TOTAL DECOUPLING OF GENERAL QUADRATIC PENCILS, PART II: STRUCTURE PRESERVING ISOSPECTRAL FLOWS

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Abstract. Quadratic pencils, $\lambda^2 M + \lambda C + K$, where M, C, and K are $n \times n$ real matrices with or without some additional properties such as symmetry, connectivity, bandedness, or positive definiteness, arise in many important applications. Recently an existence theory has been established, showing that almost all n-degree-of-freedom second order systems can be reduced to n totally independent single-degree-of-freedom second order subsystems by real-valued isospectral transformations. In contrast to the common knowledge that generally no three matrices can be diagonalized simultaneously by equivalence transformations, these isospectral transformations endeavor to maintain a special linearization form called the Lancaster structure and do break down M, C and K into diagonal matrices simultaneously. However, these transformations depend on the matrices in a rather complicated way and, hence, are difficult to construct directly. In this paper, a second part of a continuing study, a closed-loop control system that preserves both the Lancaster structure and the isospectrality is proposed as a means to achieve the diagonal reduction. Consequently, these transformations are acquired.

Key words. quadratic pencil, Lancaster structure, structure preserving, multiple-degree-of-freedom system, equivalence transformation, closed-loop control, simultaneous diagonalization

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1. Introduction.

Given $n \times n$ real matrices M_0, C_0 and K_0, the task of finding scalars $\lambda \in \mathbb{C}$ and nonzero vectors $\mathbf{u} \in \mathbb{C}^n$ satisfying

$$Q(\lambda)\mathbf{u} = 0, \tag{1.1}$$

where

$$Q(\lambda) := Q(\lambda; M_0, C_0, K_0) = \lambda^2 M_0 + \lambda C_0 + K_0, \tag{1.2}$$

is known as the *quadratic eigenvalue problem* (QEP). The scalars λ and the corresponding vectors $\mathbf{x}$ are called, respectively, eigenvalues and eigenvectors of the quadratic pencil $Q(\lambda)$. It is known that the QEP possesses $2n$ eigenvalues over the complex field, provided the leading coefficient matrix M is nonsingular. The eigeninformation $(\lambda, \mathbf{u})$ is critical to the understanding of the dynamical system

$$M_0\ddot{\mathbf{x}} + C_0\dot{\mathbf{x}} + K_0\mathbf{x} = f(t), \tag{1.3}$$

which arises frequently in many important applications, including applied mechanics, electrical oscillations, vibro-acoustics, fluid mechanics, and signal processing.

There are extensive discussions about the QEPs. Both the theory and the numerical methods are fairly complete. See, for example, the review article [15], the books [9, 13] and the references contained therein. One principal tool used for analyzing QEPs is to linearize a quadratic pencil to a linear pencil. The linearization may appear in several different forms among which one is of particular interest to us — the so called *Lancaster structure* in the linear pencil

$$L(\lambda) := L(\lambda; M_0, C_0, K_0) = \begin{bmatrix} C_0 & M_0 \\ M_0 & 0 \end{bmatrix} \lambda + \begin{bmatrix} K_0 & 0 \\ 0 & -M_0 \end{bmatrix}. \tag{1.4}$$

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The equivalence between $Q(\lambda)$ and $L(\lambda)$ can be seen from the fact that

$$\left(\begin{bmatrix} C_0 & M_0 \\ M_0 & 0 \end{bmatrix}\lambda + \begin{bmatrix} K_0 & 0 \\ 0 & -M_0 \end{bmatrix}\right)\begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix} = 0 \tag{1.5}$$

if and only if

$$\left\{\begin{array}{rcl} (\lambda C_0 + K_0)\mathbf{u} + \lambda M_0\mathbf{v} & = & 0, \\ \lambda M_0\mathbf{u} - M_0\mathbf{v} & = & 0. \end{array}\right. \tag{1.6}$$

Indeed, if M is nonsingular, then we know further that $\mathbf{v} = \lambda\mathbf{u}$. Obviously, the Lancaster structure implies that if $Q(\lambda)$ is self-adjoint, then so is $L(\lambda)$.

The main reason that the Lancaster structure is important to us is because it has been proved recently that for almost all quadratic pencils there exists real-valued $2n \times 2n$ real matrices Π_ℓ and Π_r such that

$$\Pi_\ell^\top L(\lambda)\Pi_r = L(\lambda; M_D, C_D, K_D) = \begin{bmatrix} C_D & M_D \\ M_D & 0 \end{bmatrix}\lambda + \begin{bmatrix} K_D & 0 \\ 0 & -M_D \end{bmatrix}, \tag{1.7}$$

where M_D, C_D, K_D are all real-valued $n \times n$ diagonal matrices [2, 5, 6]. In other words, there exists a real-valued *equivalence transformation* which not only preserves the Lancaster structure but also transforms the pencil $L(\lambda)$ *isospectrally* into a pencil with diagonal blocks. Note that the eigenstructure is equivalent in the sense that

$$\left(\lambda^2 M_D + \lambda C_D + K_D\right)\mathbf{z} = 0 \Leftrightarrow \begin{bmatrix} \mathbf{u} \\ \lambda\mathbf{u} \end{bmatrix} = \Pi_r \begin{bmatrix} \mathbf{z} \\ \lambda\mathbf{z} \end{bmatrix}.$$

Such a transformation is significant in that it links the dynamical behavior of a multiple-degree-of-freedom system directly to that of a system consisting of n independent single-degree-of-freedom subsystems. It breaks down the interlocking connectivity in the original system into totally disconnected subsystems while preserving the entire spectral properties. Thus it will be of great value in practice if the transformations Π_ℓ and Π_r can be found from any given pencil. The theory of existence of Π_ℓ and Π_r in [2, 5] was established on the basis of the complete spectral information of $L(\lambda)$. To construct Π_ℓ and Π_r from the availability of spectral information certainly is impractical. The focus of this paper is to construct Π_ℓ and Π_r numerically by structure preserving isospectral flows without knowing the spectral information.

The isospectral transformation from the triplet (M_0, C_0, K_0) to the triplet (M_D, C_D, K_D) is not an ordinary equivalence transformation. It depends nonlinearly on matrices (M_0, C_0, K_0). To see this relationship, denote

$$\Pi_\ell = \begin{bmatrix} \ell_{11} & \ell_{12} \\ \ell_{21} & \ell_{22} \end{bmatrix}, \quad \Pi_r = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix}, \tag{1.8}$$

where each ℓ_{ij} or r_{ij} is an $n \times n$ matrices. In order to maintain the Lancaster structure in the product $\Pi_\ell^\top L(\lambda)\Pi_r$, it is necessary that the following five equations hold:

$$\begin{aligned} -\ell_{11}^\top K_0 r_{12} + \ell_{21}^\top M_0 r_{22} &= 0, \\ -\ell_{12}^\top K_0 r_{11} + \ell_{22}^\top M_0 r_{21} &= 0, \\ \ell_{12}^\top C_0 r_{12} + \ell_{22}^\top r_{12} + \ell_{12}^\top M_0 r_{22} &= 0, \\ \ell_{11}^\top C_0 r_{12} + \ell_{21}^\top M_0 r_{12} + \ell_{11}^\top M_0 r_{22} &= \ell_{12}^\top C_0 r_{11} + \ell_{22}^\top M_0 r_{11} + \ell_{12} M_0 r_{21}^\top \\ &= -\ell_{12}^\top K_0 r_{12} + \ell_{22}^\top M_0 r_{22}. \end{aligned} \tag{1.9}$$

Additionally, the matrices Π_ℓ and Π_r must be such that the left-hand sides of the following three expressions,

$$\begin{aligned} -\ell_{12}^\top K_0 r_{12} + \ell_{22}^\top M_0 r_{22} &= M_D, \\ \ell_{11}^\top C_0 r_{11} + \ell_{21}^\top M_0 r_{11} + \ell_{11}^\top M_0 r_{21} &= C_D, \\ \ell_{11}^\top K_0 r_{11} - \ell_{21}^\top M_0 r_{21} &= K_D, \end{aligned} \tag{1.10}$$

are diagonal matrices. The conditions (1.9) and (1.10) together constitute a homogeneous second-degree polynomial system of $8n^2 - 3n$ equations in $8n^2$ unknowns. It is not obvious how the system could be solved analytically. The underdetermined system does suggest, however, that there is plenty of leeway to choose the transformation matrices Π_ℓ and Π_r. In particular, the "orbit" of $L(\lambda)$ under (Lancaster) structure preserving equivalence transformations is a nontrivial manifold on which perhaps a smooth path connecting (M_0, C_0, K_0) to (M_D, C_D, K_D) can be defined.

A special kind of isospectral flow preserving the Lancaster structure has been proposed in [7]. What is needed is a more specific control of the flow so that it starts from (M_0, C_0, K_0) and moves toward (M_D, C_D, K_D). Our contribution in this paper is that we describe a closed-loop feedback control system to drive such a flow. The resulting dynamical system can be tracked numerically.

2. Isospectral flow. Our closed-loop feedback control system is built upon the structure preserving isospectral flows proposed in [7]. For later reference, we briefly review what has been introduced in [7]. It is important to note that the flows described in this section can only maintain the Lancaster structure and the isospectrality. The flows will have to be modified in order to acquire the additional capability of reducing matrices to diagonals.

For convenience, denote the Lancaster pair in (1.4) by (A_0, B_0), that is,

$$A_0 = \begin{bmatrix} K_0 & 0 \\ 0 & -M_0 \end{bmatrix}, \qquad B_0 = \begin{bmatrix} C_0 & M_0 \\ M_0 & 0 \end{bmatrix}. \tag{2.1}$$

We are interested in characterizing two one-parameter families of structured preserving transformations $T_L(t), T_R(t) \in \mathbb{R}^{2n\times 2n}$, with $T_L(0) = T_R(0) = I_{2n}$. Let the actions of these families of transformations on (A_0, B_0) be denoted by

$$A(t) = T_L^\top(t) A_0 T_R(t), \qquad B(t) = T_L^\top(t) B_0 T_R(t), \tag{2.2}$$

respectively. Clearly, regardless how $T_L(t)$ and $T_R(t)$ are defined, $(A(t), B(t))$ is isospectral to (A_0, B_0) for any t. A special class of transformations is to require that matrices $T_L(t)$ and $T_R(t)$ satisfy, respectively, the following differential systems:

$$\frac{dT_L(t)}{dt} = T_L(t)\mathcal{L}(t) = T_L(t) \begin{bmatrix} L_{11}(t) & L_{12}(t) \\ L_{21}(t) & L_{22}(t) \end{bmatrix}, \tag{2.3}$$

$$\frac{dT_R(t)}{dt} = T_R(t)\mathcal{R}(t) = T_R(t) \begin{bmatrix} R_{11}(t) & R_{12}(t) \\ R_{21}(t) & R_{22}(t) \end{bmatrix}, \tag{2.4}$$

where each $L_{ij}(t)$ or $R_{ij}(t)$, $i, j = 1, 2$, is a $n \times n$ real one-parameter matrix yet to be defined. The task now is to impose conditions on the matrices $\mathcal{L}(t)$ and $\mathcal{R}(t)$ so that the resulting $(A(t), B(t))$ maintains the Lancaster structure for every t.

For convenience, denote the differentiation $\frac{dg}{dt}$ of any function $g(t)$ by the symbol $\dot{g}$. It is easy to see from (2.2) that

$$\begin{aligned} \dot{A} &= \dot{T}_L^\top A_0 T_R + T_L A_0 \dot{T}_R = \mathcal{L}^\top A + A\mathcal{R}, \\ \dot{B} &= \dot{T}_L^\top B_0 T_R + T_L B_0 \dot{T}_R = \mathcal{L}^\top B + B\mathcal{R}. \end{aligned}$$

It is interesting to note that these differential equations are similar to those discussed in [1] which leads to a Lie-Poisson system. By insisting that $(A(t), B(t))$ maintains the Lancaster structure, that is,

$$A(t) = \begin{bmatrix} K(t) & 0 \\ 0 & -M(t) \end{bmatrix}, \qquad B(t) = \begin{bmatrix} C(t) & M(t) \\ M(t) & 0 \end{bmatrix}, \tag{2.5}$$

we see that

$$\begin{bmatrix} \dot{K} & 0 \\ 0 & -\dot{M} \end{bmatrix} = \begin{bmatrix} L_{11}^\top K + KR_{11} & -L_{21}^\top M + KR_{12} \\ L_{12}^\top K - MR_{21} & -L_{22}^\top M - MR_{22} \end{bmatrix}, \tag{2.6}$$

$$\begin{bmatrix} \dot{C} & \dot{M} \\ \dot{M} & 0 \end{bmatrix} = \begin{bmatrix} L_{11}^\top C + CR_{11} + L_{21}^\top M + MR_{21} & L_{11}^\top M + MR_{22} + CR_{12} \\ L_{12}^\top C + L_{22}^\top M + MR_{11} & L_{12}^\top M + MR_{12} \end{bmatrix}. \tag{2.7}$$

It follows that the following five equations must be satisfied by the matrices L_{ij} and R_{ij}, $i, j = 1, 2$:

$$\begin{aligned} &KR_{12} - L_{21}^\top M = 0, \\ &L_{12}^\top K - MR_{21} = 0, \\ &L_{12}^\top M + MR_{12} = 0, \\ &L_{11}^\top M - L_{22}^\top M + CR_{12} = 0, \\ &MR_{11} - MR_{22} + L_{12}^\top C = 0. \end{aligned} \tag{2.8}$$

The conditions in (2.8) constitute a homogeneous linear system of $5n^2$ for the $8n^2$ entries in the matrices L_{ij} and R_{ij}, $i, j = 1, 2$. It is a much easier system than the nonlinear system (1.9) and (1.10). Its solution space contains $3n^2$ free parameters which we can identify as three $n \times n$ matrix parameters. The transformations $T_L(t)$ and $T_R(t)$ can now be characterized in terms of these three free matrix parameters.

In fact, by assuming that the matrix $M(t)$ is invertible, we may set forth the first matrix parameter $D(t) \in \mathbb{R}^{n\times n}$ by requiring that the relationship

$$R_{12}(t) = -D(t)M(t)$$

holds between $R_{12}(t)$ and $M(t)$. It is not difficult to derive after some algebraic manipulations that the solutions to the system (2.8) can now be identified as follows:

$$R_{12} = -DM, \tag{2.9}$$

$$R_{21} = DK, \tag{2.10}$$

$$L_{12} = D^\top M^\top, \tag{2.11}$$

$$L_{21} = -D^\top K^\top, \tag{2.12}$$

$$L_{11} - L_{22} = D^\top C^\top, \tag{2.13}$$

$$R_{11} - R_{22} = -DC. \tag{2.14}$$

Now that we have obtained these formulas, it is worth mentioning in retrospect that even without the assumption that $M(t)$ is nonsingular the matrices defined by (2.9) to (2.14) satisfy the system (2.8). Note also that implicit in (2.13) and (2.14) are the other two free matrix parameters. There are several possible ways to arrange the diagonal blocks of $\mathcal{L}(t)$ and $\mathcal{R}(t)$. The choice suggested in [7] is to define $\mathcal{L}(t)$ and $\mathcal{R}(t)$ according to the following formulas:

$$\mathcal{L} = \begin{bmatrix} D^\top & 0 \\ 0 & D^\top \end{bmatrix} \begin{bmatrix} \frac{C^\top}{2} & -M^\top \\ -K^\top & -\frac{C^\top}{2} \end{bmatrix} + \begin{bmatrix} N_L^\top & 0 \\ 0 & N_L^\top \end{bmatrix}, \tag{2.15}$$

$$\mathcal{R} = \begin{bmatrix} D & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} -\frac{C}{2} & -M \\ K & \frac{C}{2} \end{bmatrix} + \begin{bmatrix} N_R & 0 \\ 0 & N_R \end{bmatrix}, \tag{2.16}$$

where the matrices $D(t)$, $N_L(t)$ and $N_R(t)$ are free matrix parameters in $\mathbb{R}^{n\times n}$. Upon substituting the blocks of $\mathcal{L}$ and $\mathcal{R}$ into the differential system (2.6) and (2.7), we obtain a flow of the triplet $(M(t), C(t), K(t))$ which is governed by the autonomous system:

$$\begin{aligned}\dot{K} &= \frac{1}{2}(CDK - KDC) + N_L^\top K + KN_R,\\ \dot{C} &= (MDK - KDM) + N_L^\top C + CN_R,\\ \dot{M} &= \frac{1}{2}(MDC - CDM) + N_L^\top M + MN_R.\end{aligned} \tag{2.17}$$

We could also choose to define $\mathcal{L}(t)$ and $\mathcal{R}(t)$, for example, in the following way,

$$\mathcal{L} = \begin{bmatrix} D^\top & 0 \\ 0 & D^\top \end{bmatrix}\begin{bmatrix} 0 & -M^\top \\ -K^\top & C^\top \end{bmatrix} + \begin{bmatrix} N_L & 0 \\ 0 & N_L \end{bmatrix}, \tag{2.18}$$

$$\mathcal{R} = \begin{bmatrix} D & 0 \\ 0 & D \end{bmatrix}\begin{bmatrix} 0 & -M \\ K & C \end{bmatrix} + \begin{bmatrix} N_R & 0 \\ 0 & N_R \end{bmatrix}. \tag{2.19}$$

The corresponding differential system for $(M(t), C(t), K(t))$ becomes somewhat simpler:

$$\begin{aligned}\dot{K} &= N_L^\top K + KN_R\\ \dot{C} &= (MDK - KDM) + N_L^\top C + CN_R\\ \dot{M} &= (MDC - CDM) + N_L^\top M + MN_R\end{aligned} \tag{2.20}$$

It is interesting to note that simplicity does not necessarily mean benefit because the first equation in (2.20) implies $K(t)$ is an equivalent transformation of K_0, which might limit the way $K(t)$ can change. We shall concentrate on the system defined in (2.17) henceforth. The main question now is how to exploit the three free matrix parameters D, N_L, and N_R so that the resulting flow behaves in some desirable ways.

3. Selecting free parameters. In this section we demonstrate some of the possible choices of the three parameters in the system (2.17). Ultimately, we want to control the free parameters D, N_L and N_R in such a way that the isospectral flow $(M(t), C(t), K(t))$ is driven into a block diagonal triplet (M_D, C_D, K_D).

3.1. Maintaining symmetry. In additional to the Lancaster structure, it might be desirable to maintain the symmetry in the initial value (M_0, C_0, K_0), if there is any, throughout the flow $(M(t), C(t), K(t))$ for all t. This task can be accomplished by selecting the free matrix parameters with proper symmetric properties. Several sufficient conditions have already been mentioned in [5]. For example, by assuming $N_R(t) = N_L(t)$, the symmetry specified for the matrix parameter D in Table 3.1 will preserve the symmetry for the flow $(M(t), K(t), C(t))$ defined by the dynamical system (2.17).

$D(t)$	$M(t)$	$C(t)$	$K(t)$
skew-symmetric	symmetric	symmetric	symmetric
symmetric	symmetric	skew-symmetric	symmetric
symmetric	skew-symmetric	skew-symmetric	skew-symmetric
skew-symmetric	skew-symmetric	symmetric	skew-symmetric

TABLE 3.1
Preserving symmetries of $(M(t), C(t), K(t))$ by $D(t)$, if $N_R(t) = N_L(t)$.

3.2. Nahm equations. Another choice of the free matrix parameters might be worth mentioning because it makes a remarkable connection to the Nahm equations [17]. More specifically, if we choose $N_R(t) = N_L(t) = N(t)$ to be an arbitrary one-parameter flow and define

$$D(t) := S(t)T(t), \tag{3.1}$$

where $S(t)$ and $T(t)$ are solution flows to the linear system

$$\dot{S} = -NS, \tag{3.2}$$
$$\dot{T} = -TN^\top, \tag{3.3}$$

then $D(t)$ satisfies the differential equation

$$\dot{D} = -ND - DN^T \tag{3.4}$$

and remains skew-symmetric if $D(0)$ is skew-symmetric. In this way, we know from Table 3.1 that the triplet $(M(t), C(t), K(t))$ defined by (2.17) remains symmetric if (M_0, C_0, K_0) is symmetric to begin with. Using $S(t)$ and $T(t)$ to define the equivalent transformation:

$$\widetilde{M}(t) = T(t)M(t)S(t), \quad \widetilde{C}(t) = T(t)C(t)S(t), \quad \widetilde{K}(t) = T(t)K(t)S(t), \tag{3.5}$$

we find from straightforward substitution that the transformed triplet $(\widetilde{M}(t), \widetilde{C}(t), \widetilde{K}(t))$ satisfies the differential system

$$\begin{aligned} \dot{\widetilde{M}} &= [\widetilde{M}, \frac{1}{2}\widetilde{C}], \\ \dot{\widetilde{C}} &= [\widetilde{M}, \widetilde{K}], \\ \dot{\widetilde{K}} &= [\frac{1}{2}\widetilde{C}, \widetilde{K}], \end{aligned} \tag{3.6}$$

where $[X, Y] := XY - YX$ represents the Lie bracket operator. The system (3.6) bears considerable resemblance to the system known as the Nahm equations arising in the study of Yang-Hills theory. On the other hand, observe that the solution flow $(\widetilde{M}(t), \widetilde{C}(t), \widetilde{K}(t))$ in (3.6) depends only on the initial value $(\widetilde{M}(0), \widetilde{C}(0), \widetilde{K}(0))$ and is independent of how $S(t)$ and $T(t)$ at any other t. In other words, the selection of $N(t)$ will not affect the dynamics of the system (3.6). Though interesting, this choice of free matrix parameters might not be helpful in diagonalizing the triplet (M_0, C_0, K_0).

3.3. Gradient flow. One possible way to force the flow $(M(t), C(t), K(t))$ to converge to the diagonal form (M_D, C_D, K_D) is to construct the structure preserving isospectral flow $(A(t), B(t))$ defined in (2.5) in such a way that it is also a gradient flow for a certain properly selected objective function. To see how this can be achieved, we outline the idea below.

Consider the following open-loop optimal control problem:

$$\begin{aligned} \min_{\mathbf{x} \in \mathbb{R}^n} \quad & f(\mathbf{x}), \\ \text{subject to} \quad & \dot{\mathbf{x}} = g(\mathbf{x})\mathbf{u}, \quad \mathbf{x}(0) = \mathbf{x}_0, \end{aligned} \tag{3.7}$$

where the objective function $f : \mathbb{R}^n \to \mathbb{R}$ is sufficient smooth, $g : \mathbb{R}^n \to \mathbb{R}^{n \times p}$ is piecewise continuous with $\operatorname{rank}(g(\mathbf{x})) = p$, and $\mathbf{u} = \mathbf{u}(t) \in \mathbb{R}^p$ is the control. The problem as is given is not well-posed in that there are infinitely many ways to administer the control. Some additional constraints must be imposed on the admissible control $\mathbf{u}(t)$. We shall not pursue that avenue in this discussion. For our application, however, we notice that without the differential equation constraint, a natural direction for $\mathbf{x}$ to move to minimize $f(\mathbf{x})$ is the steepest descent direction. Now that the motion of $\mathbf{x}$ is governed by (3.7), perhaps one way we can do is to choose the control $\mathbf{u}$ so that the vector $\dot{\mathbf{x}}$ is as close to $-\nabla f(\mathbf{x})$ as possible. This amounts to the selection of the least squares solution $\mathbf{u}$ defined by

$$\mathbf{u}(t) = -g(\mathbf{x}(t))^{\dagger}\nabla f(\mathbf{x}(t)), \tag{3.8}$$

where $g(\mathbf{x})^{\dagger}$ stands for the Moore-Penrose generalized inverse of $g(\mathbf{x})$. In this way, the closed-loop dynamical system,

$$\dot{\mathbf{x}} = -g(\mathbf{x})g(\mathbf{x})^{\dagger}\nabla f(\mathbf{x}), \tag{3.9}$$

defines a descent flow $\mathbf{x}(t)$ for the objective function $f(\mathbf{x})$.

In our setting, we seek matrix parameters N_R, N_L and D to minimize the following objective function

$$\begin{aligned} f(K,C,M) := \frac{1}{2}\left\{\|\mathrm{offdiag}(M)\|_F^2 + \|\mathrm{offdiag}(C)\|_F^2 + \|\mathrm{offdiag}(K)\|_F^2\right\} \\ + \ \delta h(\mathrm{diag}(M), \mathrm{diag}(C), \mathrm{diag}(K)), \end{aligned} \tag{3.10}$$

subject to the condition that $(M(t), C(t), K(t))$ is governed by the differential system (2.17). In the above, $\|\cdot\|_F$ denotes the Frobenius matrix norm, $\mathrm{diag}(M)$ denotes the diagonal matrix of M, $\mathrm{offdiag}(M)$ denotes the complementary part of $\mathrm{diag}(M)$ in M, and h is a scalar function depending upon the diagonal entries of M, C and K. Our idea is to minimize the off-diagonal entries of (M, C, K) while using the function h to monitor the behavior of diagonal entries by a factor of δ. Such a monitoring is sometimes important because our structure preserving isospectral flows are not norm preserving. Our experience indicates that the diagonal entries can evolve to fairly large or small numbers. By choosing, for example,

$$h_1(\mathrm{diag}(M), \mathrm{diag}(C), \mathrm{diag}(K)) = \frac{1}{(\min(\mathrm{diag}(M)))^2} + \frac{1}{(\min(\mathrm{diag}(C)))^2} + \frac{1}{(\min(\mathrm{diag}(K)))^2}, \tag{3.11}$$

where $\min(\mathrm{diag}(M))$ denote the minimum entry in the diagonal of M, we can penalize small diagonal entries in M, C and K and, hence, avoid singular pencils. Likewise, by choosing

$$h_2(\mathrm{diag}(M), \mathrm{diag}(C), \mathrm{diag}(K)) = \|\mathrm{diag}(M)\|_F^2 + \|\mathrm{diag}(C)\|_F^2 + \|\mathrm{diag}(K)\|_F^2, \tag{3.12}$$

we can damp the growth of diagonal entries. At the moment, the choice of h is on an ad hoc basis which varies from problem to problem. We do not know of a general rule by which h should be used, but we do want to point out that modifying the definition of h and, hence, the objection function f with the hope to effect the behavior of the isospectral flow is not difficult to do. We even can modify the objective function adaptively during the integration and, hence, offer a dynamical control of the flow. The free matrix parameters D, N_L and N_R are used as controls to direct the flow. It is important to note that the dynamical system (2.17) is linear in the matrix parameters D, N_L and N_R. So our situation fits well to the model described in (3.8). In particular, the "controls" D, N_L and N_R can be obtained as the least squares solution to the equation

$$\begin{bmatrix} \frac{1}{2}(K\otimes C - C\otimes K) & K\otimes I & I\otimes K \\ K\otimes M - M\otimes K & C\otimes I & I\otimes C \\ \frac{1}{2}(C\otimes M - M\otimes C) & M\otimes I & I\otimes M \end{bmatrix} \begin{bmatrix} \mathbf{vec}(D) \\ \mathbf{vec}(N_L^{\top}) \\ \mathbf{vec}(N_R) \end{bmatrix} = \begin{bmatrix} \mathbf{vec}(-\mathrm{offdiag}(K) - \delta\frac{\partial h}{\partial K}) \\ \mathbf{vec}(-\mathrm{offdiag}(C) - \delta\frac{\partial h}{\partial C}) \\ \mathbf{vec}(-\mathrm{offdiag}(M) - \delta\frac{\partial h}{\partial M}) \end{bmatrix}, \tag{3.13}$$

where $\mathbf{vec}(X)$ denotes the vectorization of the matrix X by columns and $\frac{\partial h}{\partial K}$ denotes the partial gradient of h with respect to K. Once these controls are calculated, they are fed to (2.17) to define the flow $(M(t), C(t), K(t))$.

4. Self-adjoint pencils. Thus far, we have been considering general quadratic pencils. In applications, often we are facing a self-adjoint quadratic pencil, that is, the matrix coefficients M, C and K are all symmetric. Many of the discussions above can be simplified due to the fact established in [2] that a self-adjoint quadratic pencil can be totally decoupled by congruence transformations.

To exploit this congruence transformation, we may take $T_R(t) = T_L(t) := T(t)$ and reduce the transformations (2.2) to merely

$$A(t) = T^\top(t)A_0T(t), \qquad B(t) = T^\top(t)B_0T(t), \tag{4.1}$$

while $T(t)$ satisfies (2.4). The algebraic system (2.8) which is necessary for maintaining the Lancaster structure is reduced to

$$\begin{aligned} KR_{12} - R_{21}^\top M &= 0, \\ R_{12}^\top M + MR_{12} &= 0, \\ MR_{11} - MR_{22} + R_{12}^\top C &= 0. \end{aligned} \tag{4.2}$$

Note the second equation in (4.2) is symmetric, so the conditions (4.2) constitute a linear algebraic system of $n(n+1)/2 + 2n^2$ equations in the $4n^2$ unknowns matrices R_{ij}. In other words, in the self-adjoint case, a total of $n(n-1)/2 + n^2$ parameters can be chosen arbitrarily for the structure preserving isospectral flows. Motivated by the fact that MR_{12} has to be skew-symmetric, we set forth the first matrix parameters $D(t)$ by assuming that the matrix $M(t)$ is invertible and that

$$R_{12} = -DM$$

for some skew-symmetric matrix $D \in \mathbb{R}^{n\times n}$. Upon substitution, we find that the solution to the system (4.2) can be parameterized as follows:

$$R_{12} = -DM, \qquad \text{with } D^\top = -D, \tag{4.3}$$

$$R_{21} = DK, \tag{4.4}$$

$$R_{11} - R_{22} = -DC. \tag{4.5}$$

We choose to define the $\mathcal{R}$ matrix in (2.4) in exactly the same way as in (2.16), i.e.,

$$\mathcal{R} = \begin{bmatrix} D & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} -\frac{C}{2} & -M \\ K & \frac{C}{2} \end{bmatrix} + \begin{bmatrix} N & 0 \\ 0 & N \end{bmatrix} \tag{4.6}$$

where the matrices $N \in \mathbb{R}^{n\times n}$ is the second free matrix parameter. The corresponding differential equations for $(M(t), C(t), K(t))$ are given by

$$\begin{aligned} \dot{K} &= \frac{1}{2}(CDK - KDC) + N^\top K + KN, \\ \dot{C} &= (MDK - KDM) + N^\top C + CN, \\ \dot{M} &= \frac{1}{2}(MDC - CDM) + N^\top M + MN, \end{aligned} \tag{4.7}$$

respectively. The system (4.7) is a special case of (2.17) under the additional conditions that $N_L = N_R$ and $D^\top = -D$.

In the same spirit as that proposed in Section 3.3, we may choose the controls D and N to formulate structure preserving isospectral gradient flow with the hope that the self-adjoint triplet $(M(t), C(t), K(t))$ will converge to a diagonal triplet (M_D, C_D, K_D). Obviously, the symmetry has the advantage that the size of the optimal control problem is nearly halved. The implementation, however, requires some extra efforts to reflect that D is skew-symmetric and that only the upper triangular part of $(M(t), C(t), K(t))$ is needed for the computation. We shall report some numerical experiments in the next section without elaborating upon the programming details.

5. Coordinate transformation flows. We have developed a gradient flow for the triplet $(M(t), C(t), K(t))$ which in theory preserves the Lancaster structure and maintains the isospectrality. In practice, however, we have to caution that traditional ODE integrators generally cannot preserve these properties in the long run. Isospectral flows need to be solved by using special integration techniques. The recently developed research area, known as the geometric integration, is for that purpose. In geometric integration the underlying geometric structure which influences the qualitative nature of the phenomena are built into the numerical method, which gives the method markedly superior performance and accuracy. See for example, the web site [4], the review paper [12] and the many linkages or references contained therein. In this paper we have not investigated the applicability of any geometric integration method to our flow yet. We do want to point out, however, that for our application maybe it is sufficient to consider only the flow for the transformations $T_L(t)$ and $T_R(t)$ instead of the flow for the triplet $(M(t), C(t), K(t))$. Our idea is that the geometric structure imposed on the transformation matrices are automatically satisfied by the way we define the matrices $\mathcal{L}$ and $\mathcal{R}$. We apply the transformations to (A_0, B_0) as in (2.2) only at the end of integration to obtain the reduced form. In this way, we are saved from worrying about the loss of isospectrality in the course of integration.

Using the self-adjoint pencils to illustrate the approach and writing

$$T(t) = \begin{bmatrix} T_{11}(t) & T_{12}(t) \\ T_{21}(t) & T_{22}(t) \end{bmatrix},$$

we see that the differential system governing the transformation flow $T(t)$ is given by

$$\begin{aligned}
\dot{T}_{11} &= T_{11}\left(N - \frac{D\left(T_{11}^\top C_0 T_{11} + T_{11}^\top M_0 T_{21} + T_{21}^\top M_0 T_{21}\right)}{2}\right) + T_{12} D\left(T_{11}^\top K_0 T_{11} - T_{21}^\top M_0 T_{21}\right), \\
\dot{T}_{12} &= T_{11} D\left(T_{12}^\top K_0 T_{12} - T_{22}^\top M_0 T_{22}\right) + T_{12}\left(N + \frac{D\left(T_{11}^\top C_0 T_{11} + T_{11}^\top M_0 T_{21} + T_{21}^\top M_0 T_{21}\right)}{2}\right), \\
\dot{T}_{21} &= T_{21}\left(N - \frac{D\left(T_{11}^\top C_0 T_{11} + T_{11}^\top M_0 T_{21} + T_{21}^\top M_0 T_{21}\right)}{2}\right) + T_{22} D\left(T_{11}^\top K_0 T_{11} - T_{21}^\top M_0 T_{21}\right), \\
\dot{T}_{22} &= T_{21} D\left(T_{12}^\top K_0 T_{12} - T_{22}^\top M_0 T_{22}\right) + T_{22}\left(N + \frac{D\left(T_{11}^\top C_0 T_{11} + T_{11}^\top M_0 T_{21} + T_{21}^\top M_0 T_{21}\right)}{2}\right),
\end{aligned}$$

which makes no reference to the intermediate values $M(t)$, $C(t)$ and $K(t)$. The computation is not as complicated as it appears because many of the matrices repeatedly occur. We can rewrite the objective function (3.10) in terms of T whose gradient can easily be calculated. Since $\dot{T}$ depends linearly on D and N, the model (3.7) remains applicable. The idea of closed loop control described in Section 3.3 can now be used to obtain the matrix parameters D and N for a gradient flow $T(t)$.

6. Numerical experiments. In this section we report some experimental results from using the above-mentioned dynamical system of gradient flow. At the moment, our primary concern is not so much on the efficiency of the method. Rather, our goal is to establish some numerical evidence showing that the proposed structure preserving isospectral gradient flows work. To make this approach computationally effective requires additional ruminations, such as a specially designed geometric integrator, which are not investigated in this paper.

For demonstration purpose, we shall employ existing routines in Matlab as the ODE integrator. It is understood that many other ODE solvers can be used as well, although none of these packages are designed to preserve our geometric properties for a long period of time. The ODE Suite [14] in Matlab contains in particular a Klopfenstein-Shampine, quasi-constant step size stiff system solver

ode15s and the classical Adams-Bashforth-Molton solver **ode113**. We find that the isospectrality is deteriorated quicker by **ode15s** than by **ode113**, though neither solver can maintain the isospectrality in the long run. We set both local tolerance $AbsTol = RelTol = 10^{-12}$ while maintaining all other parameters at the default values of the Matlab codes. The numerical tests have been conducted using randomly generated initial values for the dynamical system matrices in M, C, K. The choice of the penalty function h depends on how we want to effect the flow. In the first two examples below, h_2 is used where the penalty factor is taken to be $\delta = 10^{-1}$. In the third example, h_1 with $\delta = 10^{-8}$ is used to keep the flow from converging to a singular pencil. For the ease of running text, we shall report all numerals in 5 digits only.

Example 1. Consider the three-degree of freedom mass-spring system depicted in Figure 6.1.

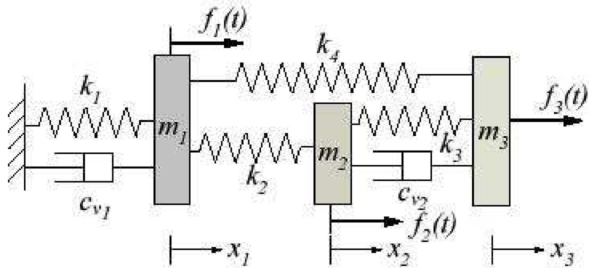

Fig. 6.1. *A three-degree-of-freedom system (from www.efunda.com).*

Assume that the mass, damping and stiffness matrices given by

$$M_0 = \begin{bmatrix} 0.5056 & 0 & 0 \\ 0 & 0.9198 & 0 \\ 0 & 0 & 0.9440 \end{bmatrix} \quad C_0 = \begin{bmatrix} 0.9814 & 0 & 0 \\ 0 & 0.9602 & -0.4582 \\ 0 & -0.4582 & 1.0794 \end{bmatrix}, \quad K_0 = \begin{bmatrix} 1.1550 & -0.4673 & -0.2788 \\ -0.4673 & 0.5849 & -0.1176 \\ -0.2788 & -0.1176 & 0.3964 \end{bmatrix}$$

Our theory asserts that this interlocking self-adjoint system can be decoupled into three single-degree-of-freedom subsystems which bear exactly the same spectral information. To obtain these subsystems, we integrate our gradient flow (4.7) with D and N being defined in the same spirit of (3.8), that is, the free matrix parameters D and N are selected in such a way that the resulting vector field $(\dot{M}, \dot{C}, \dot{K})$ is the least squares approximation to the negative gradient of the objection function (3.10). At $t \approx 16$ we find that the triplet (M, C, K) has evolved into

$$M = \begin{bmatrix} 2.6362e{+}00 & 1.5263e{-}08 & 1.7655e{-}08 \\ 1.5263e{-}08 & 3.8736e{+}00 & 2.4375e{-}08 \\ 1.7655e{-}08 & 2.4375e{-}08 & 4.8376e{+}00 \end{bmatrix},$$

$$C = \begin{bmatrix} 4.0251e{+}00 & 5.0428e{-}08 & -9.0397e{-}09 \\ 5.0428e{-}08 & 5.8081e{+}00 & -7.6755e{-}08 \\ -9.0397e{-}09 & -7.6755e{-}08 & 5.3330e{+}00 \end{bmatrix},$$

$$K = \begin{bmatrix} 5.0903e{+}00 & -1.3823e{-}07 & -3.9939e{-}08 \\ -1.3823e{-}07 & 2.9065e{+}00 & -5.9834e{-}09 \\ -3.9939e{-}08 & -5.9834e{-}09 & 1.1113e{+}00 \end{bmatrix},$$

suggesting that the matrices are being diagonalized as t goes to infinity.

Example 2. To demonstrate that our gradient flow works reasonably well in general, in our second experiment we generate three 8×8 self-adjoint initial matrices randomly with no concern of whether the resulting quadratic pencil is physically realizable or not. The initial matrices are given as follows.

$$
M_0 = \begin{bmatrix}
5.3235e+00 & -7.7083e-01 & -1.7021e+00 & 5.2962e-01 & 6.3322e-01 & -9.2241e-01 & -6.3554e-01 & 5.8556e-01 \\
-7.7083e-01 & 9.2417e+00 & -1.7353e+00 & 5.1740e+00 & -3.7535e+00 & 2.6184e-02 & -3.2625e+00 & 1.2013e+00 \\
-1.7021e+00 & -1.7353e+00 & 9.2284e+00 & -6.4989e+00 & -1.8649e+00 & 3.0272e+00 & 7.9076e+00 & 2.6297e-01 \\
5.2962e-01 & 5.1740e+00 & -6.4989e+00 & 9.4077e+00 & 2.1911e-01 & -4.3342e+00 & -7.3370e+00 & 3.840e+00 \\
6.3322e-01 & -3.7535e+00 & -1.8649e+00 & 2.1911e-01 & 6.0898e+00 & -9.8487e-01 & 2.0184e+00 & -7.3590e-01 \\
-9.2241e-01 & 2.6184e-02 & 3.0272e+00 & -4.3342e+00 & -9.8487e-01 & 9.1233e+00 & 6.2714e+00 & -2.9034e+00 \\
-6.3554e-01 & -3.2625e+00 & 7.9076e+00 & -7.3370e+00 & 2.0184e+00 & 6.2714e+00 & 1.4541e+01 & -1.7151e+00 \\
5.8556e-01 & 1.2013e+00 & 2.6297e-01 & 3.840e+00 & -7.3590e-01 & -2.9034e+00 & -1.7151e+00 & 4.5468e+00
\end{bmatrix},
$$

$$
C_0 = \begin{bmatrix}
1.0777e+01 & 5.0180e-01 & 2.6108e-01 & 4.6112e+00 & -2.8607e+00 & -3.6377e+00 & -2.3615e+00 & 2.9215e+00 \\
5.0180e-01 & 3.8908e+00 & 2.8436e-01 & 1.8618e-01 & 8.6016e-01 & 3.0443e-01 & -9.6059e-01 & 4.4675e-01 \\
2.6108e-01 & 2.8436e-01 & 1.1905e+01 & 2.2132e-01 & -4.3222e+00 & -2.9963e+00 & 4.2085e+00 & 5.4883e+00 \\
4.6112e+00 & 1.8618e-01 & 2.2132e-01 & 4.7964e+00 & 5.1232e-01 & -1.108e+00 & -8.7565e-01 & 1.6242e+00 \\
-2.8607e+00 & 8.6016e-01 & -4.3222e+00 & 5.1232e-01 & 9.4669e+00 & 4.3679e+00 & 8.9368e-01 & -1.5694e+00 \\
-3.6377e+00 & 3.0443e-01 & -2.9963e+00 & -1.108e+00 & 4.3679e+00 & 6.6132e+00 & -6.5587e-01 & 1.1057e+00 \\
-2.3615e+00 & -9.6059e-01 & 4.2085e+00 & -8.7565e-01 & 8.9368e-01 & -6.5587e-01 & 5.4341e+00 & 4.2017e+00 \\
2.9215e+00 & 4.4675e-01 & 5.4883e+00 & 1.6242e+00 & -1.5694e+00 & 1.1057e+00 & 4.2017e+00 & 1.1347e+01
\end{bmatrix},
$$

$$
K_0 = \begin{bmatrix}
5.095e+00 & -3.1576e+00 & -1.2604e+00 & -3.4265e+00 & 8.6552e-01 & 2.7751e+00 & 1.1638e+00 & -5.3069e+00 \\
-3.1576e+00 & 9.6212e+00 & -1.4043e+00 & 8.5129e-01 & 3.9745e-01 & -1.0149e+00 & 3.1666e+00 & 6.4578e+00 \\
-1.2604e+00 & -1.4043e+00 & 4.2248e+00 & -1.4245e+00 & -1.9488e+00 & 1.5413e+00 & -2.5609e+00 & -6.7258e-01 \\
-3.4265e+00 & 8.5129e-01 & -1.4245e+00 & 7.0133e+00 & 2.3293e+00 & -5.8867e+00 & 6.2474e-01 & 4.2150e+00 \\
8.6552e-01 & 3.9745e-01 & -1.9488e+00 & 2.3293e+00 & 7.1340e+00 & -4.0990e+00 & 3.7037e+00 & 1.8451e+00 \\
2.7751e+00 & -1.0149e+00 & 1.5413e+00 & -5.8867e+00 & -4.0990e+00 & 7.9290e+00 & -9.4201e-01 & -5.7685e+00 \\
1.1638e+00 & 3.1666e+00 & -2.5609e+00 & 6.2474e-01 & 3.7037e+00 & -9.4201e-01 & 7.2946e+00 & 1.3111e+00 \\
-5.3069e+00 & 6.4578e+00 & -6.7258e-01 & 4.2150e+00 & 1.8451e+00 & -5.7685e+00 & 1.3111e+00 & 9.0110e+00
\end{bmatrix}.
$$

Again, we use the gradient flow (4.7) to seek the diagonalization of the above three matrices. As the integration marches on, we find that at $t = 18$ the triplet $(M(t), C(t), K(t))$ has evolved into the following matrices:

$$
M = \begin{bmatrix}
9.8696e+00 & 1.8872e-06 & -2.9059e-06 & -1.7974e-05 & 2.1019e-07 & -1.2046e-05 & -1.3155e-06 & -3.6771e-07 \\
1.8872e-06 & 3.9277e+01 & -3.3041e-07 & 2.3073e-06 & -4.2645e-07 & 2.9113e-06 & 4.6964e-07 & -3.3942e-07 \\
-2.9059e-06 & -3.3041e-07 & 5.0117e+01 & -9.2381e-07 & 2.3446e-07 & -1.0293e-07 & 1.0538e-06 & 6.2995e-07 \\
-1.7974e-05 & 2.3073e-06 & -9.2381e-07 & 1.1038e+01 & 7.5763e-07 & -6.3820e-05 & 6.170e-07 & 1.0678e-06 \\
2.1019e-07 & -4.2645e-07 & 2.3446e-07 & 7.5763e-07 & 1.5349e+01 & 3.4752e-07 & -7.0162e-09 & 1.8576e-07 \\
-1.2046e-05 & 2.9113e-06 & -1.0293e-07 & -6.3820e-05 & 3.4752e-07 & 8.2547e+00 & 1.1788e-06 & 1.9319e-07 \\
-1.3155e-06 & 4.6964e-07 & 1.0538e-06 & 6.170e-07 & -7.0162e-09 & 1.1788e-06 & 3.8841e+01 & 2.4136e-07 \\
-3.6771e-07 & -3.3942e-07 & 6.2995e-07 & 1.0678e-06 & 1.8576e-07 & 1.9319e-07 & 2.4136e-07 & 9.3683e-02
\end{bmatrix},
$$

$$
C = \begin{bmatrix}
3.1910e+01 & -4.3944e-07 & 5.2498e-06 & 4.3910e-06 & 2.9588e-07 & 9.6416e-07 & 2.0965e-07 & 1.0194e-06 \\
-4.3944e-07 & 9.1654e+00 & 2.8050e-07 & -1.2528e-06 & 1.0689e-07 & -1.0687e-06 & -6.8847e-07 & -5.1075e-08 \\
5.2498e-06 & 2.8050e-07 & 4.9507e+01 & 2.8435e-07 & -4.2462e-07 & -5.540e-07 & -2.8281e-07 & -9.6140e-07 \\
4.3910e-06 & -1.2528e-06 & 2.8435e-07 & 1.1911e+01 & -8.5491e-08 & 1.3644e-05 & -6.8026e-07 & 3.8892e-08 \\
2.9588e-07 & 1.0689e-07 & -4.2462e-07 & -8.5491e-08 & 6.0835e+01 & 5.380e-07 & -1.3286e-07 & -2.7331e-07 \\
9.6416e-07 & -1.0687e-06 & -5.540e-07 & 1.3644e-05 & 5.380e-07 & 1.8138e+01 & 9.3527e-07 & 5.4394e-07 \\
2.0965e-07 & -6.8847e-07 & -2.8281e-07 & -6.8026e-07 & -1.3286e-07 & 9.3527e-07 & 3.2757e+01 & 1.3056e-07 \\
1.0194e-06 & -5.1075e-08 & -9.6140e-07 & 3.8892e-08 & -2.7331e-07 & 5.4394e-07 & 1.3056e-07 & 5.9449e+01
\end{bmatrix},
$$

$$
K = \begin{bmatrix}
1.6621e+00 & -9.6927e-07 & -1.3292e-05 & 1.1156e-06 & -9.4317e-06 & 6.8078e-07 & 1.4116e-06 & -1.5211e-06 \\
-9.6927e-07 & 2.1574e+01 & -6.0898e-08 & -2.9165e-07 & 1.3859e-07 & -6.8694e-07 & 1.7096e-08 & 8.0610e-07 \\
-1.3292e-05 & -6.0898e-08 & 1.5674e+01 & -2.2511e-07 & -9.8188e-07 & 4.0419e-07 & -3.7487e-07 & 1.6344e-06 \\
1.1156e-06 & -2.9165e-07 & -2.2511e-07 & 3.4844e+01 & -2.6431e-07 & 1.2303e-05 & -1.0540e-07 & -1.7776e-07 \\
-9.4317e-06 & 1.3859e-07 & -9.8188e-07 & -2.6431e-07 & 2.6028e-01 & -7.705e-07 & 5.3461e-07 & -2.8792e-07 \\
6.8078e-07 & -6.8694e-07 & 4.0419e-07 & 1.2303e-05 & -7.705e-07 & 2.3699e+01 & -1.5434e-06 & -1.0864e-06 \\
1.4116e-06 & 1.7096e-08 & -3.7487e-07 & -1.0540e-07 & 5.3461e-07 & -1.5434e-06 & 4.9072e+01 & -3.5570e-07 \\
-1.5211e-06 & 8.0610e-07 & 1.6344e-06 & -1.7776e-07 & -2.8792e-07 & -1.0864e-06 & -3.5570e-07 & 1.0512e+01
\end{bmatrix}.
$$

It might be more illustrative to represent the data in the initial triplet (M_0, C_0, K_0) and the triplet $(M(t), C(t), K(t))$ graphically in Figure 6.2 where entries of each matrix are plotted as z-values over a rectangle grid.

The dynamical behavior of the corresponding flow is depicted in Figure 6.3 where we plot the sums of norms of the three diagonal matrices (dashed line), the three off-diagonal matrices (dotted line), and the absolute value of the objective function (solid line), respectively, versus the independent variable t. The dip in the solid line for the objective function is a resolution artifact due to $|f(K, C, M)| \approx 0$ or $\ln(|f(K, C, M)|) \approx -\infty$ at that particular point. The near parallelism of the solid line and the dashed line when $t > 8$ shows that the objective value the the norm of

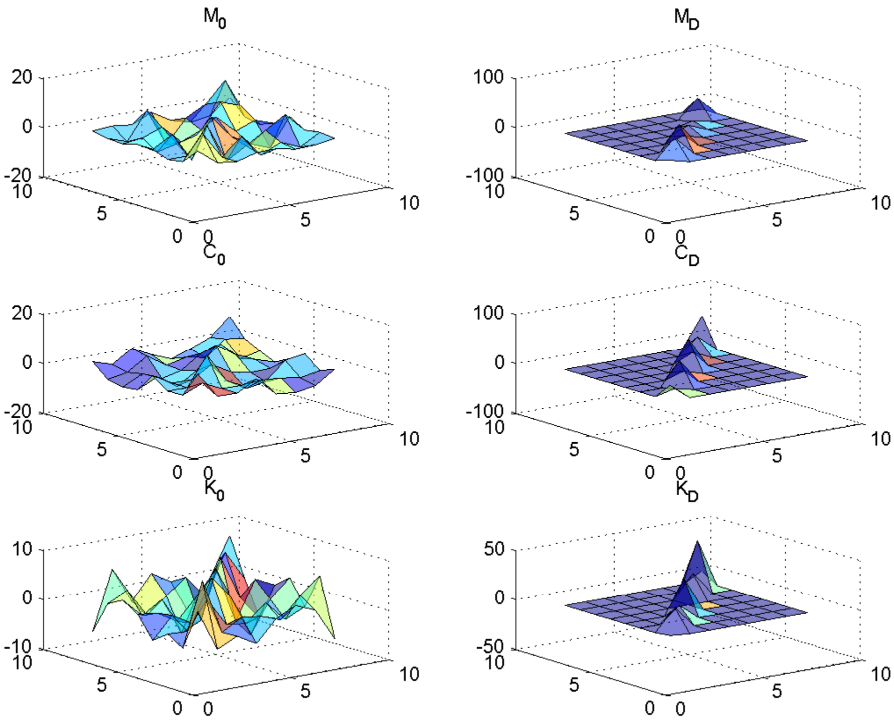

FIG. 6.2. *Graphical representation of* (M_0, C_0, K_0) *and the triplet* $(M(18), C(18), K(18))$ *in Example 2.*

the diagonal entries differ nearly by a scalar multiplication, i.e., $\delta = 10^{-1}$, and that the off-diagonal entries converge to zero.

Example 3. In our third experiment, we first use the same objective function as that in Example 2 with the following randomly generated initial matrices of size 7×7. That is, we penalize only the growth of diagonal entries and take no action to prevent the degenerateness of the these entries. The initial matrices are given by

$$M_0 = \begin{bmatrix} 6.4206e{+}00 & 2.7013e{+}00 & -4.0538e{+}00 & -3.9093e{+}00 & 3.5144e{+}00 & -1.6145e{-}01 & -1.1971e{+}00 \\ 2.7013e{+}00 & 1.0326e{+}01 & -5.4895e{+}00 & -4.2758e{+}00 & -4.5101e{+}00 & 6.2040e{+}00 & -2.1748e{-}01 \\ -4.0538e{+}00 & -5.4895e{+}00 & 8.7637e{+}00 & 1.0725e{+}00 & 1.5145e{+}00 & -4.6242e{-}01 & 1.0867e{+}00 \\ -3.9093e{+}00 & -4.2758e{+}00 & 1.0725e{+}00 & 4.8558e{+}00 & -2.1136e{+}00 & -2.6535e{+}00 & -1.8584e{-}01 \\ 3.5144e{+}00 & -4.5101e{+}00 & 1.5145e{+}00 & -2.1136e{+}00 & 1.8306e{+}01 & 2.9484e{+}00 & 1.6960e{+}00 \\ -1.6145e{-}01 & 6.2040e{+}00 & -4.6242e{-}01 & -2.6535e{+}00 & 2.9484e{+}00 & 1.0953e{+}01 & -7.6732e{-}01 \\ -1.1971e{+}00 & -2.1748e{-}01 & 1.0867e{+}00 & -1.8584e{-}01 & 1.6960e{+}00 & -7.6732e{-}01 & 6.4739e{+}00 \end{bmatrix},$$

$$C_0 = \begin{bmatrix} 2.8355e{+}00 & -3.5329e{+}00 & -9.0051e{-}01 & 2.2403e{+}00 & 2.3916e{-}01 & -1.8860e{+}00 & 1.1730e{+}00 \\ -3.5329e{+}00 & 9.1601e{+}00 & 3.1358e{-}01 & 1.6880e{+}00 & 7.7742e{-}01 & -3.3175e{-}02 & -2.0576e{+}00 \\ -9.0051e{-}01 & 3.1358e{-}01 & 5.8714e{+}00 & -2.2298e{+}00 & -2.1827e{-}01 & -1.2010e{+}00 & 6.5279e{-}01 \\ 2.2403e{+}00 & 1.6880e{+}00 & -2.2298e{+}00 & 8.3197e{+}00 & 1.7075e{-}01 & -2.8330e{+}00 & -1.2423e{+}00 \\ 2.3916e{-}01 & 7.7742e{-}01 & -2.1827e{-}01 & 1.7075e{-}01 & 3.0995e{+}00 & 1.2724e{+}00 & 1.8257e{+}00 \\ -1.8860e{+}00 & -3.3175e{-}02 & -1.2010e{+}00 & -2.8330e{+}00 & 1.2724e{+}00 & 8.7520e{+}00 & -8.4060e{-}02 \\ 1.1730e{+}00 & -2.0576e{+}00 & 6.5279e{-}01 & -1.2423e{+}00 & 1.8257e{+}00 & -8.4060e{-}02 & 2.3845e{+}00 \end{bmatrix},$$

$$K_0 = \begin{bmatrix} 5.7454e{+}00 & -5.5223e{-}01 & -1.3115e{+}00 & 2.1641e{+}00 & 2.3034e{-}01 & -9.2219e{-}01 & 8.3982e{-}02 \\ -5.5223e{-}01 & 2.3549e{+}00 & -2.7189e{-}01 & 1.9810e{+}00 & 1.9808e{-}01 & -6.1442e{-}01 & 7.5386e{-}01 \\ -1.3115e{+}00 & -2.7189e{-}01 & 9.5935e{+}00 & -1.0742e{-}01 & -5.5547e{+}00 & -5.4112e{+}00 & 4.4360e{-}01 \\ 2.1641e{+}00 & 1.9810e{+}00 & -1.0742e{-}01 & 8.0620e{+}00 & -6.8328e{-}02 & 3.2041e{-}01 & -1.1730e{+}00 \\ 2.3034e{-}01 & 1.9808e{-}01 & -5.5547e{+}00 & -6.8328e{-}02 & 5.2168e{+}00 & 2.5000e{+}00 & -1.4050e{+}00 \\ -9.2219e{-}01 & -6.1442e{-}01 & -5.4112e{+}00 & 3.2041e{-}01 & 2.5000e{+}00 & 7.2703e{+}00 & 7.9278e{-}01 \\ 8.3982e{-}02 & 7.5386e{-}01 & 4.4360e{-}01 & -1.1730e{+}00 & -1.4050e{+}00 & 7.9278e{-}01 & 7.2693e{+}00 \end{bmatrix}.$$

What we have observed is that at approximately $t = 16$, the triplet $(M(t), C(t), K(t))$ has evolved

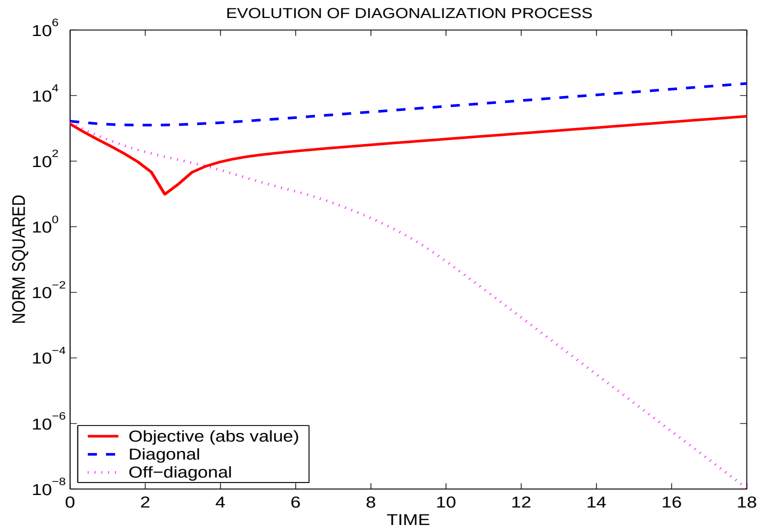

Fig. 6.3. *Behavior of the objective function*

into the following matrices:

$$M = \begin{bmatrix} 2.2153e-13 & 2.1177e-06 & -6.8153e-07 & -1.8384e-07 & 1.1511e-06 & -7.0594e-07 & -4.8310e-07 \\ 2.1177e-06 & 2.7777e+01 & -8.5516e-07 & -2.4470e-06 & -9.9352e-07 & 1.7505e-06 & 3.8014e-07 \\ -6.8153e-07 & -8.5516e-07 & 3.1307e+01 & 3.1516e-07 & 3.2599e-07 & 2.9865e-07 & -1.7375e-07 \\ -1.8384e-07 & -2.4470e-06 & 3.1516e-07 & 2.8390e+00 & -4.9293e-07 & -1.3284e-07 & 1.4923e-07 \\ 1.1511e-06 & -9.9352e-07 & 3.2599e-07 & -4.9293e-07 & 8.8150e+01 & 5.4243e-07 & 3.7747e-07 \\ -7.0594e-07 & 1.7505e-06 & 2.9865e-07 & -1.3284e-07 & 5.4243e-07 & 4.8753e+01 & -3.1072e-07 \\ -4.8310e-07 & 3.8014e-07 & -1.7375e-07 & 1.4923e-07 & 3.7747e-07 & -3.1072e-07 & 3.1905e+01 \end{bmatrix},$$

$$C = \begin{bmatrix} -3.1703e-13 & -1.2169e-06 & -4.0357e-07 & -1.6317e-06 & -6.7147e-08 & -4.3025e-07 & -1.6132e-08 \\ -1.2169e-06 & 4.2699e+01 & 1.9723e-07 & 1.3068e-06 & 1.1515e-07 & -3.7094e-07 & -3.0443e-07 \\ -4.0357e-07 & 1.9723e-07 & 2.2568e+01 & -3.4856e-07 & 5.2178e-08 & -2.2798e-07 & 6.8085e-09 \\ -1.6317e-06 & 1.3068e-06 & -3.4856e-07 & 3.3206e+01 & 2.1429e-07 & -5.4923e-07 & -2.2280e-07 \\ -6.7147e-08 & 1.1515e-07 & 5.2178e-08 & 2.1429e-07 & 1.2109e+01 & 2.9003e-08 & 2.0453e-07 \\ -4.3025e-07 & -3.7094e-07 & -2.2798e-07 & -5.4923e-07 & 2.9003e-08 & 4.5150e+01 & 2.8697e-08 \\ -1.6132e-08 & -3.0443e-07 & 6.8085e-09 & -2.2280e-07 & 2.0453e-07 & 2.8697e-08 & 5.6642e+00 \end{bmatrix},$$

$$K = \begin{bmatrix} 2.1121e-13 & -2.5786e-07 & -2.1259e-08 & 2.6425e-06 & 2.2480e-07 & -3.9657e-07 & -1.3054e-07 \\ -2.5786e-07 & 4.5693e+00 & 1.6895e-07 & -3.3515e-07 & 5.4747e-07 & -1.0080e-07 & 3.1232e-07 \\ -2.1259e-08 & 1.6895e-07 & 4.3968e+01 & -3.6565e-07 & -1.0213e-06 & -1.1184e-06 & 1.5553e-07 \\ 2.6425e-06 & -3.3515e-07 & -3.6565e-07 & 4.1656e+01 & -2.3364e-07 & 4.4752e-07 & -2.2251e-07 \\ 2.2480e-07 & 5.4747e-07 & -1.0213e-06 & -2.3364e-07 & 6.1404e+00 & 8.2303e-07 & -2.8470e-07 \\ -3.9657e-07 & -1.0080e-07 & -1.1184e-06 & 4.4752e-07 & 8.2303e-07 & 1.5933e+01 & 1.5514e-07 \\ -1.3054e-07 & 3.1232e-07 & 1.5553e-07 & -2.2251e-07 & -2.8470e-07 & 1.5514e-07 & 3.3756e+01 \end{bmatrix}.$$

This example demonstrates that the triplet $(M(t), C(t), K(t))$ may converge to a singular pencil, which is not desirable.

A remedy might come if we penalize the decaying of diagonal entries to zero by adding in the penalty function g_1 defined in (3.11). However, the penalty factor δ has to be chosen carefully. If δ is too large, the flow tends to put more emphasis on discouraging the decay of the diagonal entries at the price of slowing down the convergence of the off-diagonal entries. If δ is too small, the flow converges to a near-singular pencil. For our experiment, we adaptively use $\delta = 10^{-8}$ to discourage the diagonal entries from going to zero and $\delta = 0$ to encourage the off-diagonal entries to converge to zero. At the moment, the adaptive scheme is inserted into the integration process manually and subjectively. We are able to improve the convergence to the following matrices.

$$M = \begin{bmatrix} 1.4618e{-}01 & 8.4229e{-}21 & -7.1889e{-}20 & -2.2598e{-}20 & -3.1102e{-}20 & 2.2559e{-}19 & -1.6235e{-}20 \\ -1.4672e{-}15 & 5.2520e{+}00 & 4.4173e{-}20 & 8.5578e{-}20 & 8.7294e{-}21 & 2.7976e{-}05 & -1.8820e{-}07 \\ 5.0709e{-}15 & 4.5057e{-}16 & 3.9248e{+}00 & -4.7978e{-}20 & -1.2110e{-}20 & 1.2027e{-}19 & 3.3292e{-}21 \\ 1.9132e{-}15 & 1.1803e{-}14 & -8.9042e{-}15 & 1.7497e{-}01 & 1.0212e{-}19 & 3.5002e{-}20 & 2.0289e{-}20 \\ 1.4079e{-}15 & 2.8080e{-}15 & -6.7490e{-}15 & 2.6045e{-}16 & 9.6442e{-}01 & -3.2220e{-}19 & -6.3030e{-}21 \\ -1.2395e{-}18 & 2.7976e{-}05 & -2.3740e{-}19 & 5.4627e{-}19 & -2.0220e{-}19 & 5.9730e{-}08 & -2.5479e{-}05 \\ 4.3913e{-}15 & -1.8820e{-}07 & 1.3186e{-}15 & -2.8604e{-}15 & -1.5211e{-}14 & -2.5479e{-}05 & 9.1884e{+}00 \end{bmatrix},$$

$$C = \begin{bmatrix} 4.9848e{+}00 & 8.3472e{-}23 & 4.2650e{-}21 & 3.3450e{-}20 & 3.1178e{-}20 & -5.3475e{-}19 & 2.2401e{-}20 \\ -3.8580e{-}14 & 6.9063e{+}00 & 1.4369e{-}20 & 4.7488e{-}20 & 6.4548e{-}20 & -3.4666e{-}05 & -4.3162e{-}07 \\ -5.9428e{-}15 & -6.6258e{-}15 & 7.5498e{-}01 & -3.7208e{-}20 & 1.7218e{-}21 & -2.3996e{-}20 & 6.0364e{-}21 \\ 1.4816e{-}14 & 3.6117e{-}15 & 3.5201e{-}16 & 1.0704e{+}01 & 1.6483e{-}20 & -1.9994e{-}19 & -4.5170e{-}20 \\ -2.3898e{-}14 & 6.6073e{-}15 & 3.4875e{-}14 & 2.9369e{-}14 & 4.5047e{+}00 & 1.0176e{-}19 & -3.0356e{-}20 \\ -4.9065e{-}19 & -3.4666e{-}05 & -2.6450e{-}18 & -1.3797e{-}19 & 2.4698e{-}18 & 6.1934e{-}08 & -4.4886e{-}05 \\ -6.2580e{-}15 & -4.3162e{-}07 & 1.9122e{-}15 & -3.4434e{-}14 & -2.4793e{-}14 & -4.4886e{-}05 & 6.1427e{+}00 \end{bmatrix},$$

$$K = \begin{bmatrix} 8.9073e{-}04 & -6.7678e{-}20 & -6.0650e{-}20 & -7.9906e{-}20 & -3.3974e{-}21 & 3.9575e{-}19 & 7.8452e{-}21 \\ -9.1039e{-}16 & 4.9464e{+}00 & -3.5549e{-}20 & 3.5156e{-}20 & 6.6650e{-}20 & 1.9716e{-}05 & 6.0100e{-}07 \\ 1.0159e{-}14 & 8.8236e{-}15 & 4.6964e{+}00 & -6.7415e{-}20 & 2.3783e{-}22 & -1.2507e{-}19 & -3.4392e{-}20 \\ -8.1779e{-}16 & 3.8584e{-}16 & -2.4487e{-}15 & 4.2010e{+}00 & 6.4197e{-}21 & 2.1525e{-}19 & 6.5043e{-}20 \\ 4.7487e{-}15 & -9.4770e{-}15 & -3.7846e{-}15 & 1.6717e{-}14 & 1.6497e{+}01 & -4.4318e{-}20 & 1.0721e{-}19 \\ -1.3417e{-}19 & 1.9716e{-}05 & -6.3702e{-}19 & 1.2723e{-}19 & -7.8421e{-}19 & 8.5473e{-}09 & 8.9375e{-}05 \\ -2.4348e{-}15 & 6.0100e{-}07 & 1.0354e{-}15 & -2.5474e{-}15 & -1.0384e{-}14 & 8.9375e{-}05 & 5.6434e{+}00 \end{bmatrix}.$$

7. Conclusion. In an earlier study, we have shown in theory that all most all quadratic pencils $\lambda^2 M + \lambda C + K$ can be transformed isospectrally into pencils with diagonal matrix coefficients. This result has two significant implications: First, it shows that the conventional persuasion that no three general matrices can be simultaneously diagonalized is perhaps because the question of diagonalization of a quadratic pencil has not been posed in an appropriate context. Perhaps a right way to ask the question is how to block diagonalizing the Lancaster structure. Secondly, it asserts that the dynamical behavior of almost all n-degree-of-freedom second order systems can be identified from that of n independent single-degree-of-freedom second-order subsystems. Despite the importance, the transformations involved in the reduction are rather complicated and difficult to realize numerically. The theoretical proof requires the knowledge of the entire spectral information. Without using the spectral information, there does not seem to have any numerical algorithm in the literature for this purpose.

In this paper, we exploit the free matrix parameters in the structure preserving isospectral flows. In particular, we propose a simple closed-loop control that amends the structure preserving isospectral flow into a gradient flow. The gradient flow intends to reduce the magnitude of off-diagonal elements. Since the gradient flow can be tracked by available ODE integrator, it is feasible for numerical computation. Computer simulations seem to suggest the working of this approach.

Lot of room remains for further study. The most imperative topic is to develop special geometric integrator for the isospectral flow described in this paper. Since the structure preserving equivalence transformations do not form a group, we do not think that current Lie group methods are applicable. Also, it remains to be studied on whether the system (1.9) and (1.10) could be tackled by some structure preserving iterative schemes.

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