

TOTAL DECOUPLING OF GENERAL QUADRATIC PENCILS, PART I: THEORY

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Abstract. The notion of quadratic pencils, $\lambda^2 M + \lambda C + K$, where M , C , and K are $n \times n$ real matrices with or without some additional properties such as symmetry or positive definiteness, plays critical roles in many important applications. It has been long desirable, yet with very limited success, to reduce a complicated high-degree-of-freedom system to some simpler low-degree-of-freedom subsystems. Recently Garvey et al. [J. Sound Vibration, 258(2002), pp. 885-909] proposed a promising approach by which, under some mild assumptions, a general quadratic pencils can be converted by real-valued isospectral transformations into a totally decoupled system. This approach, if numerically feasible, would reduce the original n -degree-of-freedom second order system to n totally independent single-degree-of-freedom second order subsystems. Such a claim would be a striking breakthrough in the common knowledge that generally no three matrices M , C , and K can be simultaneously diagonalized. This paper intends to serve three purposes: to clarify some of the ambiguities in the original proposition, to simplify some of the computational details and, most importantly, to complete the theory of existence by matrix polynomial factorization tactics.

Key words. quadratic pencil, Lancaster structure, structure preserving, multiple-degree-of-freedom system, model reduction, matrix polynomial factorization, equivalence transformation, canonical form, simultaneous diagonalization

AMS subject classifications. 15A22, 70H15, 93B10

1. Introduction. The quadratic eigenvalue problem (QEP) involves finding scalars λ and nonzero vectors \mathbf{u} to satisfy the equation

$$Q(\lambda)\mathbf{u} = 0, \tag{1.1}$$

where $Q(\lambda)$ is the quadratic pencil

$$Q(\lambda) := Q(\lambda; M, C, K) = \lambda^2 M + \lambda C + K \tag{1.2}$$

defined by three given $n \times n$ matrices M , C and K . The scalars λ and the corresponding vectors \mathbf{u} are called, respectively, eigenvalues and eigenvectors of the quadratic pencil $Q(\lambda)$. It is known that the QEP has $2n$ finite eigenvalues over the complex field, provided that the leading matrix coefficient M is nonsingular. We shall consider only real-valued coefficient matrices in this paper.

The QEP has been studied extensively because its formation arises frequently in wide ranging disciplines, including applied mechanics, electrical oscillation, vibro-acoustics, fluid mechanics, and signal processing. In a recent treatise, Tisseur and Meerbergen [9] surveyed a good many applications, mathematical properties, and a variety of numerical techniques for the QEP. The QEP arising in practice often entails some additional conditions on the coefficient matrices. Consider, for example, the three-degree-of-freedom mass-spring system depicted in Figure 1.1, where m_j , c_{ν_j} , and k_j represent the mass, damping and stiffness parameters, respectively. It is not difficult to see that the corresponding equation of motion has the following specifically structured matrix coefficients [2]

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \begin{bmatrix} c_{\nu_1} & 0 & 0 \\ 0 & c_{\nu_2} & -c_{\nu_2} \\ 0 & -c_{\nu_2} & c_{\nu_2} \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} + \begin{bmatrix} k_1 + k_2 + k_4 & -k_2 & -k_4 \\ -k_2 & k_2 + k_3 & -k_3 \\ -k_4 & -k_3 & k_3 + k_4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \end{bmatrix}.$$

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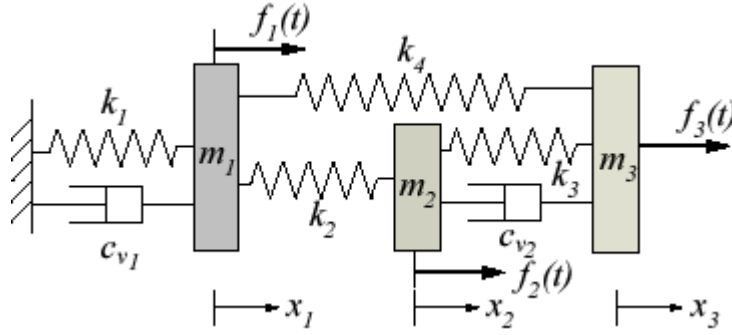


FIG. 1.1. A three-degree-of-freedom system (from www.efunda.com).

In general, the second-order dynamical system with n -degree-of-freedom is of the form

$$M\ddot{\mathbf{x}} + (C + G)\dot{\mathbf{x}} + (K + N)\mathbf{x} = F(t), \quad (1.3)$$

where the coefficient matrices may have the following interpretations and structures:

$$\begin{aligned} M &:= \text{Mass matrix; } M = M^\top > 0. \\ C &:= \text{Damping matrix; } C = C^\top. \\ K &:= \text{Stiffness matrix; } K = K^\top > 0. \\ G &:= \text{Gyroscopic matrix; } G^\top = -G. \\ N &:= \text{Dissipation matrix; } N^\top = -N. \\ F &:= \text{External force.} \end{aligned}$$

It is the homogeneous solution of (1.3) that is most critical in the long-term behavior of the system. Assume the homogeneous solution $\mathbf{x}(t)$ in the form

$$\mathbf{x}(t) = e^{\lambda t} \mathbf{u}.$$

Upon substitution (and without causing ambiguity, denoting the linear and the constant coefficients by the same notation C and K , respectively), we find that λ and \mathbf{u} are the nontrivial solution to the QEP:

$$Q(\lambda)\mathbf{u} := (\lambda^2 M + \lambda C + K)\mathbf{u} = 0.$$

An undamped QEP, that is, when $C = 0$, is equivalent to a generalized eigenvalue problem of which the spectral analysis is well established. In particular, there are modal coordinates under which the undamped QEP can be represented by diagonal coefficient matrices. In other words, the undamped QEP can be totally decoupled. This reduction of the original multiple-degree-of-freedom system to totally decoupled single-degree-of-freedom second order systems is very desirable from a practical point of view. Considerable efforts have been given to study the simultaneous diagonalization of two matrices by congruence or equivalence transformations. See, for example, the discussion in the book [8, Section 4.5] and the paper [10]. In a more realistic environment, however, it is more often that the QEP is damped. It is therefore desired to generalize the notion of reduction to three matrices. In the literature, a quadratic pencil is said to be *proportionally* or *classically damped* if all three matrices M , C and K can be simultaneously diagonalized. It is commonly accepted, however, that no equivalence or congruence coordinate transformations can simultaneously diagonalize the three coefficient matrices in a general quadratic system.

In two recent papers [3, 4], Garvey et al. proposed a different way to perceive the simultaneous diagonalization of three matrices. We outline their ideas below. It is easy to show that the QEP (1.1) is equivalent to the generalized eigenvalue problem

$$L(\lambda) \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix} = 0, \quad (1.4)$$

where

$$L(\lambda) := L(\lambda; M, C, K) = \begin{bmatrix} C & M \\ M & 0 \end{bmatrix} \lambda + \begin{bmatrix} K & 0 \\ 0 & -M \end{bmatrix}. \quad (1.5)$$

Clearly, if M is nonsingular, then $\mathbf{v} = \lambda \mathbf{u}$. The arrangement in the *symmetrically* linearized pencil $L(\lambda)$ is referred to as the *Lancaster structure*. If there exist nonsingular $2n \times 2n$ matrices Π_ℓ and Π_r such that the equivalence transformation applied to (1.5) maintains the Lancaster structure, that is,

$$\Pi_\ell L(\lambda) \Pi_r = \begin{bmatrix} C_D & M_D \\ M_D & 0 \end{bmatrix} \lambda + \begin{bmatrix} K_D & 0 \\ 0 & -M_D \end{bmatrix}, \quad (1.6)$$

and such that M_D , C_D and K_D are all diagonal matrices, then the QEP (1.1) is equivalent to the totally decoupled system

$$(\lambda^2 M_D + \lambda C_D + K_D) \mathbf{z} = 0. \quad (1.7)$$

In this case, the eigenvectors \mathbf{u} and \mathbf{z} are related by

$$\begin{bmatrix} \mathbf{u} \\ \lambda \mathbf{u} \end{bmatrix} = \Pi_r \begin{bmatrix} \mathbf{z} \\ \lambda \mathbf{z} \end{bmatrix}, \quad (1.8)$$

provided that M_D and M are nonsingular. Most importantly, the two quadratic systems (1.1) and (1.7) are isospectral.

The notion outlined above is different from the usual task of simultaneously diagonalizing the coefficient matrices in the linearized pencil $L(\lambda)$. Rather, by maintaining the Lancaster structure, the approach links a multiple-degree-of-freedom system directly to a single-degree-of-freedom system. For the idea to work, the following questions must be addressed:

1. Do the structure preserving transformations Π_ℓ and Π_r exist?
2. Can the transformations Π_ℓ and Π_r be real-valued so that the resulting diagonal matrices M_D , C_D and K_D remains to be real-valued?
3. Is there any relationship between Π_ℓ and Π_r , say, $\Pi_\ell = \Pi_r^\top$?
4. How to find the real-valued transformations Π_ℓ and Π_r numerically?

According to Garvey et al. [3], the answers to Questions 1 and 2 are affirmative. The proof was delineated in an appendix of [3] which, in our view, contains some ambiguities. We also think the instructions suggested in [3] for the construction of matrices Π_ℓ and Π_r contain some errors and are unnecessarily complicated. The goal of this paper is to reconfirm the fact that the canonical form described in (1.6) is achievable by offering a clearer and simpler proof. Along the way, we can answer Question 3 which is an open problem speculated in [3]. We are able to advance to the completion of the theory by applying a useful notion from matrix polynomial factorization described in [7].

This paper contains two main parts: The first part addresses general pencils. In section 3 we prove the existence of equivalence transformations by which almost all general quadratic pencils can be totally decoupled. Some of the original arguments by Garvey et al. are fixed and remarkably simplified [3]. Assuming the availability of spectral decomposition, the proof itself is constructive and can be converted into an algorithm. The second part addresses self-adjoint pencils. In section 4 we

begin with an outline of matrix polynomial factorization, showing that the real-valued eigenvalues of self-adjoint quadratic pencils are necessarily divided into two categories. We then show that the total decoupling can be achieved by congruence transformations. We believe these results are innovative and the theory is now complete.

Needless to say, it would be of great theoretical and practical significance if almost all n -degree-of-freedom systems can be completely decoupled into n single-degree-of-freedom subsystems. This paper establishes the theoretical foundation that such a reduction is possible.

2. Nonlinear Relationship. It has to be made clear that the procedure offered either in this paper or from [3] begins with the spectral decomposition of the pencil $L(\lambda)$, so the proof itself cannot serve as a numerical means to answer Question 4. We do have a numerical way working on the triplet (M, C, K) to reduce it to the triplet (M_D, C_D, K_D) , but the details will have to be discussed in a separate paper [1]. See also [5]. It is worth noting that the isospectral transformation from the triplet (M, C, K) to the triplet (M_D, C_D, K_D) is not the conventional equivalence transformation. Rather, it is a nonlinear relationship among all three matrices (M, C, K) .

Indeed, denoting

$$\Pi_\ell = \begin{bmatrix} \ell_{11} & \ell_{12} \\ \ell_{21} & \ell_{22} \end{bmatrix}, \quad \Pi_r = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix}, \quad (2.1)$$

where each ℓ_{ij} or r_{ij} is an $n \times n$ matrices, in order to maintain the Lancaster structure in the product $\Pi_\ell L(\lambda) \Pi_r$ it is necessary that the following five equations hold:

$$\begin{aligned} -\ell_{11}Kr_{12} + \ell_{12}Mr_{22} &= 0, \\ -\ell_{21}Kr_{11} + \ell_{22}Mr_{21} &= 0, \\ \ell_{21}Cr_{12} + \ell_{22}Mr_{12} + \ell_{21}Mr_{22} &= 0, \\ \ell_{11}Cr_{12} + \ell_{12}Mr_{12} + \ell_{11}Mr_{22} &= \ell_{21}Cr_{11} + \ell_{22}Mr_{11} + \ell_{21}Mr_{21} \\ &= -\ell_{21}Kr_{12} + \ell_{22}Mr_{22}. \end{aligned} \quad (2.2)$$

Additionally, we are seeking Π_ℓ and Π_r so that

$$\begin{aligned} -\ell_{21}Kr_{12} + \ell_{22}Mr_{22} &= M_D, \\ \ell_{11}Cr_{11} + \ell_{12}Mr_{11} + \ell_{11}Mr_{21} &= C_D, \\ \ell_{11}Kr_{11} - \ell_{12}Mr_{21} &= K_D, \end{aligned} \quad (2.3)$$

are diagonal matrices. The conditions (2.2) and (2.3) together constitute a nonlinear algebraic system of $8n^2 - 3n$ equations in $8n^2$ unknowns, but the system is not easy to solve.

The extra degrees of freedom in the underdetermined system (2.2) and (2.3) suggest that both (M, C, K) and (M_D, C_D, K_D) reside on some nontrivial manifold. A structure preserving isospectral flow, that is, a differentiable path, starting from (M, C, K) is characterized in [5]. We shall describe a closed-loop feedback control system in the forthcoming paper [1] to drive such a flow to (M_D, C_D, K_D) numerically. The remaining of this paper shall concentrate on the theoretical issues.

3. General Pencil and Equivalence Transformation. In this section, we detail steps toward proving the existence of the canonical form (1.6) for a quadratic pencil with general coefficient matrices M, C and K in $\mathbb{R}^{n \times n}$. For convenience, define

$$A := \begin{bmatrix} -K & 0 \\ 0 & M \end{bmatrix}, \quad B := \begin{bmatrix} C & M \\ M & 0 \end{bmatrix}. \quad (3.1)$$

Let $(\lambda_j, \mathbf{x}_j)$, $j = 1, \dots, 2n$, denote the j -th *right* eigenpair of the pencil $\lambda B - A$, that is, assume

$$A\mathbf{x}_j = \lambda_j B\mathbf{x}_j. \quad (3.2)$$

In general, the spectrum will be a mix of complex-valued and real-valued eigenvalues. Recall that the corresponding eigenvector \mathbf{u}_j of the original quadratic pencil $Q(\lambda; M, C, K)$ can be recovered from the fact that

$$\mathbf{x}_j = \begin{bmatrix} \mathbf{u}_j \\ \lambda_j \mathbf{u}_j \end{bmatrix}. \quad (3.3)$$

To fix the idea, we shall begin with a spectral decomposition where the following scheme concerning real and complex eigenvalues and the associated eigenvectors holds:

- Each pair of complex conjugated eigenvalues and the corresponding eigenvectors is placed next to each other, and
- The eigenvectors corresponding to real-valued eigenvalues are real-valued.

This scheme will be changed along our later development. Define

$$\Lambda := \text{diag} \{ \lambda_1, \lambda_2, \dots, \lambda_{2n} \} \in \mathbb{C}^{2n \times 2n}, \quad (3.4)$$

$$X := [\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{2n}] \in \mathbb{C}^{2n \times 2n}, \quad (3.5)$$

respectively. Likewise, let $(\lambda_j, \mathbf{y}_j)$, $j = 1, \dots, 2n$, denote the j -th *left* eigenpair of the pencil $\lambda B - A$, that is, assume

$$\mathbf{y}_j^\top A = \lambda_j \mathbf{y}_j^\top B. \quad (3.6)$$

Denote the corresponding matrix of left eigenvectors by $Y \in \mathbb{C}^{2n \times 2n}$. Be aware that we use “transpose” rather than “conjugate transpose” for the left eigenvectors.

For simplicity, we shall assume henceforth that all eigenvalues of $L(\lambda)$ are simple, that is, we shall assume that the diagonal matrix Λ has distinct diagonal entries. We think our argument below can be generalized in a straightforward yet tedious way to the case where nontrivial Jordan chains occur, but we shall not elaborate the details in this paper.

Observe from (3.2) and (3.6) that the relationship

$$Y^\top B X \Lambda = \Lambda Y^\top B X = Y^\top A X \quad (3.7)$$

holds. The first equality in (3.7) indicates that $Y^\top A X$ commutes with the diagonal matrix Λ which has distinct entries. It follows that the two matrices A_1 and B_1 defined by

$$\begin{aligned} A_1 &:= Y^\top A X, \\ B_1 &:= Y^\top B X, \end{aligned} \quad (3.8)$$

must also be diagonal. Assume further that A_1^{-1} exists. Clearly, the scaled columns

$$X^{[2]} := X A_1^{-1/2}, \quad (3.9)$$

$$Y^{[2]} := Y A_1^{-1/2}, \quad (3.10)$$

where diagonal entries of $A_1^{1/2}$ are the principal square roots of those of A_1 , are still the right and left eigenvectors of $L(\lambda)$, respectively. By using this set of scaled eigenvectors, we see that

$$\begin{aligned} A_2 &:= Y^{[2]\top} A X^{[2]} = I_{2n}, \\ B_2 &:= Y^{[2]\top} B X^{[2]} = \Lambda^{-1}, \end{aligned} \quad (3.11)$$

where I_{2n} stands for the $2n \times 2n$ identity matrix. In order to achieve (3.11), it is important to note that the scaled eigenvectors $\mathbf{x}_i^{[2]}$ and $\mathbf{y}_i^{[2]}$ corresponding to the real eigenvalues λ_i can become purely imaginary, if $\mathbf{y}_i^\top A \mathbf{x}_i < 0$.

3.1. Classification. It is useful to identify the special characteristics of eigenvalues by their locations in the spectrum. Let S_1 , S_2 and S_3 denote, respectively, the index (locality) sets of complex-valued, real-valued, and purely imaginary-valued eigenvectors in $X^{[2]}$. It is possible that S_k is empty. Let elements in S_k be denoted by k_j , $j = 1, \dots, |S_k|$, where $|S_k|$ stands for the cardinality of the set S_k . Write

$$S_1 = S_1^+ \cup S_1^-,$$

where S_1^+ and S_1^- are the index sets of complex-valued eigenvalues with positive and negative imaginary parts, respectively. Let $p := |S_1^+|$ and $\rho := \min\{|S_2|, |S_3|\}$. We now further regroup the real eigenvalues into six subcategories by following the rules described in [3]:

1. If the end index is less than the start index in any of the the following "for" statements, define the corresponding set to be empty.
2. For $j = 1, \dots, \rho$,

$$\begin{aligned} \{2_j, 3_j\} \in C_a &\Leftrightarrow \lambda_{2_j} \lambda_{3_j} > 0, \\ \{2_j, 3_j\} \in C_f &\Leftrightarrow \lambda_{2_j} \lambda_{3_j} < 0. \end{aligned}$$

3. For $j = 1, \dots, \frac{|S_2| - \rho}{2}$,

$$\begin{aligned} \{2_{\rho+2j-1}, 2_{\rho+2j}\} \in C_b &\Leftrightarrow \lambda_{2_{\rho+2j-1}} \lambda_{2_{\rho+2j}} < 0, \\ \{2_{\rho+2j-1}, 2_{\rho+2j}\} \in C_d &\Leftrightarrow \lambda_{2_{\rho+2j-1}} \lambda_{2_{\rho+2j}} > 0. \end{aligned}$$

4. For $j = 1, \dots, \frac{|S_3| - \rho}{2}$,

$$\begin{aligned} \{3_{\rho+2j-1}, 3_{\rho+2j}\} \in C_c &\Leftrightarrow \lambda_{3_{\rho+2j-1}} \lambda_{3_{\rho+2j}} < 0, \\ \{3_{\rho+2j-1}, 3_{\rho+2j}\} \in C_e &\Leftrightarrow \lambda_{3_{\rho+2j-1}} \lambda_{3_{\rho+2j}} > 0. \end{aligned}$$

Note that elements in each category $\nu = a, b, \dots, f$, are indices and appear in pairs. Collect all first indices from these pairs in C_ν into a subset C_ν^+ and all second indices into C_ν^- . We have

$$C_\nu = C_\nu^+ \cup C_\nu^-.$$

Denote $q_\nu := \frac{|C_\nu|}{2} = |C_\nu^+| = |C_\nu^-|$. For convenience, we further collect the indices into two sets:

$$\begin{aligned} T^+ &:= [C_a^+, \dots, C_f^+], \\ T^- &:= [C_a^-, \dots, C_f^-]. \end{aligned}$$

Define the permutation matrix $P \in \mathbb{R}^{2n \times 2n}$ by rearranging the columns of the identity matrix according to the index vector

$$\boldsymbol{\delta} = [S_1^+, T^+, S_1^-, T^-]. \quad (3.12)$$

Note that $|T^+| = |T^-| = n - p$. By rearranging the eigenvectors accordingly,

$$X^{[3]} := X^{[2]}P = XA_1^{-1/2}P, \quad (3.13)$$

$$Y^{[3]} := Y^{[2]}P = YA_1^{-1/2}P, \quad (3.14)$$

we obtain from (3.11)

$$\begin{aligned} A_3 &:= Y^{[3]\top} AX^{[3]} = I_{2n}, \\ B_3 &:= Y^{[3]\top} BX^{[3]} = \begin{bmatrix} \Lambda_{S_1^+}^{-1} & 0 & & & & \\ 0 & \Lambda_{T^+}^{-1} & & & & \\ & & \Lambda_{S_1^-}^{-1} & 0 & & \\ & & & & & \\ & & & & & \Lambda_{T^-}^{-1} \end{bmatrix}, \end{aligned} \quad (3.15)$$

where each diagonal block in B_3 is composed of the reciprocals of eigenvalues from the indicated subset of indices. Note that generally $X^{[3]}$ and $Y^{[3]}$ are complex-valued. We remark that the eigenvectors in $X^{[3]}$ (and, likewise, those of $Y^{[3]}$) have been rearranged in such a way that

$$X^{[3]} = \left[X_{S_1^+}^{[3]}, X_{T^+}^{[3]}, X_{S_1^-}^{[3]}, X_{T^-}^{[3]} \right].$$

For example,

$$X_{T^+}^{[3]} = \left[X_{C_a^+}^{[3]}, X_{C_b^+}^{[3]}, X_{C_c^+}^{[3]}, X_{C_d^+}^{[3]}, X_{C_e^+}^{[3]}, X_{C_f^+}^{[3]} \right],$$

where columns in $X_{C_a^+}^{[3]}$, $X_{C_b^+}^{[3]}$, $X_{C_d^+}^{[3]}$, and $X_{C_f^+}^{[3]}$ are real-valued while those in $X_{C_c^+}^{[3]}$ and $X_{C_e^+}^{[3]}$ are purely imaginary.

Up to now we have been following along the procedures suggested in [3], except that we are providing more specific classifications by the indices. Now we begin to deviate. For each $\nu = a, b, \dots, f$, let E_ν denote the identity matrix I_{q_ν} . Define the following four block diagonal matrices:

$$\begin{aligned} L^+ &:= \text{diag}\{E_a, E_b, iE_c, E_d, -iE_e, E_f\}, \\ L^- &:= \text{diag}\{iE_a, E_b, iE_c, -E_d, iE_e, -iE_f\}, \\ R^+ &:= \text{diag}\{E_a, E_b, iE_c, E_d, iE_e, E_f\}, \\ R^- &:= \text{diag}\{iE_a, E_b, iE_c, E_d, iE_e, iE_f\}, \end{aligned} \quad (3.16)$$

where $i = \sqrt{-1}$. It is important to note that columns in the products

$$\left[X_{T^+}^{[4]}, X_{T^-}^{[4]} \right] := \left[X_{T^+}^{[3]}, X_{T^-}^{[3]} \right] \begin{bmatrix} R^+ & 0 \\ 0 & R^- \end{bmatrix}, \quad (3.17)$$

$$\left[Y_{T^+}^{[4]}, Y_{T^-}^{[4]} \right] := \left[Y_{T^+}^{[3]}, Y_{T^-}^{[3]} \right] \begin{bmatrix} L^+ & 0 \\ 0 & L^- \end{bmatrix}, \quad (3.18)$$

are all real-valued now. Observe the facts that

$$\begin{bmatrix} L^+ & 0 \\ 0 & L^- \end{bmatrix} \begin{bmatrix} R^+ & 0 \\ 0 & R^- \end{bmatrix} = \text{diag}\{\hat{I}_{n-p}, \tilde{I}_{n-p}\}, \quad (3.19)$$

where

$$\hat{I}_{n-p} := \text{diag}\{E_a, E_b, -E_c, E_d, E_e, E_f\}, \quad (3.20)$$

$$\tilde{I}_{n-p} := \text{diag}\{-E_a, E_b, -E_c, -E_d, -E_e, E_f\}, \quad (3.21)$$

and that both diagonal matrices Ω_+ and Ω_- defined by

$$\Omega^+ := L^+ \Lambda_{T^+}^{-1} R^+ = \text{diag}\{\Lambda_{C_a^+}^{-1}, \Lambda_{C_b^+}^{-1}, -\Lambda_{C_c^+}^{-1}, \Lambda_{C_d^+}^{-1}, \Lambda_{C_e^+}^{-1}, \Lambda_{C_f^+}^{-1}\} \quad (3.22)$$

$$\Omega^- := L^- \Lambda_{T^-}^{-1} R^- = \text{diag}\{-\Lambda_{C_a^-}^{-1}, \Lambda_{C_b^-}^{-1}, -\Lambda_{C_c^-}^{-1}, -\Lambda_{C_d^-}^{-1}, -\Lambda_{C_e^-}^{-1}, \Lambda_{C_f^-}^{-1}\} \quad (3.23)$$

remain real-valued.

We summarize the procedure thus far as follows.

LEMMA 3.1. *Define the realization matrices*

$$J_\ell := \begin{bmatrix} \frac{1}{\sqrt{2}}I_p & 0 & \frac{-i}{\sqrt{2}}I_p & 0 \\ 0 & L^+ & 0 & 0 \\ \frac{1}{\sqrt{2}}I_p & 0 & \frac{i}{\sqrt{2}}I_p & 0 \\ 0 & 0 & 0 & L^- \end{bmatrix}, \quad J_r := \begin{bmatrix} \frac{1}{\sqrt{2}}I_p & 0 & \frac{-i}{\sqrt{2}}I_p & 0 \\ 0 & R^+ & 0 & 0 \\ \frac{1}{\sqrt{2}}I_p & 0 & \frac{i}{\sqrt{2}}I_p & 0 \\ 0 & 0 & 0 & R^- \end{bmatrix}, \quad (3.24)$$

then the matrices

$$X^{[4]} := X^{[3]}J_r = XA_1^{-1/2}PJ_r, \quad (3.25)$$

$$Y^{[4]} := Y^{[3]}J_\ell = YA_1^{-1/2}PJ_\ell, \quad (3.26)$$

are real-valued. In this case, the pencil $\lambda B - A$ can be transformed isospectrally into $\lambda B_4 - A_4$, where

$$A_4 := Y^{[4]\top}AX^{[4]} = \begin{bmatrix} I_p & 0 & 0 & 0 \\ 0 & \hat{I}_{n-p} & 0 & 0 \\ 0 & 0 & -I_p & 0 \\ 0 & 0 & 0 & \tilde{I}_{n-p} \end{bmatrix}, \quad (3.27)$$

$$B_4 := Y^{[4]\top}BX^{[4]} = \begin{bmatrix} \Re(\Lambda_{S_1^+}^{-1}) & 0 & \Im(\Lambda_{S_1^+}^{-1}) & 0 \\ 0 & \Omega^+ & 0 & 0 \\ \Im(\Lambda_{S_1^+}^{-1}) & 0 & -\Re(\Lambda_{S_1^+}^{-1}) & 0 \\ 0 & 0 & 0 & \Omega^- \end{bmatrix}. \quad (3.28)$$

We need to further reduce A_4 and B_4 into the form (1.6).

3.2. Elimination. Our next step is to employ equivalence transformation by a matrix of the form

$$F := \begin{bmatrix} \Phi & 0 & I_p & 0 \\ 0 & \Psi^+ & 0 & I_{n-p} \\ I_p & 0 & \Phi & 0 \\ 0 & I_{n-p} & 0 & \Psi^- \end{bmatrix}, \quad (3.29)$$

where Φ , Ψ^+ and Ψ^- are real-valued diagonal matrices, to eliminate the lower right $n \times n$ block of B_4 while maintaining the diagonal form of A_4 . Since all blocks involved in (3.27), (3.28), and (3.29) are diagonal, it suffices to consider the elimination entry by entry. We consider the entries of Φ , Ψ^+ and Ψ^- separately.

To define the k -th diagonal entry ϕ_k in Φ , let $\alpha_k + i\beta_k$ represent the k -th diagonal entry in $\Lambda_{S_1^+}^{-1}$. Observe that

$$\begin{bmatrix} \phi_k & 1 \\ 1 & \phi_k \end{bmatrix} \begin{bmatrix} \alpha_k & \beta_k \\ \beta_k & -\alpha_k \end{bmatrix} \begin{bmatrix} \phi_k & 1 \\ 1 & \phi_k \end{bmatrix} = \begin{bmatrix} \alpha_k \phi_k^2 + 2\beta_k \phi_k - \alpha_k & \beta_k + \beta_k \phi_k^2 \\ \beta_k + \beta_k \phi_k^2 & \alpha_k + 2\beta_k \phi_k - \alpha_k \phi_k^2 \end{bmatrix}.$$

The (2, 2) entry of the 2×2 matrix on the right hand side above can be eliminated by choosing

$$\phi_k := \frac{\beta_k + \sqrt{\alpha_k^2 + \beta_k^2}}{\alpha_k}. \quad (3.30)$$

In matrix form, this choice can be written as

$$\Phi := \left(\Im(\Lambda_{S_1^+}^{-1}) + \sqrt{\left(\Re(\Lambda_{S_1^+}^{-1})\right)^2 + \left(\Im(\Lambda_{S_1^+}^{-1})\right)^2} \right) \left(\Re(\Lambda_{S_1^+}^{-1})\right)^{-1}. \quad (3.31)$$

Likewise, let k -th entries of Ω^+ and Ω^- be denoted as ω_k^+ and ω_k^- , respectively. Observe that

$$\begin{bmatrix} \psi_k^+ & 1 \\ 1 & \psi_k^- \end{bmatrix} \begin{bmatrix} \omega_k^+ & 0 \\ 0 & \omega_k^- \end{bmatrix} \begin{bmatrix} \psi_k^+ & 1 \\ 1 & \psi_k^- \end{bmatrix} = \begin{bmatrix} \omega_k^+ (\psi_k^+)^2 + \omega_k^- & \omega_k^+ \psi_k^+ + \omega_k^- \psi_k^- \\ \omega_k^+ \psi_k^+ + \omega_k^- \psi_k^- & \omega_k^+ + \omega_k^- (\psi_k^-)^2 \end{bmatrix}.$$

In order to eliminate the (2,2) entry, we must choose

$$\psi_k^- := \pm \sqrt{-\frac{\omega_k^+}{\omega_k^-}}. \quad (3.32)$$

One immediate concern is whether (3.32) is a real number, but that concern is perfectly addressed in the specific way we choose signs when defining the four matrices L^\pm and R^\pm .

LEMMA 3.2. *By the way the index subsets C_ν , $\nu = a, b, \dots, f$ are defined and, subsequently, the diagonal matrices Ω^+ and Ω^- are arranged, ω_k^+ and ω_k^- have opposite signs.*

At its the first glance, the choice of sign in (3.32) does not seem important. But if we want to maintain the diagonal form when the same equivalence transformation is applied to A_4 , the signs of ψ_k^+ and ψ_k^- must be selected more carefully. The reason can be seen from the following two calculations:

$$\begin{aligned} \begin{bmatrix} \psi_k^+ & 1 \\ 1 & \psi_k^- \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \psi_k^+ & 1 \\ 1 & \psi_k^- \end{bmatrix} &= \begin{bmatrix} (\psi_k^+)^2 + 1 & \psi_k^+ + \psi_k^- \\ \psi_k^+ + \psi_k^- & 1 + (\psi_k^-)^2 \end{bmatrix}, \\ \begin{bmatrix} \psi_k^+ & 1 \\ 1 & \psi_k^- \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \psi_k^+ & 1 \\ 1 & \psi_k^- \end{bmatrix} &= \begin{bmatrix} (\psi_k^+)^2 - 1 & \psi_k^+ - \psi_k^- \\ \psi_k^+ - \psi_k^- & 1 - (\psi_k^-)^2 \end{bmatrix}. \end{aligned}$$

Depending on whether the k -th diagonal entry in the product $\hat{I}_{n-p} \tilde{I}_{n-p}$ is positive or negative one, we have to choose $\psi_k^+ = -\psi_k^-$ or $\psi_k^+ = \psi_k^-$, accordingly, in order to keep the off-diagonal entries of the 2×2 matrices on the right hand side above zero. In matrix form, if we choose

$$\Psi^+ := \sqrt{-\Omega^+(\Omega^-)^{-1}}, \quad (3.33)$$

then

$$\Psi^- := -\hat{I}_{n-p} \tilde{I}_{n-p} \Psi^+. \quad (3.34)$$

The elimination process is summarized as follows.

LEMMA 3.3. *Form the real-valued matrix F in (3.29) with diagonal matrices Φ , Ψ^+ and Ψ^- given by (3.31), (3.33) and (3.34), respectively. Define*

$$X^{[5]} := X^{[4]} F = X A_1^{-1/2} P J_r F, \quad (3.35)$$

$$Y^{[5]} := Y^{[4]} F = Y A_1^{-1/2} P J_\ell F, \quad (3.36)$$

then the two matrices

$$A_5 := Y^{[5]\top} A X^{[5]} = \begin{bmatrix} A_5^{[11]} & 0 \\ 0 & A_5^{[22]} \end{bmatrix}, \quad (3.37)$$

$$B_5 := Y^{[5]\top} B X^{[5]} = \begin{bmatrix} B_5^{[11]} & B_5^{[12]} \\ B_5^{[12]} & 0 \end{bmatrix}, \quad (3.38)$$

where all $A_5^{[kj]}$ and $B_5^{[kj]}$, $j, k = 1, 2$, are diagonal matrices, form a real-valued pencil isospectral to the original $\lambda B - A$.

3.3. Scaling. Our last step is to scale the pair (A_5, B_5) back to the Lancaster structure. This can be accomplished by a diagonal matrix of the form

$$\Gamma := \begin{bmatrix} I_n & 0 \\ 0 & \Theta \end{bmatrix}. \quad (3.39)$$

Indeed, observe that

$$\begin{aligned} \begin{bmatrix} I_n & 0 \\ 0 & \Theta \end{bmatrix} \begin{bmatrix} A_5^{[11]} & 0 \\ 0 & A_5^{[22]} \end{bmatrix} \begin{bmatrix} I_n & 0 \\ 0 & \Theta \end{bmatrix} &= \begin{bmatrix} A_5^{[11]} & 0 \\ 0 & \Theta A_5^{[22]} \Theta \end{bmatrix}, \\ \begin{bmatrix} I_n & 0 \\ 0 & \Theta \end{bmatrix} \begin{bmatrix} B_5^{[11]} & B_5^{[12]} \\ B_5^{[12]} & 0 \end{bmatrix} \begin{bmatrix} I_n & 0 \\ 0 & \Theta \end{bmatrix} &= \begin{bmatrix} B_5^{[11]} & B_5^{[12]} \Theta \\ \Theta B_5^{[12]} & 0 \end{bmatrix}, \end{aligned}$$

and that

$$A_5^{[22]} = \text{diag}\{I_p - \Phi^2, I_{n-p} + \tilde{I}_{n-p}(\Psi^+)^2\}. \quad (3.40)$$

Assuming the generic condition that $A_5^{[22]}$ is nonsingular in general, the choice of the diagonal matrix

$$\Theta := B_5^{[22]} A_5^{[22]^{-1}} \quad (3.41)$$

is sufficient to restore the Lancaster structure. Through the sequence of pencils $\lambda B_i - A_i$, $i = 1, \dots, 5$, all of which are isospectral to the original pencil $\lambda B - A$, the final step of scaling completes the construction of the equivalence transformation.

Our major result is stated below.

THEOREM 3.4. *Given a quadratic pencil $Q(\lambda; M, C, K) = \lambda^2 M + \lambda C + K$ with real-valued matrix coefficients, let columns of X and Y denote the right and the left eigenvectors of the linear pencil $\lambda B - A$, respectively, where A and B are defined in (3.1). Assume that*

1. *All eigenvalues are simple,*
2. *The matrix A_1 defined in (3.8) is invertible,*
3. *The matrix $A_5^{[22]}$ involved in (3.40) is invertible.*

Define

$$X^{[6]} := X^{[5]} \Gamma = X A_1^{-1/2} P J_r F \Gamma, \quad (3.42)$$

$$Y^{[6]} := Y^{[5]} \Gamma = Y A_1^{-1/2} P J_\ell F \Gamma, \quad (3.43)$$

where A_1 is the diagonal matrix given by (3.8), P is the permutation matrix determined by (3.12), J_r and J_ℓ are the realization matrices defined by (3.24), F is the elimination matrix defined by (3.29), and Γ is the scaling matrix given by (3.39). Then the original pencil $\lambda B - A$ is isospectral to the pencil $\lambda B_6 - A_6$, where

$$A_6 := Y^{[6]\top} A X^{[6]} = \begin{bmatrix} -K_D & 0 \\ 0 & M_D \end{bmatrix}, \quad (3.44)$$

$$B_6 := Y^{[6]\top} B X^{[6]} = \begin{bmatrix} C_D & M_D \\ M_D & 0 \end{bmatrix}, \quad (3.45)$$

and M_D , C_D and K_D are real-valued diagonal matrices. In other words, the quadratic pencil $\lambda^2 M + \lambda C + K$ is totally decoupled by isospectral transformations into $\lambda^2 M_D + \lambda C_D + K_D$.

The desirable structure preservation transformations Π_ℓ and Π_r are given by

$$\Pi_\ell := Y^{[6]\top}, \quad (3.46)$$

$$\Pi_r := X^{[6]}, \quad (3.47)$$

which by our construction described above are real-valued.

3.4. Numerical Example. We conclude this section with a numerical example. Up to this point, note that the theory is for general quadratic pencils where the matrix coefficients are general matrices in $\mathbb{R}^{n \times n}$. We emphasize that the construction of the equivalence transformation begins with the availability of the complete spectral information, the so called *Jordan triplet* in [6].

For the easy running of text, we report only four numeric digits even though all calculations involved are accurate to the machine accuracy. Consider the case

$$M = \begin{bmatrix} 0.7621 & 0.4447 & 0.7382 & 0.9169 \\ 0.4565 & 0.6154 & 0.1763 & 0.4103 \\ 0.0185 & 0.7919 & 0.4057 & 0.8936 \\ 0.8214 & 0.9218 & 0.9355 & 0.0579 \end{bmatrix}, C = \begin{bmatrix} 0.3710 & -1.0226 & 0.3155 & 0.5045 \\ 0.7283 & 1.0378 & 1.5532 & 1.8645 \\ 2.1122 & -0.3898 & 0.7079 & -0.3398 \\ -1.3573 & -1.3813 & 1.9574 & -1.1398 \end{bmatrix},$$

$$K = \begin{bmatrix} -0.2111 & -0.6014 & -2.0046 & 1.2366 \\ 1.1902 & 0.5512 & -0.4931 & -0.6313 \\ -1.1162 & -1.0998 & 0.4620 & -2.3252 \\ 0.6353 & 0.0860 & -0.3210 & -1.2316 \end{bmatrix}.$$

Its spectrum in ascending order of moduli is given by

$$\{0.0348, -0.0877, 0.5453, -2.0162 \pm 1.4196i, 0.1815 \pm 2.5083i, 3.0425\},$$

among which the four real eigenvalues are classified into

$$C_d = \{3, 8\}, \quad C_f = \{2, 1\}.$$

Be aware of the order in C_f because the eigenvector corresponding to the eigenvalue 0.0348 is purely imaginary. The structure preserving transformation matrices Π_ℓ and Π_r are given by

$$\Pi_\ell = \begin{bmatrix} 0.4189 & 0.9173 & 0.0945 & -0.4584 & -2.0559 & -1.9378 & -0.0730 & 1.2765 \\ -6.9347 & 12.4232 & -0.9207 & -2.4408 & 3.2216 & -7.7792 & -14.4484 & -8.8834 \\ -0.5439 & 0.0331 & 1.9456 & -2.3711 & -0.4656 & -0.4208 & 2.0679 & -1.4933 \\ -0.3465 & -10.1901 & -3.9018 & 12.2692 & -0.0115 & 0.0231 & -0.0563 & 0.0023 \\ 0.3381 & 0.3187 & 0.0120 & -0.2100 & -0.9446 & -0.3678 & 0.0461 & 0.3881 \\ -0.5094 & 1.2300 & 2.2846 & 1.4046 & -7.1196 & 12.8698 & -0.0912 & -1.9307 \\ 0.2807 & 0.2536 & -1.2465 & 0.9001 & 0.4631 & 0.9430 & -2.5265 & 0.8583 \\ -3.7728 & 7.5521 & -18.4169 & 0.7416 & -0.1468 & -10.5898 & -2.9272 & 12.2299 \end{bmatrix},$$

$$\Pi_r = \begin{bmatrix} 0.3340 & -3.7885 & 0.5893 & -8.7045 & 0.0300 & 0.9596 & -0.8007 & -8.7108 \\ -0.2226 & -14.6629 & 1.0132 & 12.2692 & -0.0866 & 3.9812 & -0.2769 & 0.7416 \\ -0.8990 & 10.1808 & -2.4537 & -4.3424 & -0.4661 & 2.5332 & 1.0515 & -13.8562 \\ 0.9328 & 6.0803 & -1.6896 & -2.5127 & 0.3928 & -3.4286 & 0.8571 & -9.4260 \\ -0.1822 & -6.0687 & 1.3284 & -0.0266 & 0.2132 & -3.4401 & -2.2834 & -8.2435 \\ 0.5264 & -25.1784 & 0.4594 & 0.0023 & 0.1265 & -13.2174 & 0.0197 & 12.2299 \\ 2.8341 & -16.0210 & -1.7445 & -0.0423 & 0.9806 & 11.1005 & 1.3190 & -3.6091 \\ -2.3881 & 21.6836 & -1.4220 & -0.0288 & -0.6509 & 4.8354 & 1.3855 & -2.0139 \end{bmatrix}.$$

The totally decoupled system has matrix coefficients given by

$$M_D = \begin{bmatrix} -0.4463 & 0.0000 & -0.0000 & -0.0000 \\ -0.0000 & -120.8951 & -0.0000 & -0.0000 \\ -0.0000 & -0.0000 & -2.7605 & -0.0000 \\ 0.0000 & -0.0000 & 0.0000 & 457.1029 \end{bmatrix}, C_D = \begin{bmatrix} -1.7998 & 0.0000 & -0.0000 & -0.0000 \\ -0.0000 & 43.8949 & -0.0000 & -0.0000 \\ -0.0000 & 0.0000 & 9.9041 & 0.0000 \\ 0.0000 & -0.0000 & 0.0000 & 24.1913 \end{bmatrix},$$

$$K_D = \begin{bmatrix} -2.7138 & -0.0000 & -0.0000 & -0.0000 \\ 0.0000 & -764.5811 & 0.0000 & -0.0000 \\ -0.0000 & -0.0000 & -4.5797 & 0.0000 \\ -0.0000 & 0.0000 & 0.0000 & -1.3969 \end{bmatrix}.$$

If so desired, further scaling by ± 1 can change the diagonal entries of M_D into nonnegative numbers. Regardless, the five single-degree-of-freedom subsystems are easy to read by now.

4. Self-Adjoint Pencil and Congruence Transformation. Thus far, we have shown that a general quadratic pencil $Q(\lambda; M, C, K)$ can be totally decoupled by means of equivalence transformation. In this section, we argue further that a self-adjoint quadratic pencil, that is, when the matrix coefficients M , C and K are all symmetric, can be totally decoupled by congruence transformations.

Observe first that we may choose both left and right eigenvectors of the symmetric pencil $\lambda B - A$ to be the same. Thus we begin with $X = Y$ in (3.7). Let (λ, \mathbf{x}) denote a real-valued eigenpair for the pencil $\lambda B - A$. Note that

$$\lambda = \frac{\mathbf{x}^\top A \mathbf{x}}{\mathbf{x}^\top B \mathbf{x}}. \quad (4.1)$$

Our goal in this section is to show that the real eigenvalues can be regrouped in such a way that all pairs lie within categories C_a , C_b and C_c only. This assertion, claimed by Garvey et al, has never been proved in [3].

The key ingredient in our construction of a congruence transformation is the notion of factorization for self-adjoint matrix polynomials [7, Chapter 11]. The machinery used to develop the general theory for matrix polynomials of arbitrary degree is quite involved. For simplicity and for completeness, we outline its idea for quadratic pencils via the following theorem.

THEOREM 4.1. *Assume that the leading coefficient M in the self-adjoint quadratic pencil $Q(\lambda; M, C, K)$ is nonsingular and that the spectrum is simple. Let $(\lambda, \mathbf{x}(\lambda))$ denote a real-valued eigenpair for the associated pencil $\lambda B - A$ defined in (3.1). Then, for almost all self-adjoint pencils $Q(\lambda; M, C, K)$, the real eigenvalues can be divided evenly into two groups: half of the eigenvalues are such that the product $\mathbf{x}(\lambda)^\top B \mathbf{x}(\lambda) > 0$ and the other half are such that $\mathbf{x}(\lambda)^\top B \mathbf{x}(\lambda) < 0$.*

Proof. Using (3.7), we already know $B_1 = Y^\top B X$ is a diagonal matrix. Write

$$B_1 = \text{diag} \left\{ b^{[1]}, \dots, b^{[2n]} \right\}.$$

Similar to what we have done in Section 3.1, let \mathcal{S}_1 , \mathcal{S}_2 and \mathcal{S}_3 denote, respectively, the index sets of complex-valued, positive, and negative diagonal entries in B_1 . Write

$$\mathcal{S}_1 = \mathcal{S}_1^+ \cup \mathcal{S}_1^-,$$

where \mathcal{S}_1^+ and \mathcal{S}_1^- are the index subsets of diagonal entries of B_1 with positive and negative imaginary parts, respectively. We want to show that $|\mathcal{S}_1^+ \cup \mathcal{S}_2| = |\mathcal{S}_1^- \cup \mathcal{S}_3| = n$.

One of the two sets $\mathcal{S}_1^+ \cup \mathcal{S}_2$ and $\mathcal{S}_1^- \cup \mathcal{S}_3$ will have cardinality at least n . We may assumed, without loss of generality, that $|\mathcal{S}_1^+ \cup \mathcal{S}_2| \geq n$. The other case can be argued similarly. Denote $\rho_1 := |\mathcal{S}_1^+|$ and $\rho_2 := |\mathcal{S}_2|$. Consider the matrix

$$Z := X_{\mathcal{S}_1^+ \cup \mathcal{S}_2} = \begin{bmatrix} \mathbf{x}_{1_1^+}, \dots, \mathbf{x}_{1_{\rho_1}^+} & \mathbf{x}_{2_1}, \dots, \mathbf{x}_{2_{\rho_2}} \end{bmatrix} \in \mathbb{C}^{2n \times \rho_1} \oplus \mathbb{R}^{2n \times \rho_2},$$

whose columns are selected from those of X with column indices given by $\mathcal{S}_1^+ \cup \mathcal{S}_2$. By assumption, the matrix Z is of full column rank $\rho_1 + \rho_2$. Using the relationship (3.3), write

$$Z = \begin{bmatrix} U \\ U\Upsilon \end{bmatrix},$$

where

$$U := U_{\mathcal{S}_1^+ \cup \mathcal{S}_2} = \begin{bmatrix} \mathbf{u}_{1_1^+}, \dots, \mathbf{u}_{1_{\rho_1}^+} & \mathbf{u}_{2_1}, \dots, \mathbf{u}_{2_{\rho_2}} \end{bmatrix} \in \mathbb{C}^{n \times \rho_1} \oplus \mathbb{R}^{n \times \rho_2}, \quad (4.2)$$

denote the upper half of the matrix Z and

$$\Upsilon := \Lambda_{\mathcal{S}_1^+ \cup \mathcal{S}_2} \quad (4.3)$$

denotes the corresponding portion from the diagonal matrix Λ defined in (3.4). We want to show that U is of full column rank. If this is true, then because we have assumed $\rho_1 + \rho_2 \geq n$, it must be $\rho_1 + \rho_2 = n$.

By construction, we know that $Z^\top BZ$ is a diagonal matrix with some complex-valued entries. If the adjoint Z^* , that is, the transpose of the complex conjugate of Z , is used instead, we observe that

$$D := Z^* BZ = \text{diag} \left\{ 0, \dots, 0, b^{[2_1]}, \dots, b^{[2_{\rho_2}]} \right\}$$

has nonnegative entries. Suppose $U\mathbf{z} = 0$ for some $\mathbf{z} \in \mathbb{C}^{\rho_1 + \rho_2}$. It is easy to see that

$$\mathbf{z}^* D \mathbf{z} = \begin{bmatrix} 0 & (U\Upsilon\mathbf{z})^* \end{bmatrix} \begin{bmatrix} C & M \\ M & 0 \end{bmatrix} \begin{bmatrix} 0 \\ U\Upsilon\mathbf{z} \end{bmatrix} = 0.$$

It follows that $D\mathbf{z} = 0$. This is equivalent to

$$U^* M U \Upsilon \mathbf{z} = 0 \quad (4.4)$$

and, hence $(U\Upsilon\mathbf{z})^* M U \Upsilon \mathbf{z} = 0$.

The set of symmetric matrices (M, C, K) such that

$$\{\mathbf{u} \in \mathbb{C}^{n \times n} \mid \mathbf{u}^* M \mathbf{u} = 0\} \cap \{U\Upsilon\mathbf{z} \in \mathbb{C}^{n \times n} \mid \mathbf{z} \in \mathbb{C}^{\rho_1 + \rho_2}, M U \Upsilon^2 + C U \Upsilon + K U = 0\} \neq \emptyset$$

forms an algebraic variety in the space of $\mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n}$. This variety has measure zero. That is, for almost all self-adjoint quadratic pencils $Q(\lambda; M, C, K)$, the condition (4.4) implies that $U\Upsilon\mathbf{z} = 0$. We have thus proved that \mathbf{z} is in the null space of Z . Since Z is of full column rank, it must be $\mathbf{z} = 0$ and, hence, U is of full column rank. \square

We remark that in [6] these two sets $\mathcal{S}_1^+ \cup \mathcal{S}_2$ and $\mathcal{S}_1^- \cup \mathcal{S}_3$ are said to be B -nonnegative and B -nonpositive, respectively. It is also worth pointing out that the proof in [6, 7] is for the factorization of *monic* self-adjoint matrix polynomials, that is, the leading matrix coefficient is the identity matrix. The proof can easily be modified if the leading matrix coefficient is positive definite. What we have proved, however, is that the factorization remains true for almost all regular self-adjoint quadratic pencils. We have observed this phenomenon numerically.

4.1. With Positive Definite Coefficients. One of the most important quadratic pencils in application is the class when M is symmetric and positive definite and both C and K are symmetric and positive semidefinite. It is known in this case that all eigenvalues of $Q(\lambda)$ lie in the left half-plane. In particular, all real eigenvalues are nonnegative. The following result therefore is of great consequence in practice.

THEOREM 4.2. *Any self-adjoint pencil $Q(\lambda; M, C, K)$ with positive definite matrix coefficients can be totally decoupled by congruence transformations.*

Proof. It suffices to show that $|S_2| = |S_3|$. If this is true, then all real-value eigenvalues are categorized into the single group C_a . It follows that $L^+ = R^+$, $L^- = R^-$ and, hence, $J_\ell = J_r$. In other words, the equivalence transformation involved in (3.44) and (3.45) becomes congruence transformation because $Y^{[6]} = X^{[6]}$.

The classification of S_2 and S_3 is based on the signs of the products $\mathbf{x}^\top \mathbf{A} \mathbf{x}$ where \mathbf{x} is a real-valued eigenvector associated with a real-valued eigenvalue. By Theorem 4.1, we have already known that there precisely half of these eigenvectors are such that $\mathbf{x}^\top \mathbf{B} \mathbf{x}$ is positive and the other half eigenvectors give rise to negative products. Since all real-values eigenvalues are of one sign, by (4.1) we know that precisely half of these eigenvectors are such that $\mathbf{x}^\top \mathbf{A} \mathbf{x}$ is positive and the other half leads to negative $\mathbf{x}^\top \mathbf{A} \mathbf{x}$. This shows $|S_2| = |S_3|$. \square

4.2. Without Positive Definite Coefficients. There is no specific pattern for real-valued eigenvalues of a general self-adjoint quadratic pencil. However, for any given real-valued eigenpair (λ, \mathbf{x}) , the signs of λ , $\mathbf{x}^\top \mathbf{A} \mathbf{x}$, and $\mathbf{x}^\top \mathbf{B} \mathbf{x}$ are related through the relationship

$$\operatorname{sgn}(\mathbf{x}^\top \mathbf{A} \mathbf{x}) = \operatorname{sgn}(\lambda) \operatorname{sgn}(\mathbf{x}^\top \mathbf{B} \mathbf{x}). \quad (4.5)$$

This relationship together with Theorem 4.1 enables us to refine the classification described in Section 3.1 as follows.

Case 1. Suppose that $\rho = \min\{|S_2|, |S_3|\} \neq 0$. Assume $\lambda_{2_s} > 0$ for a certain s . Then necessarily $\mathbf{x}_{2_s}^\top \mathbf{A} \mathbf{x}_{2_s} > 0$. By (4.5), we must have the corresponding $\mathbf{x}_{2_s}^\top \mathbf{B} \mathbf{x}_{2_s} > 0$. By definition, we always have $\mathbf{x}_{3_j}^\top \mathbf{A} \mathbf{x}_{3_j} < 0$ among all possible eigenvalues λ_{3_j} . By Theorem 4.1, it cannot be so that $\mathbf{x}_{3_j}^\top \mathbf{B} \mathbf{x}_{3_j} > 0$ for all j . There must be a certain t such that $\mathbf{x}_{3_t}^\top \mathbf{B} \mathbf{x}_{3_t} < 0$. By (4.5) again, we have the corresponding $\lambda_{3_t} > 0$. We therefore can pair $\{2_s, 2_t\} \in C_a$. Likewise, a given $\lambda_{2_s} < 0$ can be paired with a certain $\lambda_{3_t} < 0$. Repeating this argument, we see that $C_f = \emptyset$ since no pairs of eigenvalues should be placed in C_f . Only C_a is needed and contains ρ pairs.

Case 2. After we have exhausted as many pairs in C_a as possible, suppose there are some leftovers in S_2 but none in S_3 . Denote this remaining subset of S_2 by \hat{S}_2 . Note that $\mathbf{x}_{2_j}^\top \mathbf{A} \mathbf{x}_{2_j} > 0$ for all $2_j \in \hat{S}_2$. Assume that one such eigenvalue $\lambda_{2_s} > 0$. Then $\mathbf{x}_{2_s}^\top \mathbf{B} \mathbf{x}_{2_s} > 0$. However, by Theorem 4.1, it cannot be so that $\mathbf{x}_{2_j}^\top \mathbf{B} \mathbf{x}_{2_j} > 0$ for all the remaining $2_j \in \hat{S}_2$. There must be a certain $2_t \in \hat{S}_2$ such that $\mathbf{x}_{2_t}^\top \mathbf{B} \mathbf{x}_{2_t} < 0$. By (4.5), we must have the corresponding $\lambda_{2_t} > 0$. Therefore, the pair $(2_s, 2_t)$ is categorized into C_b . Likewise, a given $\lambda_{2_s} < 0$ from the remaining subset \hat{S}_2 can be paired with a certain $\lambda_{2_t} < 0$ with $2_t \in \hat{S}_2$. Repeating this argument, we see that no pairs of eigenvalues from \hat{S}_2 should be places in C_d . Only C_b is needed and $C_d = \emptyset$.

Case 3. Suppose that Case 2 does not happen. Rather, suppose that after we have exhausted as many pairs in C_a as possible, there are some leftovers in S_3 but none in S_2 . Denote this remaining subset of S_3 by \hat{S}_3 . Note that $\mathbf{x}_{3_j}^\top \mathbf{A} \mathbf{x}_{3_j} < 0$ for all $3_j \in \hat{S}_3$. Assume that one such eigenvalue $\lambda_{3_s} > 0$. Then $\mathbf{x}_{3_s}^\top \mathbf{B} \mathbf{x}_{3_s} < 0$. However, by Theorem 4.1, it cannot be so that $\mathbf{x}_{3_j}^\top \mathbf{B} \mathbf{x}_{3_j} < 0$ for all the remaining $3_j \in \hat{S}_3$. There must be a certain $3_t \in \hat{S}_3$ such that $\mathbf{x}_{3_t}^\top \mathbf{B} \mathbf{x}_{3_t} > 0$. By (4.5), we must have the corresponding $\lambda_{3_t} < 0$. Therefore, the pair $(3_s, 3_t)$ is categorized into C_c . Likewise, a given $\lambda_{3_s} < 0$ from the remaining subset \hat{S}_3 can be paired with a certain $\lambda_{3_t} > 0$ with $3_t \in \hat{S}_3$. Repeating this argument, we see that no pairs of eigenvalues from \hat{S}_3 should be places in C_e . Only C_c is needed and $C_e = \emptyset$.

By now, we have generalized Theorem 4.2 to the general self-adjoint quadratic pencil.

THEOREM 4.3. *Almost all self-adjoint pencil $Q(\lambda; M, C, K)$ with simple spectrum can be totally decoupled by congruence transformations.*

Proof. The arguments in the above three cases show that the real-valued eigenvalues can be regrouped in such a way that all pairs lie with the categories C_a , C_b and C_c only. An examination of the structure defined in (3.16) shows that $L^+ = R^+$, $L^- = R^-$ and, hence, $J_\ell = J_r$. \square

5. Conclusion. It is commonly accepted that there is no equivalence transformation that can simultaneously diagonalize the three coefficient matrices in a general quadratic pencil. Following the ideas by Garvey et al., however, we have shown that real-valued transformations that decouples the original n -degree-of-freedom system as the direct sum of n single-degree-of-freedom subsystems do exist for all most all quadratic pencils. The turning point that makes this advancement is that the notion of diagonalization is replaced by the Lancaster structure.

Our contributions in this paper are threefold: We offer a complete mathematical procedure showing how a given spectral decomposition which usually is complex-valued can be converted into real-valued equivalence transformations. Our approach, particularly in the steps of elimination and scaling, is easier and more straightforward than that suggested in [3]. In case of self-adjoint quadratic pencils, we employ the notion of matrix polynomial factorization to prove a conjecture in [3] that the equivalence transformation is indeed a congruence transformation. This fact has never been justified before.

A prerequisite of the maneuver outlined in this paper is the availability of a spectral decomposition of the original quadratic pencil, which makes the process numerically infeasible. To develop a more effective numerical method to realize such a decoupling process is an interesting open problem.

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