil $Q(\lambda)$ created from our scheme.

THEOREM 2.4. *Let $(\Lambda, X) \in \mathbb{R}^{k\times k} \times \mathbb{R}^{n\times k}$ in the form of (1.3) and (1.5) denote the partial eigeninformation and $Q(\lambda)$ be the quadratic pencil defined by coefficients (2.2), (2.3) and (2.4). Assume that X has full column rank k.*

1. *If $k = n$, then $Q(\lambda)$ has double eigenvalue λ_j for each $\lambda_j \in \sigma(\Lambda)$;*
2. *If $k < n$, then $Q(\lambda)$ has double eigenvalue λ_j for each $\lambda_j \in \sigma(\Lambda)$. The remaining $2(n-k)$ eigenvalues of $Q(\lambda)$ are all complex conjugate with nonzero imaginary parts. In addition, if the matrices U and V in (2.1) are chosen from an orthogonal basis of the null space of Ω, then the remaining $2(n-k)$ eigenvalues are only $\pm i$ with corresponding eigenvectors $\mathbf{z} \pm i\mathbf{z}$ where $X^\top \mathbf{z} = 0$.*

Proof. The case $k = n$ is easy. The matrices $U^\top$ and $V^\top$ involved in (2.1) forming the null space of Ω are square matrices of size n. We also know from Theorem 2.2 that $V^\top$ is nonsingular. Observe that

$$V^{-1}U = -X\Lambda X^{-1}. \tag{2.7}$$

Using the factorization (2.6), we see that

$$\det(Q(\lambda)) = (\det(\lambda V + U))^2 .$$

It is clear that $Q(\lambda)$ has double eigenvalue λ_j at every $\lambda_j \in \sigma(\Lambda)$.

We now consider the case when $k < n$. Since $X^\top \in \mathbb{R}^{k\times n}$ is of full row rank, there exists an orthogonal matrix $P_1 \in \mathbb{R}^{n\times n}$ such that

$$X^\top P_1^\top = \begin{bmatrix} X_{11}^\top & 0_{n\times(n-k)} \end{bmatrix}, \tag{2.8}$$

where $X_{11}^\top \in \mathbb{R}^{k\times k}$ is nonsingular. There also exists an orthogonal matrix $Q_1 \in \mathbb{R}^{m\times m}$ such that

$$P_1 V^\top Q_1 = \begin{bmatrix} V_{11}^\top & 0_{k\times(n-k)} & 0_{k\times(m-n)} \\ V_{21}^\top & \mathcal{A} & 0_{(n-k)\times(m-n)} \end{bmatrix} \in \mathbb{R}^{n\times m}, \tag{2.9}$$

with appropriate sizes for the other three submatrices. In particular, note that both $V_{11}^\top \in \mathbb{R}^{k\times k}$ and $\mathcal{A} \in \mathbb{R}^{(n-k)\times(n-k)}$ are nonsingular matrices, because $V^\top$ is of full row rank by Theorem 2.2. From the fact that

$$\begin{bmatrix} X^\top & \Lambda^\top X^\top \end{bmatrix} \begin{bmatrix} P_1^\top & 0 \\ 0 & P_1^\top \end{bmatrix} \begin{bmatrix} P_1 & 0 \\ 0 & P_1 \end{bmatrix} \begin{bmatrix} U^\top \\ V^\top \end{bmatrix} Q_1 = 0, \tag{2.10}$$

we conclude that the structure of $P_1 U^\top Q_1$ must be of the form

$$P_1 U^\top Q_1 = \begin{bmatrix} U_{11}^\top & 0_{k\times(n-k)} & 0_{k\times(m-n)} \\ U_{21}^\top & \Delta & \mathcal{B} \end{bmatrix} \in \mathbb{R}^{n\times m}, \tag{2.11}$$

where $\mathcal{B} \in \mathbb{R}^{(n-k)\times(n-k)}$ is nonsingular. Because $\begin{bmatrix} U^\top \\ V^\top \end{bmatrix}$ is of full column rank, together with the fact that both $\mathcal{A}$ and $\mathcal{B}$ in (2.9) and (2.11) are nonsingular, it follows that $\begin{bmatrix} U_{11}^\top \\ V_{11}^\top \end{bmatrix}$ must be of full column rank. Note that $U_{11}^\top$ is nonsingular if and only if Λ has no zero eigenvalue. Using $V_{11}^\top$ as a pivot matrix to eliminate $V_{21}^\top$ in (2.9), we may claim that there exits a nonsingular matrix P_2 such that

$$\tilde{U}^\top := P_2 P_1 U^\top Q_1 = \begin{bmatrix} U_{11}^\top & 0 & 0 \\ \tilde{U}_{21}^\top & \Delta & \mathcal{B} \end{bmatrix},$$
$$\tilde{V}^\top := P_2 P_1 V^\top Q_1 = \begin{bmatrix} V_{11}^\top & 0 & 0 \\ 0 & \mathcal{A} & 0 \end{bmatrix}.$$

Compute the three matrices

$$\tilde{M} := \tilde{V}^\top \tilde{V} = \begin{bmatrix} V_{11}^\top V_{11} & 0 \\ 0 & \mathcal{A}\mathcal{A}^\top \end{bmatrix},$$
$$\tilde{C} := \tilde{U}^\top \tilde{V} + \tilde{V}^\top \tilde{U} = \begin{bmatrix} U_{11}^\top V_{11} + V_{11}^\top U_{11} & V_{11}^\top \tilde{U}_{21} \\ \tilde{U}_{21}^\top V_{11} & \mathcal{A}\Delta^\top + \Delta\mathcal{A}^\top \end{bmatrix},$$
$$\tilde{K} := \tilde{U}^\top \tilde{U} = \begin{bmatrix} U_{11}^\top U_{11} & U_{11}^\top \tilde{U}_{21} \\ \tilde{U}_{21}^\top U_{11} & \tilde{U}_{21}^\top \tilde{U}_{21} + \mathcal{B}\mathcal{B}^\top + \Delta\Delta^\top \end{bmatrix},$$

and define the quadratic pencil $\tilde{Q}(\lambda) := \lambda^2 \tilde{M} + \lambda \tilde{C} + \tilde{K}$. By construction, it is clear that $\tilde{Q}(\lambda) = (P_2 P_1)\, Q(\lambda)(P_2 P_1)^\top$. This congruence relation ensures that $\tilde{Q}(\lambda)$ preserves the same eigenvalue information as $Q(\lambda)$. Define

$$Q_{11}(\lambda) := \lambda^2 (V_{11}^\top V_{11}) + \lambda (V_{11}^\top U_{11} + U_{11}^\top V_{11}) + U_{11}^\top U_{11}, \tag{2.12}$$
$$P_3 := \begin{bmatrix} I & 0 \\ -\tilde{U}_{21}^\top (\lambda V_{11} + U_{11}) Q_{11}^{-1}(\lambda) & I \end{bmatrix}. \tag{2.13}$$

It is further seen that $\tilde{Q}(\lambda)$ can be factorized as

$$P_3 \begin{bmatrix} Q_{11}(\lambda) & 0 \\ 0 & (\lambda\mathcal{A} + \Delta)(\lambda\mathcal{A}^\top + \Delta^\top) + \mathcal{B}\mathcal{B}^\top \end{bmatrix} P_3^\top. \tag{2.14}$$

We thus have effectively decomposed the quadratic pencil $\tilde{Q}(\lambda)$ into two subpencils.

By construction, we see from (2.8), (2.10) and (2.12) that the quadratic subpencil $Q_{11}(\lambda)$ in (2.12) solves exactly the ISQEP with spectral data (Λ, X_{11}). For this problem, we have already proved in the first part that $Q_{11}(\lambda)$ must have double eigenvalue λ_j for each $\lambda_j \in \sigma(\Lambda)$. It only remains to check the eigenvalues for the subpencil $(\mu\mathcal{A} + \Delta)(\mu\mathcal{A}^\top + \Delta^\top) + \mathcal{B}\mathcal{B}^\top$. Recall that the matrix $\mathcal{B}$ in (2.11) is nonsingular. The matrix $(\mu\mathcal{A} + \Delta)(\mu\mathcal{A}^\top + \Delta^\top) + \mathcal{B}\mathcal{B}^\top$ is positive definite for every $\mu \in \mathbb{R}$. In particular, its determinant cannot be zero for any real μ. Therefore, the remaining eigenvalues of $Q(\lambda)$ must be all complex conjugate with nonzero imaginary parts.

If, in addition, the columns of $\begin{bmatrix} U^\top \\ V^\top \end{bmatrix}$ in (2.10) are orthogonal to begin with, then both $\mathcal{A}$ and $\mathcal{B}$ are $(n-k) \times (n-k)$ orthogonal matrices and the submatrix Δ in (2.11) must be a zero matrix. By (2.14), the remaining eigenvalues of $Q(\lambda)$ can only be $\pm i$. Observe further that there exists a nonsingular $W \in \mathbb{R}^{k \times k}$ such that

$$\begin{bmatrix} I & 0 \\ 0 & W \end{bmatrix} \begin{bmatrix} U & V \\ X^\top & \Lambda^\top X^\top \end{bmatrix} \begin{bmatrix} U^\top & X \\ V^\top & X\Lambda \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & W^\top \end{bmatrix} = \begin{bmatrix} I_{2n-k} & 0 \\ 0 & I_k \end{bmatrix}. \tag{2.15}$$

It follows that

$$\begin{aligned} U^\top U + X W^\top W X^\top &= I_n, \\ U^\top V + X W^\top W \Lambda^\top X^\top &= 0, \\ V^\top U + X \Lambda W^\top W X^\top &= 0, \\ V^\top V + X \Lambda W^\top W \Lambda^\top X^\top &= I_n. \end{aligned} \tag{2.16}$$

For any $\mathbf{z}$ satisfying $X^\top \mathbf{z} = 0$, we see from the above equations that

$$\begin{aligned} U^\top U\mathbf{z} &= \mathbf{z}, \\ V^\top V\mathbf{z} &= \mathbf{z}, \\ U^\top V\mathbf{z} + V^\top U\mathbf{z} &= 0. \end{aligned}$$

This show that $Q(\pm i)(\mathbf{z} \pm i\mathbf{z}) = 0$. ☐

Theorem 2.4 is significant in several fronts. First, if $k = n$, then *all* eigenvalues of $Q(\lambda)$ are completely counted. Secondly, if $k < n$ and if the basis of null space $\mathcal{N}(\Omega)$ are selected to be mutually orthogonal (as we normally would do by using, say, MATLAB), then again all eigenvalues of $Q(\lambda)$ are completed determined. In other words, we are *not* allowed to supplement any additional $n-k$ eigenpairs to simplify this ISQEP. The solution of our method for ISQEP is the most natural way for $k(<n)$ prescribed pairs of eigenvalues and eigenvectors. In Section 2.3, we shall study how non-orthogonal basis of $\mathcal{N}(\Omega)$ can help to improve the ISQEP approximation.

We can further calculate the geometric multiplicity of the double roots characterized in Theorem 2.5.

Theorem 2.5. *Let (Λ, X) in the form of (1.3) and (1.5) denote the prescribed eigenpair of the quadratic pencil $Q(\lambda)$ defined before. Assume that Λ has distinct spectrum and X has full column rank. Then*

1. *Each real-valued $\lambda_j \in \sigma(\Lambda)$ has an elementary divisor of degree 2, that is, the dimension of the null space $\mathcal{N}(Q(\lambda_j))$ is 1.*
2. *The dimension of $\mathcal{N}(Q(\lambda_j))$ of a complex-valued eigenvalue $\lambda_j \in \sigma(\Lambda)$ is generically 1. That is, pairs of matrices (Λ, X) of which a complex-valued eigenvalue has a linear elementary divisor forms a measure zero set.*

Proof. Real-valued eigenvalues correspond to those $\lambda_j \in \sigma(\Lambda)$ with $j = 2\ell+1, \ldots, k$. We have already seen in Theorem 2.1 that $Q(\lambda_j)\mathbf{x}_j = 0$, where $\mathbf{x}_j$ is the j-th column of X. Suppose that the $\mathcal{N}(Q(\lambda_j))$ has dimension greater than 1. From (2.6), it must be that

$$\operatorname{rank}\left(\lambda_j V^\top + U^\top\right) \leq n-2. \tag{2.17}$$

Rewrite (2.1) as

$$\begin{bmatrix} X^\top & \Lambda^\top X^\top \end{bmatrix} \begin{bmatrix} I & -\lambda_j I \\ 0 & I \end{bmatrix} \begin{bmatrix} I & \lambda_j I \\ 0 & I \end{bmatrix} \begin{bmatrix} U^\top \\ V^\top \end{bmatrix} = 0, \tag{2.18}$$

which is equivalent to

$$\begin{bmatrix} X^\top & (\Lambda^\top - \lambda_j I)X^\top \end{bmatrix} \begin{bmatrix} \lambda_j V^\top + U^\top \\ V^\top \end{bmatrix} = 0. \tag{2.19}$$

Note that, since Λ has distinct spectrum and $X^\top$ has full row-rank,

$$\operatorname{rank}\left((\Lambda^\top - \lambda_j I)X^\top\right) = k-1,$$

or equivalently,

$$\dim\left(\mathcal{N}\left((\Lambda^\top - \lambda_j I)X^\top\right)\right) = n-k+1. \tag{2.20}$$

On the other hand, there exists an orthogonal $G_j \in \mathbb{R}^{m\times m}$ such that

$$\begin{bmatrix} \lambda_j V^\top + U^\top \\ V^\top \end{bmatrix} G_j = \begin{bmatrix} U_{j1}^\top & 0 \\ V_{j1}^\top & V_{j2}^\top \end{bmatrix}, \tag{2.21}$$

where, due to (2.17), $V_{j2}^\top$ has at least $m-(n-2)=n-k+2$ linearly independent columns. We then see from (2.19) that

$$(\Lambda^\top - \lambda_j I)X^\top V_{j2}^\top = 0,$$

implying that $\dim\left(\mathcal{N}\left((\Lambda^\top - \lambda_j I)X^\top\right)\right) \geq n-k+2$. This contradicts with (2.20).

To examine the complex-valued case, notice that (1.9) can be rewritten as

$$M(XR)(R^H\Lambda^2 R) + C(XR)(R^H\Lambda R) + KXR = 0,$$

where R is defined in (1.6). In particular from (1.7) and (1.8), for $1 \leq j \leq 2\ell$, we have

$$Q(\lambda_j)\mathbf{x}_j = 0.$$

We first consider the case $k=n$. Two observations are due at the moment. First, the matrix V in the basis $\begin{bmatrix} U^\top \\ V^\top \end{bmatrix}$ for the null space $\mathcal{N}\left(\left[X^\top, \Lambda^\top X^\top\right]\right)$ can be an arbitrary nonsingular matrix. Secondly, if there exists another vector $\mathbf{z} \in \mathbb{C}^n$ independent of $\mathbf{x}_j$ such that $Q(\lambda_j)\mathbf{z} = 0$, we claim that for this kind of eigenvalue the matrix $(V^\top V)^{-1}$ must satisfy some kind of algebraic varieties in $\mathbb{R}^{n\times n}$. Putting these two facts together, we conclude that any complex-valued eigenvalue having a linear elementary divisor must come from a set of measure zero.

To see the claim concerning the algebraic varieties for $(V^\top V)^{-1}$, we use (2.7) and (1.7) to rewrite $\lambda_j V + U$ as

$$\lambda_j V + U = VXR(\lambda_j I - \Lambda)R^H X^{-1},$$

and thus factorize $Q(\lambda_j)$ as

$$\begin{aligned} Q(\lambda_j) &= (\lambda_j V^\top + U^\top)(\lambda_j V + U) \\ &= X^{-\top}\bar{R}(\lambda_j I - \Lambda)R^\top X^\top V^\top VXR(\lambda_j I - \Lambda)R^H X^{-1}. \end{aligned} \tag{2.22}$$

If $Q(\lambda_j)\mathbf{z} = 0$, from (2.22) we have

$$R^\top X^\top V^\top VXR(\lambda_j I - \Lambda)R^H X^{-1}\mathbf{z} = \tau \mathbf{e}_j, \tag{2.23}$$

where $\mathbf{e}_j$ is the jth standard unit vector and τ is some scalar. Rewrite (2.23) as

$$(\lambda_j I - \Lambda)R^H X^{-1}\mathbf{z} = \tau R^H X^{-1}(V^\top V)^{-1}X^{-\top}\bar{R}\mathbf{e}_j.$$

Hence, a necessary condition for the existence of $\mathbf{z}$ is that $(V^\top V)^{-1}$ must satisfy the algebraic equation

$$\mathbf{e}_j^\top R^H X^{-1}(V^\top V)^{-1}X^{-\top}\bar{R}\mathbf{e}_j = 0. \tag{2.24}$$

We note in passing that the condition (2.24) for $(V^\top V)^{-1}$ is also sufficient since the above argument can be reversed to show the existence of a vector $\mathbf{z}$ in the null space of $Q(\lambda_j)$.

For the case $k < n$, a similar argument holds. Indeed, using the decompositions (2.12) and (2.14) given in Theorem 2.4, a sufficient and necessary condition for the existence of $\mathbf{z}$ is exactly the same as (2.24) where X and V are replaced by X_{11} and V_{11}, respectively. In either case, outside the algebraic variety, the elementary divisor of a generically prescribed complex eigenvalues therefore is of degree 2. $\square$

To further demonstrate the subtlety of the second statement in Theorem 2.5, we make an interesting observation as follows.

Corollary 2.6. *Suppose in the given (Λ, X) that X has full column rank and that Λ has distinct spectrum. Assume further that $\{\pm i\} \subset \sigma(\Lambda)$. Construct the quadratic pencil $Q(\lambda)$ by taking an orthogonal basis $[U,V]^\top$ for the null space $\mathcal{N}(\Omega)$. Then the dimension of $\mathcal{N}(Q(\pm i))$ is 2. In other words, in this non-generic case, both eigenvalues $\pm i$ have two linear elementary divisors.*

Proof. From (2.15), we have $W(X^\top X + \Lambda^\top X^\top X \Lambda)W^\top = I_n$. It follows that

$$W^\top W = (X^\top X + \Lambda^\top X^\top X\Lambda)^{-1}.$$

The last equation in (2.16) gives rise to

$$(V^\top V)^{-1} = (I - X\Lambda W^\top W \Lambda^\top X^\top)^{-1}.$$

Upon substitution, it holds that

$$\begin{aligned}
X^{-1}(V^\top V)^{-1}X^{-\top} &= X^{-1}(I - X\Lambda W^\top W\Lambda^\top X^\top)^{-1}X^{-\top}\\
&= X^{-1}(I - X\Lambda(X^\top X + \Lambda^\top X^\top X\Lambda)^{-1}\Lambda^\top X^\top)^{-1}X^{-\top}\\
&= X^{-1}(I - X\Lambda X^{-1}(I + X^{-\top}\Lambda^\top X^\top X\Lambda X^{-1})^{-1}X^\top\Lambda^\top X^\top)^{-1}X^{-\top}\\
&= X^{-1}(I + X\Lambda X^{-1}X^{-\top}\Lambda^\top X^\top)X^{-\top}\\
&= X^{-1}X^{-\top} + \Lambda X^{-1}X^{-\top}\Lambda^\top, \qquad (2.25)
\end{aligned}$$

where the forth equality is, after some algebraic manipulation, due to the Sherman-Morrison-Woodburg formula. Substituting (2.25) into (2.24) and assuming that j is the index that defines $\lambda_j = \pm i$, we find that

$$\begin{aligned}
\mathbf{e}_j^\top R^H X^{-1}(V^\top V)^{-1}X^{-\top}\bar{R}\mathbf{e}_j &= \mathbf{e}_j^\top(R^H X^{-1}X^{-\top}\bar{R} + R^H\Lambda R R^H X^{-1}X^{-\top}\bar{R}R^\top\Lambda^\top\bar{R})\mathbf{e}_j\\
&= \mathbf{e}_j^\top(\tilde{X}^{-1}\tilde{X}^{-\top} + \tilde{\Lambda}\tilde{X}^{-1}\tilde{X}^{-\top}\tilde{\Lambda}^\top)\mathbf{e}_j = 0.
\end{aligned}$$

The sufficient condition is met and, therefore, $\dim(\mathcal{N}(Q(\pm i))) = 2$. ☐

2.3. Numerical Experiment. In this section we intend to highlight two main points by numerical examples. The first example demonstrates the eigenstructure of a solution to a typical ISQEP. From the discussion in the preceding sections, we already have a pretty good idea about how the eigenstructure should look like. We now demonstrate numerically how the selection of U and V might affect the geometric multiplicity of the double eigenvalue. The second example has important meaning in applications. We demonstrate how some additional optimization constraints can be incorporated into the construction of a solution to ISQEP. These additional constraints are imposed by some logistic reasons with the hope to better approximate a real physical system. In this example, we also experiment with the effect of feeding various amount of information on eigenvalues and eigenvectors to the construction. In particular, we compare the discrepancy between a given (analytic) quadratic pencil and the resulting ISQEP approximation by varying the values of k and the optimal constraints. All calculation are done by using MATLAB in its default (double) precision. For the ease of running text, however, we shall report all numerals in 5 digits only.

Example 1. Consider the ISQEP where the partial eigenstructure $(\Lambda,\ X) \in \mathbb{R}^{5\times 5} \times \mathbb{R}^{5\times 5}$ is randomly generated. Assume

$$X = \begin{bmatrix} -0.4132 & 5.2801 & 2.9437 & -6.6098 & -9.6715\\ -4.3518 & 3.2758 & -5.1656 & 9.1024 & -9.1357\\ -0.1336 & -4.0588 & 2.5321 & 3.3049 & -4.4715\\ -5.1414 & 4.4003 & -2.2721 & 5.2872 & 6.9659\\ 8.6146 & -4.0112 & -6.9380 & 1.4345 & -4.4708 \end{bmatrix}$$

and

$$\Lambda = \begin{bmatrix} -0.2168 & -4.3159 & 0 & 0 & 0 \\ 4.3159 & -0.2168 & 0 & 0 & 0 \\ 0 & 0 & 2.0675 & -0.9597 & 0 \\ 0 & 0 & 0.9597 & 2.0675 & 0 \\ 0 & 0 & 0 & 0 & -0.3064 \end{bmatrix}.$$

Choose a basis $\begin{bmatrix} U_1^\top \\ V_1^\top \end{bmatrix}$ for the null space $\mathcal{N}\left(\begin{bmatrix} X^\top \Lambda^\top X^\top \end{bmatrix}\right)$, say,

$$U_1^\top = \begin{bmatrix} 0.26861 & 0.56448 & -0.08687 & 0.39491 & -0.24252 \\ 0.32690 & -0.24385 & 0.00804 & -0.32844 & 0.42471 \\ -0.33739 & 0.27725 & -0.15949 & -0.05883 & 0.58406 \\ -0.13374 & 0.43824 & 0.09638 & 0.28605 & 0.46936 \\ -0.42433 & 0.17867 & 0.69977 & -0.12829 & -0.16140 \end{bmatrix},$$

$$V_1^\top = \begin{bmatrix} 0.51817 & 0.09467 & 0.20341 & -0.04075 & 0.32693 \\ 0.25575 & 0.38674 & -0.09339 & -0.32830 & -0.22850 \\ 0.31749 & -0.02297 & 0.63841 & 0.01156 & 0.05987 \\ -0.02434 & -0.40196 & 0.09987 & 0.65755 & 0.09646 \\ 0.27184 & 0.02061 & -0.01859 & 0.30413 & -0.03669 \end{bmatrix}$$

and construct

$$Q_1(\lambda) = \lambda^2(V_1^\top V_1) + \lambda(V_1^\top U_1 + U_1^\top V_1) + (U_1^\top U_1).$$

This quadratic pencil has double eigenvalue λ_j for each $\lambda_j \in \sigma(\Lambda)$, according to our theory. Furthermore, we compute the singular values of each $Q(\lambda_j)$ and find that

$$\begin{aligned} \mathrm{svd}(Q_1(-0.21683 \pm 4.3159i)) &= \{17.394, 15.039, 4.3974, 2.6136, 1.2483 \times 10^{-15}\}, \\ \mathrm{svd}(Q_1(2.0675 \pm 0.95974i)) &= \{5.9380, 4.9789, 1.1788, 0.45926, 4.6449 \times 10^{-16}\}, \\ \mathrm{svd}(Q_1(-0.30635)) &= \{1.0937, 1.0346, 0.89436, 0.18528, 3.8467 \times 10^{-17}\}, \end{aligned}$$

implying that the dimension of the null space $Q(\lambda_j)$ is precisely 1 for each $\lambda_j \in \sigma(\Lambda)$.

However, suppose we choose a special basis for $\mathcal{N}\left(\begin{bmatrix} X^\top \Lambda^\top X^\top \end{bmatrix}\right)$ by

$$\begin{bmatrix} U_2^\top \\ V_2^\top \end{bmatrix} = \begin{bmatrix} U_1^\top V_1^{-\top} X^{-1} \\ X^{-1} \end{bmatrix}$$

and construct

$$Q_2(\lambda) = \lambda^2(V_2^\top V_2) + \lambda(V_2^\top U_2 + U_2^\top V_2) + (U_2^\top U_2).$$

We find that

$$\begin{aligned} \mathrm{svd}(Q_2(-0.21683 \pm 4.3159i)) &= \{15.517,\ 0.12145,\ 0.07626,\ 3.4880 \times 10^{-15},\ 7.9629 \times 10^{-16}\}, \\ \mathrm{svd}(Q_2(2.0675 \pm 0.95974i)) &= \{21.064,\ 0.16325,\ 0.02540,\ 3.2321 \times 10^{-15},\ 5.2233 \times 10^{-16}\}, \\ \mathrm{svd}(Q_2(-0.30635)) &= \{20.995,\ 0.19733,\ 0.08264,\ 0.02977,\ 1.6927 \times 10^{-15}\}. \end{aligned}$$

In this case, each of the four the complex-valued eigenvalues of $\sigma(\Lambda)$ has linear elementary divisors.

Example 2. We can further exploit the freedom in the selection of basis for the null space $\mathcal{N}(\Omega)$. In this example we first demonstrate a few ways to select the basis under some special circumstances. We then illustrate the effect of available eigeninformation on the construction.

To fix the idea, we first generate randomly a 10×10 symmetric quadratic pencil $\hat{Q}(\lambda) = \lambda^2 \hat{M} + \lambda \hat{C} + \hat{K}$, where $\hat{M}$ and $\hat{K}$ are also positive definite, as an analytic model. We then compare the effect of k on its ISQEP approximations for $k = 1, \ldots, 10$. To save the space, we shall not report the data of these test matrices $\hat{M}$, $\hat{C}$ and $\hat{K}$ in this paper, but will make them available upon request. We merely report that the spectrum of $\hat{Q}(\lambda)$ turns out to be the following 10 pairs of complex-conjugate values,

$$\begin{aligned}\{&-0.27589 \pm 1.8585i,\ -0.19201 \pm 1.5026i,\ -0.15147 \pm 1.0972i,\ -0.11832 \pm 0.54054i,\\ &-0.07890 \pm 1.3399i,\ -0.07785 \pm 0.76383i,\ -0.07716 \pm 0.86045i,\ -0.07254 \pm 1.1576i,\\ &-0.06276 \pm 0.97722i,\ -0.05868 \pm 0.18925i\}.\end{aligned}$$

These eigenvalues are not arranged in any specific order. Without loss of generality, we shall *pretend* that the first 5 pairs in the above list are the partially described eigenvalues and wish to reconstruct the quadratic pencil. For $\ell = 1, \ldots, 5$ (and hence $k = 2\ell$), denote these eigenvalues as $\alpha_\ell \pm i\beta_\ell$ Also, define partial eigenpairs $(\Lambda_{2\ell},\ X_{2\ell})$ of $\hat{Q}(\lambda)$ according to (1.3) and (1.5), that is,

$$\Lambda_{2l} = \operatorname{diag}\left\{ \begin{bmatrix} \alpha_1 & \beta_1 \\ -\beta_1 & \alpha_1 \end{bmatrix}, \ldots, \begin{bmatrix} \alpha_\ell & \beta_\ell \\ -\beta_\ell & \alpha_\ell \end{bmatrix} \right\}, \tag{2.26}$$

$$X_{2\ell} = [x_{1R}, x_{1I}, \ldots, x_{\ell R}, x_{\ell I}], \tag{2.27}$$

where $x_{\ell R} \pm i x_{\ell I}$ is the eigenvector of $\hat{Q}(\lambda)$ corresponding to $\alpha_\ell \pm i\beta_\ell$.

Let $\begin{bmatrix} U_\ell^\top \\ V_\ell^\top \end{bmatrix} \in \mathbb{R}^{2n \times (2n-2\ell)}$ be an orthogonal basis for $\mathcal{N}(\left[X_{2\ell}^\top\ \Lambda_{2\ell}^\top X_{2\ell}^\top\right])$. We now introduce three ways to select a *new* basis for $\mathcal{N}(\left[X_{2\ell}^\top\ \Lambda_{2\ell}^\top X_{2\ell}^\top\right])$, each of which is done for a different optimization purpose. The physical meaning of these optimal constraints will be explained at the end of this section.

Case 1. Suppose $\hat{K} = L_{\hat{K}} L_{\hat{K}}^\top$ and $\hat{M} = L_{\hat{M}} L_{\hat{M}}^\top$ are the Cholesky factorizations of $\hat{K}$ and $\hat{M}$ in the model pencil, respectively. Find a matrix $G_{\ell 1}^\top \in \mathbb{R}^{(2n-2\ell)\times(2n-2\ell)}$ by solving the sequence of least-square problems

$$\min \left\| \begin{bmatrix} U_\ell^\top \\ V_\ell^\top \end{bmatrix} G_{\ell 1}^\top(:, j) - \begin{bmatrix} L_{\hat{K}} & 0_{n-2\ell} \\ 0_{n-2\ell} & L_{\hat{M}} \end{bmatrix} (:, j) \right\|_2, \tag{2.28}$$

for each of its columns $G_{\ell 1}^\top(:, j)$, $j = 1, \ldots, 2n - 2\ell$. For convenience, we have adopted here the MATLAB notation $(:, j)$ to denote the jth column of a matrix.

The solution of (2.28) is intended to, not only solve the ISQEP, but also best approximate the original $\hat{K}$ and $\hat{M}$ in the sense that the quantity

$$\|U_\ell^\top G_{\ell 1}^\top G_{\ell 1} U_\ell - \hat{K}\|_F + \|V_\ell^\top G_{\ell 1}^\top G_{\ell 1} V_\ell - \hat{M}\|_F. \tag{2.29}$$

is minimized among all possible $G_{\ell 1}^\top \in \mathbb{R}^{(2n-2\ell)\times(2n-2\ell)}$. Once such a matrix $G_{\ell 1}^\top$ is found, we compute the coefficient matrices according to our recipe, that is,

$$\begin{aligned} M_{\ell 1} &= V_\ell^\top G_{\ell 1}^\top G_{\ell 1} V_\ell, \quad K_{\ell 1} = U_\ell^\top G_{\ell 1}^\top G_{\ell 1} U_\ell, \\ C_{\ell 1} &= U_\ell^\top G_{\ell 1}^\top G_{\ell 1} V_\ell + V_\ell^\top G_{\ell 1}^\top G_{\ell 1} U_\ell, \end{aligned} \tag{2.30}$$

and define the quadratic pencil

$$Q_{\ell 1}(\lambda) = \lambda^2 M_{\ell 1} + \lambda C_{\ell 1} + K_{\ell 1}, \tag{2.31}$$

according to $\ell = 1, \ldots, 5$.

Case 2. We first transform $V_\ell^\top$ to $[V_{\ell 0}^\top, 0]$ by an orthogonal transformation. Then we find a matrix $G_{\ell 2}^\top \in \mathbb{R}^{(2n-2\ell)\times(2n-2\ell)}$ in the form

$$G_{\ell 2}^\top = \begin{bmatrix} E_{\ell 2}^\top & 0 \\ 0 & F_{\ell 2}^\top \end{bmatrix}, \tag{2.32}$$

where $E_{\ell 2}^\top = V_{\ell 0}^{-\top} L_{\hat{M}}$ and $F_{\ell 2}^\top$ is an arbitrary $(n-2\ell)\times(n-2\ell)$ orthogonal matrix.

Case 3. We transform $U_\ell^\top$ to $[U_{\ell 0}^\top, 0]$ by an orthogonal transformation. Then we find a matrix $G_{\ell 3}^\top \in \mathbb{R}^{(2n-2\ell)\times(2n-2\ell)}$ in the form

$$G_{\ell 3}^\top = \begin{bmatrix} E_{\ell 3}^\top & 0 \\ 0 & F_{\ell 3}^\top \end{bmatrix}, \tag{2.33}$$

where $E_{\ell 3}^\top = U_{\ell 0}^{-\top} L_{\hat{K}}$ and $F_{\ell 3}^\top$ is an arbitrary $(n-2\ell)\times(n-2\ell)$ orthogonal matrix.

The purpose of finding $G_{\ell 2}^\top$ and $G_{\ell 3}^\top$ in the form of (2.32) and (2.33) is to, not only solve the ISQEP, but also best approximate the original $\hat{M}$ and $\hat{K}$, respectively, in the sense that

$$\|V_\ell^\top G_{\ell 2}^\top G_{\ell 2} V_\ell - \hat{M}\|_F \tag{2.34}$$

and

$$\|U_\ell^\top G_{\ell 3}^\top G_{\ell 3} U_\ell - \hat{K}\|_F \tag{2.35}$$

are minimized by $G_{\ell 2}^\top$ and $G_{\ell 3}^\top$, respectively. Once these matrices are found, we define quadratic pencils $Q_{\ell 2}(\lambda)$ and $Q_{\ell 3}(\lambda)$ in exactly the same way as we define $Q_{\ell 1}(\lambda)$.

It would be interesting to see how the reconstructed quadratic pencils for the ISQEP, with the above-mentioned optimization in mind, approximate the original pencil. Toward that end, we measure the total difference

$$d_{\ell j} = \|M_{\ell j} - \hat{M}\|_F + \|C_{\ell j} - \hat{C}\|_F + \|K_{\ell j} - \hat{K}\|_F, \tag{2.36}$$

between the original pencil and the reconstructed pencil for each $j = 1, 2, 3$ and $\ell = 1, \ldots, 5$.

In Figure 2.1 we plot the *error* $d_{\ell j}$ between $\hat{Q}(\lambda)$ and $Q_{\ell j}(\lambda)$ for the various cases. Not surprisingly, we notice that the quadratic pencil $Q_{\ell 1}(\lambda)$ constructed from $G_{\ell 1}^\top$ is superior to the other two. What might be interesting to note is that in Case 1 the amount of eigeninformation available to the ISQEP does not seem to make any significance difference in the measurement of $d_{\ell 1}$. That is, all $d_{\ell 1}$ seems to be of the same order regardless of the value of ℓ. We think a reason for this happening is because $G_{\ell 1}^\top$ has somewhat more freedom to choose so that $M_{\ell 1}$ and $K_{\ell 1}$ better approximate $\hat{M}$ and $\hat{K}$, respectively.

In real application for vibrating systems, the stiffness matrix $\hat{K}$ and the mass matrix $\hat{M}$ of a mathematical model can usually be obtained by finite element or finite difference method. It is the damping matrix $\hat{C}$ in such a system that is generally not known. If some partial eigenstructure can be measured by experiment, then the construction proposed in Case 1 might be a good way to recover the original system by best approximating the stiffness matrix and the mass matrix in the sense of minimizing (2.29).

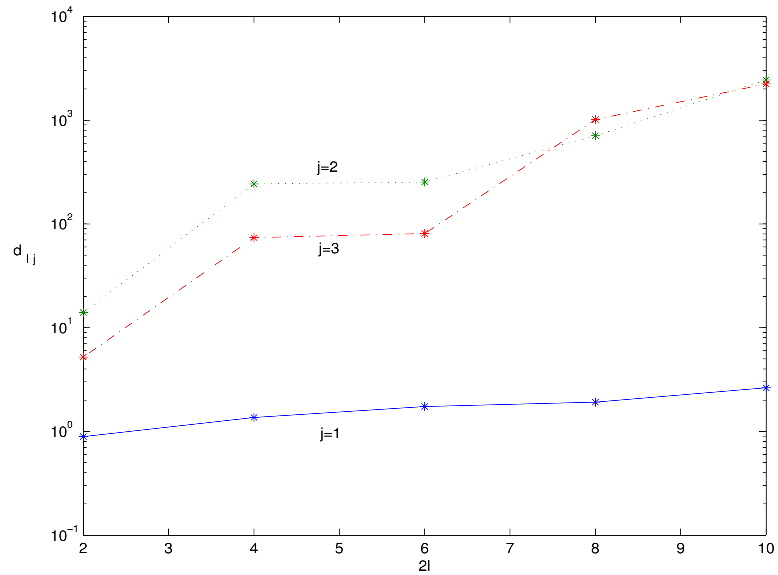

FIG. 2.1. *Errors of ISQEP approximations.*

3. Solving IMQEP. With the existence theory established in the preceding section for the ISQEP where $k = n$ plays a vital role in deciding whether the resulting quadratic pencil is singular, it is interesting to study in this section yet another scenario of the IQEP.

In the IMQEP, the leading matrix coefficient M is known and fixed and only symmetric C and K are to be determined. We have already suggested earlier by counting the cardinality of unknowns and equations that the number of prescribed eigenpair could go up to $k = n+1$. Since the prescribed eigenvectors now form a matrix X of size $n \times (n+1)$, by assuming that X is of full rank, there is at least one column in the given $n \times (n+1)$ matrix X depending linearly on the other columns. The following analysis is contingent on whether this linearly dependent column is real-valued or complex-valued. We separate the discussion into two cases. Either case shows a way to solve the IMQEP.

3.1. Real Linearly Dependent Eigenvector. Assume that the linearly dependent column vector is real-valued. By rearranging the columns if necessary, we may assume without loss of generality that this vector is $\mathbf{x}_{n+1}$. It follows that the $n \times n$ submatrix

$$X_1 := [\mathbf{x}_1, \bar{\mathbf{x}}_1, \ldots, \mathbf{x}_{2\ell-1}, \bar{\mathbf{x}}_{2\ell-1}, \mathbf{x}_{2\ell+1}, \ldots, \mathbf{x}_n]. \tag{3.1}$$

of $\tilde{X}$ defined in (1.8) is nonsingular. Let

$$\Lambda_1 := \operatorname{diag}\{\lambda_1, \bar{\lambda}_1, \ldots, \lambda_{2\ell-1}, \bar{\lambda}_{2\ell-1}, \lambda_{2\ell+1}, \ldots, \lambda_n\}, \tag{3.2}$$

be the corresponding submatrix of $\tilde{\Lambda}$ defined in (1.7). Both matrices are closed under complex conjugation in the sense defined before.

Define

$$S := X_1 \Lambda_1 X_1^{-1}. \tag{3.3}$$

Note that, due to the complex conjugation, S is a real-valued $n \times n$ matrix. Define a quadratic pencil $Q(\lambda)$ via the factorization

$$Q(\lambda) := (\lambda I_n + S + C)(\lambda I_n - S), \tag{3.4}$$

where C is yet to be determined. Upon comparing the expression of (3.4) with (1.10), we see that

$$K = -(S + C)S. \tag{3.5}$$

The first criterion for solving the IMQEP is that both matrices C and K be real-valued and symmetric. Thus the undetermined real-valued matrix C must first satisfy the following two equations simultaneously:

$$\begin{cases} C^\top = C, \\ S^\top C - CS = S^2 - (S^\top)^2. \end{cases} \tag{3.6}$$

The following result provides a partial characterization of the matrix C we are looking for.

THEOREM 3.1. *The general solution to (3.6) is given by the formula*

$$C = -(S + S^\top) + \sum_{j=1}^{n} \gamma_j \mathbf{y}_j \mathbf{y}_j^\top. \tag{3.7}$$

where vectors $\mathbf{y}_j$, $j = 1, \dots n$, are the columns of the matrix

$$Y_1 := X_1^{-\top} = [\mathbf{y}_1, \dots, \mathbf{y}_{2\ell-1}, \mathbf{y}_{2\ell}, \mathbf{y}_{2\ell+1}, \dots, \mathbf{y}_n] \tag{3.8}$$

and the scalars γ_j, $j = 1, \dots, n$, are arbitrary complex numbers.

Proof. It is easy to see that $-(S + S^\top)$ is a particular solution of (3.6). The formula thus follows from an established result ([11, Theorem 1, Section 12.5]). □

It might be worth mentioning that the columns of Y_1 are also closed under complex conjugation and, hence, C is real-valued if and only if the corresponding coefficients γ_j are complex conjugate. It only remains to determine these combination coefficients in (3.7) so that the IMQEP is solved. Toward that end, observe first that

$$X_1 \Lambda_1^2 + C X_1 \Lambda_1 + K X_1 = 0, \tag{3.9}$$

regardless how the scalars γ_j, $j = 1, \dots, n$, are chosen. In other words, n pairs of the given data have already satisfied the spectral constraint in the IMQEP. We use the fact that the last pair $(\lambda_{n+1}, \mathbf{x}_{n+1}) \in \mathbb{R} \times \mathbb{R}^n$ in the given data must also be an eigenpair of $Q(\lambda)$ in (3.4) to determine the parameters γ_j, $j = 1, \dots, n$.

Define

$$\mathbf{z} := (\lambda_{n+1} I - S)\mathbf{x}_{n+1} \in \mathbb{R}^n. \tag{3.10}$$

Plugging the eigenpair $(\lambda_{n+1}, \mathbf{x}_{n+1})$ into (3.4) and using (3.7), we obtain the equation

$$\lambda_{n+1}\mathbf{z} = S^\top \mathbf{z} - \sum_{j=1}^{n} \gamma_j \mathbf{y}_j \mathbf{y}_j^\top \mathbf{z}$$

which can be written as

$$-X_1^\top(\lambda_{n+1}\mathbf{z} - S^\top \mathbf{z}) = \operatorname{diag}\{\mathbf{y}_1^\top \mathbf{z}, \dots, \mathbf{y}_n^\top \mathbf{z}\} \begin{bmatrix} \gamma_1 \\ \vdots \\ \gamma_n \end{bmatrix}. \tag{3.11}$$

Obviously, values of $\gamma_j \mathbf{y}_j^\top \mathbf{z}$, $j = 1, \dots, n$, are uniquely determined. However, the value of γ_j is unique only if

$$\mathbf{y}_j^\top \mathbf{z} = \mathbf{e}_j^\top X_1^{-1} \mathbf{z} \neq 0, \tag{3.12}$$

where $\mathbf{e}_j$ denotes the jth standard unit vector. In terms of the original data, the condition can be written equivalently as

$$\mathbf{e}_j^\top(\lambda_{n+1}I - \Lambda_1)X_1^{-1}\mathbf{x}_{n+1} \neq 0. \tag{3.13}$$

If we assume that the condition (3.12) holds for all $j = 1, \ldots, n$, then the last step in solving the IMQEP is to show that elements in the solution $\{\gamma_1, \ldots, \gamma_n\}$ to (3.1) are closed under complex conjugation in exact the way as columns of Y_1.

For convenience, we shall denote

$$\mathbf{r} := \lambda_{n+1}\mathbf{z} - S^\top\mathbf{z} \in \mathbb{R}^n. \tag{3.14}$$

The first 2ℓ elements in the equation (3.11) are

$$-\mathbf{x}_{2j-1}^\top\mathbf{r} = \mathbf{y}_{2j-1}^\top\mathbf{z}\gamma_{2j-1}, \tag{3.15}$$

$$-\mathbf{x}_{2j}^\top\mathbf{r} = \mathbf{y}_{2j}^\top\mathbf{z}\gamma_{2j}, \quad \text{for } j = 1, \ldots, \ell. \tag{3.16}$$

Recall $\mathbf{x}_{2j-1} = \bar{\mathbf{x}}_{2j}$ and $\mathbf{y}_{2j-1} = \bar{\mathbf{y}}_{2j}$ for $j = 1, \ldots, \ell$. Upon taking the conjugation of (3.15) and comparing with (3.16), we conclude that

$$\gamma_{2j} = \bar{\gamma}_{2j-1}, \quad \text{for } j = 1, \ldots, \ell. \tag{3.17}$$

Similarly, $\gamma_k \in \mathbb{R}$, for $k = 2\ell + 1, \ldots, n$. It is now finally proved that both C and K are indeed real-valued and symmetric. We summarize our first major result as follows:

THEOREM 3.2. *Let $(\tilde{\Lambda}, \tilde{X}) \in \mathbb{C}^{(n+1)\times(n+1)} \times \mathbb{C}^{n\times(n+1)}$ be given as in (1.7) and (1.8). Assume that one eigenvector, say, $\mathbf{x}_{n+1} \in \mathbb{R}^n$, depends linearly on the remaining eigenvectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$ which are linearly independent. If the condition (3.13) is satisfied for all $j = 1, \ldots, n$, then the IMQEP has a unique solution.*

We point out quickly that the equation (3.11) is not necessarily consistent. In particular, a possible scenario is as follows.

COROLLARY 3.3. *Under the same assumptions as in Theorem 3.2, if $\mathbf{e}_j^\top X_1^{-1}\mathbf{z} = 0$ and $\mathbf{e}_j^\top(\lambda_{n+1}I - \Lambda_1^\top)X_1^\top\mathbf{z} \neq 0$ for some j, then the IMQEP has no solution.*

3.2. Complex Linearly Dependent Eigenvector. Assume that the linearly dependent column vector is complex-valued. By rearranging the columns if necessary, we may assume without loss of generality that this vector is $\mathbf{x}_{2\ell}$. It follows that the $n \times n$ matrix

$$X_1 = [\underbrace{\mathbf{x}_1, \bar{\mathbf{x}}_1, \ldots, \mathbf{x}_{2\ell-3}, \bar{\mathbf{x}}_{2\ell-3}}_{complex-conjugated}, \underbrace{\mathbf{x}_{2\ell+1}, \ldots, \mathbf{x}_{n+1}}_{real-valued}, \quad \underbrace{\mathbf{x}_{2\ell-1}}_{complex-valued} \quad]$$

is nonsingular. For convenience, we shall re-index the sequence of the above column vectors by successive integers. Without causing ambiguity, we shall use the same notation for the renumbered vectors. Specifically, we rewrite the above X_1 as

$$X_1 = [\underbrace{\mathbf{x}_1, \bar{\mathbf{x}}_1, \ldots, \mathbf{x}_{2m-1}, \bar{\mathbf{x}}_{2m-1}}_{complex-conjugated}, \underbrace{\mathbf{x}_{2m+1}, \ldots, \mathbf{x}_{n-1}}_{real-valued}, \quad \underbrace{\mathbf{x}_n}_{complex-valued} \quad], \tag{3.18}$$

column by column but only rename the indices, and define the corresponding

$$\Lambda_1 = \operatorname{diag}\{\lambda_1, \bar{\lambda}_1, \ldots, \lambda_{2m-1}, \bar{\lambda}_{2m-1}, \lambda_{2m+1}, \ldots, \lambda_{n-1}, \lambda_n\}. \tag{3.19}$$

We could further assume in (3.18) that

$$\bar{\mathbf{x}}_n \in \ \operatorname{span}\{\mathbf{x}_1, \bar{\mathbf{x}}_1, \ldots, \mathbf{x}_{2m-1}, \bar{\mathbf{x}}_{2m-1}, \mathbf{x}_n\}, \tag{3.20}$$

since otherwise one of the real-valued eigenvectors (and this is possible only if $2m + 1 < n$) must be linearly dependent and we would go back to the case in Section 3.1. The following argument is analogous to that of Section 3.1, but additional details need to be filled in.

Following (3.3) through (3.5) except that S is now complex-valued, we want to determine the matrix C in the factorization (3.4) and the corresponding K via several steps. We first require both C and K to be Hermitian. That is, the matrix C must satisfy the following equations:

$$\begin{cases} C^H = C \in \mathbb{C}^{n\times n}, \\ S^H C - CS = S^2 - (S^H)^2 \in \mathbb{C}^{n\times n}. \end{cases} \tag{3.21}$$

In contrast to Theorem 3.1, the characterization of C is a little bit more complicated.

THEOREM 3.4. *The general solution to (3.21) is given by the formula*

$$\begin{aligned} C = -(S + S^H) &+ \gamma_1 \mathbf{y}_1 \mathbf{y}_2^H + \gamma_2 \mathbf{y}_2 \mathbf{y}_1^H + \cdots + \gamma_{2m-1} \mathbf{y}_{2m-1} \mathbf{y}_{2m}^H + \gamma_{2m} \mathbf{y}_{2m} \mathbf{y}_{2m-1}^H \\ &+ \gamma_{2m+1} \mathbf{y}_{2m+1} \mathbf{y}_{2m+1}^H + \cdots + \gamma_{n-1} \mathbf{y}_{n-1} \mathbf{y}_{n-1}^H. \end{aligned} \tag{3.22}$$

where vectors $\mathbf{y}_i$, $i = 1, \ldots, n$ are the columns of the matrix

$$Y_1 := X_1^{-H} = [\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_{2m}, \mathbf{y}_{2m+1}, \ldots, \mathbf{y}_n]. \tag{3.23}$$

Proof. Again, the formula is similar to that in Theorem 3.1 using exactly the same established result ([11, Theorem 1, Section 12.5]). The slight complication is due to the fact that the first $2m$ eigenvalues of S and S^H coincide in a conjugated way and the last eigenvalues of S and S^H are distinct. ☐

Note that for $j = 1, \ldots m$, $\mathbf{y}_{2j-1}$, $\mathbf{y}_{2j} = \bar{\mathbf{y}}_{2j-1}$ are the eigenvectors of S^H with eigenvalues to $\bar{\lambda}_{2j-1}$ and λ_{2j-1}, respectively. Likewise, for $k = 2m + 1, \ldots, n - 1$, $\mathbf{y}_k \in \mathbb{R}^n$ is the eigenvector of S^H corresponding to $\lambda_j \in \mathbb{R}$. Finally, $\mathbf{y}_n \in \mathbb{C}^n$ is the eigenvector of S^H corresponding to $\bar{\lambda}_n \in \mathbb{C}$.

By construction, we already know that (3.9) is satisfied with X_1 defined by (3.18) and C defined by (3.22). It remains to determine the coefficients $\gamma_1, \ldots, \gamma_{n-1}$ so that the deleted linearly dependent vector $\bar{\mathbf{x}}_n$ (the original $\mathbf{x}_{2\ell}$ before the re-indexing) is also an eigenvector with eigenvalue $\bar{\lambda}_n$. Of course, we also need to make sure that the resulting C and K are real-valued ultimately.

Let

$$\mathbf{z} = (\bar{\lambda}_n I_n - S)\bar{\mathbf{x}}_n. \tag{3.24}$$

Substituting the eigenpair $(\bar{\lambda}_n, \bar{\mathbf{x}}_n)$ into (3.4) and using (3.22), we obtain

$$(\bar{\lambda}_n I_n - S^H)\mathbf{z} = -[\mathbf{y}_1, \ldots, \mathbf{y}_{2m}, \mathbf{y}_{2m+1}, \ldots, \mathbf{y}_{n-1}] \begin{bmatrix} \gamma_1 \mathbf{y}_2^H \mathbf{z} \\ \gamma_2 \mathbf{y}_1^H \mathbf{z} \\ \vdots \\ \gamma_{2m-1} \mathbf{y}_{2m}^H \mathbf{z} \\ \gamma_{2m} \mathbf{y}_{2m-1}^H \mathbf{z} \\ \gamma_{2m+1} \mathbf{y}_{2m+1}^H \mathbf{z} \\ \vdots \\ \gamma_{n-1} \mathbf{y}_{n-1}^H \mathbf{z} \end{bmatrix}. \tag{3.25}$$

With the assumption of (3.20), it is not difficult to see that

$$\mathbf{y}_j^H \mathbf{z} = 0, \quad \text{for } j = 2m + 1, \ldots n - 1. \tag{3.26}$$

The equation of (3.25) is equivalent to the equation

$$(\bar{\lambda}_n I_n - \Lambda_1^H) X_1^H \mathbf{z} = - \begin{bmatrix} \gamma_1 \mathbf{y}_2^H \mathbf{z} \\ \gamma_2 \mathbf{y}_1^H \mathbf{z} \\ \vdots \\ \gamma_{2m-1} \mathbf{y}_{2m}^H \mathbf{z} \\ \gamma_{2m} \mathbf{y}_{2m-1}^H \mathbf{z} \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{3.27}$$

The left-hand side of (3.27) is completely known. It is now clear that the coefficients $\gamma_1, \ldots, \gamma_{2m}$ are uniquely determined if

$$\mathbf{y}_j^H \mathbf{z} \neq 0, \quad \text{for } j = 1, \ldots, 2m, \tag{3.28}$$

whereas the coefficients $\gamma_{2m+1}, \ldots, \gamma_{n-1}$ in (3.25) (and hence in (3.22)) can be arbitrary real numbers so long as the last $n - 2m$ equations in (3.27) are consistent, that is,

$$\mathbf{x}_j^H \mathbf{z} = 0, \quad \text{for } j = 2m+1, \ldots, n-1. \tag{3.29}$$

Assuming (3.28) and (3.29), we now show that the resulting matrix C in (3.22) is Hermitian. Toward that end, it suffices to show that $\gamma_{2j-1} = \bar{\gamma}_{2j}$, for $j = 1, \ldots, m$. Based on (3.26) and (3.29), we introduction the following two vectors for convenience.

$$\mathbf{p} := X_1^H \mathbf{z} = [p_1, \ldots, p_{2m}, 0, \ldots, 0, p_n]^T \in \mathbb{C}^n, \tag{3.30}$$
$$\mathbf{q} := X_1^{-1} \bar{\mathbf{x}}_n = [q_1, \ldots, q_{2m}, 0, \ldots, 0, q_n]^T \in \mathbb{C}^n. \tag{3.31}$$

For $j = 1, \ldots m$, the $(2j-1)$-th and the $(2j)$-th components of (3.27) are, respectively,

$$\begin{aligned} (\bar{\lambda}_n - \bar{\lambda}_{2j-1}) p_{2j-1} &= -\gamma_{2j-1} \mathbf{y}_{2j}^H \mathbf{z} = -\gamma_{2j-1} (\bar{\lambda}_n - \lambda_{2j}) q_{2j}, \\ (\bar{\lambda}_n - \bar{\lambda}_{2j}) p_{2j} &= -\gamma_{2j} \mathbf{y}_{2j-1}^H \mathbf{z} = -\gamma_{2j} (\bar{\lambda}_n - \lambda_{2j-1}) q_{2j-1}. \end{aligned}$$

Since $\lambda_{2j-1} = \bar{\lambda}_{2j}$, it follows that

$$p_{2j-1} = -\gamma_{2j-1} q_{2j}, \tag{3.32}$$
$$p_{2j} = -\gamma_{2j} q_{2j-1}. \tag{3.33}$$

On the other hand, observe that

$$\mathbf{z} = (\bar{\lambda}_n I_n - S) \bar{\mathbf{x}}_n = (\bar{\lambda}_n I_n - \bar{S} + \bar{S} - S) \bar{\mathbf{x}}_n = (\bar{S} - S) \bar{\mathbf{x}}_n \tag{3.34}$$

since $\bar{\mathbf{x}}_n$ is an eigenvector of $\bar{S}$. Observe also that

$$(\bar{S} - S) \mathbf{x}_j = 0 \quad \text{for } j = 1, \ldots, n-1 \tag{3.35}$$

because of the complex conjugation. It follows that

$$(\bar{S} - S) \bar{\mathbf{x}}_n = q_n (\bar{S} - S) \mathbf{x}_n \tag{3.36}$$
$$\mathbf{x}_{2j-1}^H (\bar{S} - S) \bar{\mathbf{x}}_n = -\gamma_{2j-1} q_{2j} \tag{3.37}$$
$$\mathbf{x}_{2j}^H (\bar{S} - S) \bar{\mathbf{x}}_n = -\gamma_{2j} q_{2j-1}. \tag{3.38}$$

Taking conjugation of (3.37) and using (3.36), we obtain

$$-\bar{q}_n \bar{\mathbf{x}}_{2j-1}^H (\bar{S} - S)\bar{\mathbf{x}}_n = -\bar{\gamma}_{2j-1}\bar{q}_{2j}. \tag{3.39}$$

Comparing (3.38) and (3.39), since $\bar{\mathbf{x}}_{2j-1} = \mathbf{x}_{2j}$ for all $j = 1, \ldots, m$, we obtain a critical relationship that

$$\bar{q}_n \gamma_{2j} q_{2j-1} = -\bar{\gamma}_{2j-1}\bar{q}_{2j}, \quad \text{for} \quad j = 1, \ldots, m. \tag{3.40}$$

Now we are ready to show that $\gamma_{2j} = \bar{\gamma}_{2j-1}$, $j = 1, \ldots, m$. We rewrite $\mathbf{x}_n = \bar{X}_1 \bar{\mathbf{q}}$ from (3.31) as

$$\begin{aligned} \mathbf{x}_n &= \bar{q}_1 \bar{\mathbf{x}}_1 + \bar{q}_2 \bar{\mathbf{x}}_2 + \cdots + \bar{q}_{2m-1}\bar{\mathbf{x}}_{2m-1} + \bar{q}_{2m}\bar{\mathbf{x}}_{2m} + \bar{q}_n \bar{\mathbf{x}}_n \\ &= \bar{q}_1 \mathbf{x}_2 + \bar{q}_2 \mathbf{x}_1 + \cdots + \bar{q}_{2m-1}\mathbf{x}_{2m} + \bar{q}_{2m}\mathbf{x}_{2m-1} + \bar{q}_n \bar{\mathbf{x}}_n. \end{aligned} \tag{3.41}$$

Replacing the last term by

$$\bar{q}_n \bar{\mathbf{x}}_n = \bar{q}_n q_1 \mathbf{x}_1 + \bar{q}_n q_2 \mathbf{x}_2 + \cdots + \bar{q}_n q_{2m}\mathbf{x}_{2m} + |q_n|^2 \mathbf{x}_n,$$

we obtain the equality

$$\begin{aligned} \mathbf{x}_n = &(\bar{q}_n q_1 + \bar{q}_2)\mathbf{x}_1 + (\bar{q}_n q_2 + \bar{q}_1)\mathbf{x}_2 + \cdots + (\bar{q}_n q_{2m-1} + \bar{q}_{2m})\mathbf{x}_{2m-1} \\ &+ (\bar{q}_n q_{2m} + \bar{q}_{2m-1})\mathbf{x}_{2m} + |q_n|^2 \mathbf{x}_n. \end{aligned}$$

Since $\{\mathbf{x}_1, \ldots, \mathbf{x}_{2m}, \mathbf{x}_n\}$ are linearly independent, it holds that

$$\bar{q}_n q_{2j-1} + \bar{q}_{2j} = 0, \quad \text{for } j = 1, \ldots, m, \tag{3.42}$$

Substituting (3.42) into (3.40), we have proved that $\gamma_{2j} = \bar{\gamma}_{2j-1}$, for $j = 1, \ldots, m$.

By now, we have completed the proof that the matrix C constructed using (3.27) is Hermitian. We are ready to state our second major result.

THEOREM 3.5. *Let $(\tilde{\Lambda}, \tilde{X}) \in \mathbb{C}^{(n+1)\times(n+1)} \times \mathbb{C}^{n\times(n+1)}$ be given as in (1.7) and (1.8). Assume that one eigenvector, say, $\mathbf{x}_{2\ell} \in \mathbb{C}^n$, depends linearly on the remaining eigenvectors which are linearly independent. Then*

1. *Suppose $\ell = \frac{n+1}{2}$, that is, suppose that there is no real-valued vector at all in X. If the condition (3.28) is satisfied for $j = 1, \ldots, n-1$, then the IMQEP has a unique solution.*
2. *Suppose $\ell < \frac{n+1}{2}$ and that (3.20) holds. If the condition (3.28) is satisfied for $j = 1, \ldots, 2\ell - 2$ and the condition (3.29) is satisfied for $j = 2\ell + 1, \ldots, n+1$, then the IMQEP has infinite many solutions; otherwise it has no solution.*

Proof. Thus far, we have already shown that both matrices C and K can be constructed uniquely and are Hermitian. It only remains to show that C and K are real symmetric. It suffices to prove that $C = \bar{C}$ and $K = \bar{K}$.

Consider the IMQEP associated with the spectral data $(\bar{\tilde{\Lambda}}, \bar{\tilde{X}})$, the complex conjugate of the original data $(\tilde{\Lambda}, \tilde{X})$. Then the sufficient condition (3.28) for the problem associated with $(\tilde{\Lambda}, \tilde{X})$ applies equally well to the new problem associated with $(\bar{\tilde{\Lambda}}, \bar{\tilde{X}})$. A quadratic pencil therefore can be constructed to solve the new IMQEP. Indeed, by repeating the procedure of construction described above, it is not difficult to see that the constructed pencil for $(\bar{\tilde{\Lambda}}, \bar{\tilde{X}})$ is of the form

$$\widetilde{Q}(\lambda) = \lambda^2 I_n + \lambda \bar{C} + \bar{K}.$$

Since Λ and X are closed under complex conjugation, the spectral information $(\bar{\tilde{\Lambda}}, \bar{\tilde{X}})$ is actually a reshuffle of (Λ, X). As a matter of fact, these two IMQEPs are the same problem. In the first case where $\ell = \frac{n+1}{2}$, the solution is already unique. In the second case where $\ell < \frac{n+1}{2}$ and (3.20) holds, so long as the arbitrarily selected real coefficients $\gamma_{2m+1}, \ldots, \gamma_{n-1}$ remain fix, the complex-conjugated coefficients $\gamma_1, \ldots, \gamma_{2m}$ are also uniquely determined. In either case, we must have that $C = \bar{C} = C^H$ and $K = \bar{K} = K^H$. ∎

3.3. Numerical Examples. The argument presented in the proceeding section offers a constructive way to solve the IMQEP. In this section we use numerical examples to illustrate the two cases discussed above. For the ease of running text, we report all numbers in 5 significant digits only, though all calculations are carried out in full precision.

Example 1. To generate test data, we first randomly generate a 5×5 real symmetric quadratic pencil $Q(\lambda) = \lambda^2 I + \lambda C + K$ and compute its "exact" eigenpairs (Λ_e, X_e) numerically. We obtain that $\Lambda_e = \mathrm{diag}\{\lambda_1, \ldots, \lambda_{10}\}$, $X_e = [\mathbf{x}_1, \ldots, \mathbf{x}_{10}]$ with $\lambda_1 = -0.31828 + 0.86754i = \bar{\lambda}_2$, $\lambda_3 = -0.95669 + 0.17379i = \bar{\lambda}_4$, $\lambda_5 = -4.4955$, $\lambda_6 = 1.5135$, $\lambda_7 = -0.24119 + 0.029864i = \bar{\lambda}_8$, $\lambda_9 = 0.91800$, $\lambda_{10} = -1.7359$, and the corresponding eigenvectors

$$\mathbf{x}_1 = \bar{\mathbf{x}}_2 = \begin{bmatrix} 15.159 - 11.123i \\ -77.470 - 14.809i \\ 2.1930 - 10.275i \\ 0.38210 + 16.329i \\ 57.042 + 18.419i \end{bmatrix}, \ \mathbf{x}_3 = \bar{\mathbf{x}}_4 = \begin{bmatrix} 65.621 + 34.379i \\ 22.625 + 24.189i \\ -37.062 + 15.825i \\ -9.6496 + 14.401i \\ -0.61893 + 25.609i \end{bmatrix}, \ \mathbf{x}_5 = \begin{bmatrix} 2.2245 \\ 1.5893 \\ 2.1455 \\ 2.1752 \\ 1.6586 \end{bmatrix},$$

$$\mathbf{x}_6 = \begin{bmatrix} 34.676 \\ -5.8995 \\ 37.801 \\ -66.071 \\ -6.6174 \end{bmatrix}, \ \mathbf{x}_7 = \bar{\mathbf{x}}_8 = \begin{bmatrix} 35.257 - 0.31888i \\ -25.619 - 4.2156i \\ 98.914 - 1.0863i \\ -21.348 + 5.8290i \\ -97.711 - 1.0693i \end{bmatrix}, \ \mathbf{x}_9 = \begin{bmatrix} -97.828 \\ 10.879 \\ 100.00 \\ -4.3638 \\ 22.282 \end{bmatrix}, \ \mathbf{x}_{10} = \begin{bmatrix} -1.3832 \\ 4.4564 \\ -1.1960 \\ -4.0934 \\ 5.7607 \end{bmatrix},$$

Note that the above spectral data are not arranged in any specific order. According to our theory, any $n+1$ eigenpairs satisfying the specification of (3.1) and (3.2) and the sufficient condition (3.13) or (3.28), depending upon whether assumptions in Section 3.1 or Section 3.2 with $\ell = \frac{n+1}{2}$ are applicable, should ensure the full recovery of the original pencil.

Case 1: Suppose the prescribed partial eigeninformation is given by

$$(\tilde{\Lambda}, \tilde{X}) = (\mathrm{diag}\{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6\}, \ [\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4, \mathbf{x}_5, \mathbf{x}_6]).$$

It is easy to check that the real-valued eigenvector $\mathbf{x}_6$ depends linearly on the first five eigenvectors which are linearly independent. This fits the situation discussed in Section 3.1 where we choose to work with

$$(\hat{\Lambda}_1, \hat{X}_1) = \left(\mathrm{diag}\{\lambda_1, \bar{\lambda}_1, \lambda_3, \bar{\lambda}_3, \lambda_5\}, \ [\mathbf{x}_1, \bar{\mathbf{x}}_1, \mathbf{x}_3, \bar{\mathbf{x}}_3, \mathbf{x}_5]\right).$$

We construct the unique real symmetric quadratic pencil

$$\hat{Q}(\lambda) = \lambda^2 I_5 + \lambda \hat{C} + \hat{K}$$

by the method described in the proof of Theorem 3.2. In Tables 3.1 and 3.2, we show the residual $\|\hat{Q}(\lambda_j)\mathbf{x}_j\|_2$, where $(\lambda_j, \mathbf{x}_j)$ are the computed eigenpairs of $Q(\lambda)$, for $j = 1, \ldots, 10$,, as well as the difference $\|\hat{C} - C\|_2$ and $\|\hat{K} - K\|_2$, respectively.

Table 3.1

eigenpairs	residual $\|\hat{Q}_1(\lambda_j)\mathbf{x}_j\|_2$
$(\lambda_1, \mathbf{x}_1)$	2.2612e-015
$(\lambda_2, \mathbf{x}_2)$	2.2612e-015
$(\lambda_3, \mathbf{x}_3)$	2.9827e-015
$(\lambda_4, \mathbf{x}_4)$	2.9827e-015
$(\lambda_5, \mathbf{x}_5)$	2.0381e-015
$(\lambda_6, \mathbf{x}_6)$	1.8494e-014
$(\lambda_7, \mathbf{x}_7)$	7.9955e-014
$(\lambda_8, \mathbf{x}_8)$	7.9955e-014
$(\lambda_9, \mathbf{x}_9)$	4.4264e-014
$(\lambda_{10}, \mathbf{x}_{10})$	4.5495e-014

Table 3.2

$\|\hat{C} - C\|_2$	1.8977e-014
$\|\hat{K} - K\|_2$	7.3897e-014

Case 2: Suppose the prescribed spectral information is given by

$$(\tilde{\Lambda}, \tilde{X}) = (\mathrm{diag}\{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_7, \lambda_8\},\ [\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4, \mathbf{x}_7, \mathbf{x}_8]).$$

Note that all eigenvectors are complex-valued. This fits the situation discussed in Section 3.2 with $\ell = \frac{n+1}{2}$ where we choose to work with.

$$(\check{\Lambda}_1, \check{X}_1) = \left(\mathrm{diag}\{\lambda_1, \bar{\lambda}_1, \lambda_3, \bar{\lambda}_3, \lambda_7\},\ [\mathbf{x}_1, \bar{\mathbf{x}}_1, \mathbf{x}_3, \bar{\mathbf{x}}_3, \mathbf{x}_7]\right).$$

We construct the unique real symmetric quadratic pencil

$$\check{Q}(\lambda) = \lambda^2 I_5 + \lambda \check{C} + \check{K}$$

by the method described in the proof of Theorem 3.5. In Tables 3.3 and 3.4, we show the residual $\|\check{Q}(\lambda_j)\mathbf{x}_j\|_2$, for $j = 1, \ldots, 10$,, as well as the difference $\|\check{C} - C\|_2$ and $\|\check{K} - K\|_2$, respectively.

Table 3.3

eigenpairs	residual $\|\check{Q}(\lambda_j)\mathbf{x}_j\|_2$
$(\lambda_1, \mathbf{x}_1)$	4.5422e-016
$(\lambda_2, \mathbf{x}_2)$	4.5422e-016
$(\lambda_3, \mathbf{x}_3)$	7.8025e-016
$(\lambda_4, \mathbf{x}_4)$	7.8025e-016
$(\lambda_5, \mathbf{x}_5)$	3.7137e-014
$(\lambda_6, \mathbf{x}_6)$	2.9549e-014
$(\lambda_7, \mathbf{x}_7)$	9.4143e-016
$(\lambda_8, \mathbf{x}_8)$	9.4143e-016
$(\lambda_9, \mathbf{x}_9)$	6.0018e-014
$(\lambda_{10}, \mathbf{x}_{10})$	4.6464e-014

Table 3.4

$\|\check{C} - C\|_2$	1.9222e-014
$\|\check{K} - K\|_2$	1.7951e-014

It can be checked that both cases above satisfy the sufficient conditions (3.13) and (3.28), respectively. The errors shown in the tables seem to be quite satisfactory.

Example 2. In the previous example we demonstrate two scenarios of prescribed spectral information that give rise to the same unique solution to the IMQEP. Now we demonstrate the second situation in Theorem 3.5 when both $\ell < \frac{n+1}{2}$ and (3.20) take place. Our theory asserts that there will be either infinitely many solutions to the IMQEP or no solution at all.

Consider the case where $n = 4$ and the prescribed eigenvalues are given by $\lambda_1 = 3.3068 + 8.1301i = \bar{\lambda}_2$, $\lambda_3 = 1.8702 + 2.7268i = \bar{\lambda}_4$, $\lambda_5 = 5.4385$ with corresponding eigenvectors

$$\mathbf{x}_1 = \bar{\mathbf{x}}_2 = \begin{bmatrix} 0 \\ 9.2963 + 1.5007i \\ 2.3695 + 1.9623i \\ 3.8789 + 1.0480i \end{bmatrix},\ \mathbf{x}_3 = \bar{\mathbf{x}}_4 = \begin{bmatrix} 0 \\ 6.5809 + 8.3476i \\ 4.9742 + 8.0904i \\ 1.1356 + 5.5542i \end{bmatrix},\ \mathbf{x}_5 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

respectively. It is obvious upon inspection that the linearly dependent vector in the above X must be a complex-valued vector. Let this linearly dependent vector be $\mathbf{x}_4$. Then the real symmetric quadratic pencil

$$Q(\lambda) = \lambda^2 I + \lambda C + K,$$

where $C = -(S + S^H) + \gamma_1 \mathbf{y}_1 \mathbf{y}_2^H + \gamma_2 \mathbf{y}_2 \mathbf{y}_1^H + \gamma_3 \mathbf{y}_3 \mathbf{y}_3^H$ and $K = -(S + C)S$, can be constructed with arbitrary $\gamma_3 \in \mathbb{R}$. In Table 3.5 we show the residual $\|Q(\lambda_j)\mathbf{x}_j\|_2$, for $j = 1, \ldots, 5$ with various values of γ_3.

Table 3.5

	$\gamma_3 = 2.56$	$\gamma_3 = 40.6$	$\gamma_3 = 506$
eigenpairs	residual	residual	residual
$(\lambda_1, \mathbf{x}_1)$	2.9334e-011	2.9334e-011	2.9334e-011
$(\lambda_2, \mathbf{x}_2)$	2.9334e-011	2.9334e-011	2.9334e-011
$(\lambda_3, \mathbf{x}_3)$	7.8802e-011	7.8802e-011	7.8802e-011
$(\lambda_4, \mathbf{x}_4)$	7.8802e-011	7.8802e-011	7.8802e-011
$(\lambda_5, \mathbf{x}_5)$	1.7764e-015	2.8422e-014	4.5475e-013

Suppose we modify the first entries of the complex eigenvectors to

$$\mathbf{x}_1 = \bar{\mathbf{x}}_2 = \begin{bmatrix} 9.2963 + 1.5007i \\ 9.2963 + 1.5007i \\ 2.3695 + 1.9623i \\ 3.8789 + 1.0480i \end{bmatrix}, \; \mathbf{x}_3 = \bar{\mathbf{x}}_4 = \begin{bmatrix} 6.5809 + 8.3476i \\ 6.5809 + 8.3476i \\ 4.9742 + 8.0904i \\ 1.1356 + 5.5542i \end{bmatrix}, \; \mathbf{x}_5 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Still, we see that the linearly dependent vector in the corresponding X must be a complex-valued vector, say $\mathbf{x}_4$. However, we find that the condition (3.29) is not satisfied because

$$\mathbf{x}_3^H \mathbf{z} = \mathbf{x}_3^H (\bar{\lambda}_4 I_4 - S)\bar{\mathbf{x}}_4 = -115.54 + 600.67i \neq 0.$$

The system (3.27) being inconsistent, the real coefficient γ_3 in (3.22) is not solvable. We conclude that the prescribed vectors and the corresponding scalars λ_i, $i = 1, \ldots, 5$ indicated above cannot be part of the spectrum of any 4×4 real-valued, symmetric and monic quadratic pencil.

4. Conclusion. The quadratic eigenvalue problem arises in many important applications. Its inverse problem is equally important in practice. In a large or complicated system, often it is the case that only partial eigeninformation is available. To understand how a physical system modelled by a quadratic pencil should be modified with only partial eigeninformation in hand, it will be very helpful to first understand how the IQEP should be solved. Some general theory toward that end has been presented in this paper.

In the first part of this paper, we found that the ISQEP is solvable, provided that the number of given eigenpairs is less than or equal to the size of matrices and that the given vectors are linearly independent. A simple recipe for constructing such a matrix was described, which can serve as the basis for numerical computation. We also found that the unspecified eigenstructure of the reconstructed quadratic pencil is quite limited in the sense discussed in Section 2.2. We demonstrated three different ways for the construction that not only satisfied the spectral constraints but also best approximated the original analytical model in some least squares sense.

In the second part of this paper, we established some general existence theory for the inverse problem when the leading matrix coefficient M is known and fixed. The procedure used in the proof can also provide a basis for numerical computation.

It should be noted that the stiffness matrix K is normally more complicated than the mass matrix M. The requirement of maintaining physical feasibility also imposes constraints on the stiffness matrix, making it less flexible and more difficult to construct. Thus, one usual way of formulating an inverse eigenvalue problem is to have the stiffness matrix K determined and fixed from the existing structure, known as the static constraints, and then to find the mass matrix M so that some desired natural frequencies are achieved. This is sometimes so desired even without the damping term C. By exchanging the roles of M and K, the discussion in this paper could be applied equally well to the inverse quadratic eigenvalue problem formed with the aforementioned static constraints in mind.

The study made in this paper should have shed light on the long standing question of how much a quadratic pencil could be updated, modified, or tuned if some of its eigenvalues and eigenvectors are to be kept invariant. Finally, we should point out that there are unfinished tasks in this study. Among these, sensitivity analysis in the case of a unique solution, robustness in the case of multiple solutions, and existence theory where M or K are specially structured are just a few interesting topics that are yet to be further investigated.

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