

# STRUCTURE PRESERVING ISOSPECTRAL TRANSFORMATION FOR TOTAL OR PARTIAL DECOUPLING OF SELF-ADJOINT QUADRATIC PENCILS

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**Abstract.** Quadratic pencils  $\lambda^2 M + \lambda C + K$ , where  $M$ ,  $C$ , and  $K$  are  $n \times n$  real matrices, arise in many important applications. Its spectral properties effect the vibration behavior of the underlying system. Depending on the inter-connectivity of the elements within the system, the coefficient matrices might inherit additional properties such as symmetry, block form, zero row sum, or positive definiteness. For both theoretical considerations and practical applications, it is often desired to decompose a complicated system into dissociated subsystems without tampering with the innate vibration properties. Total decoupling refers to the extreme case where an  $n$ -degree-of-freedom system is reduced to  $n$  totally independent single-degree-of-freedom subsystems, while partial decoupling into units of independent modules is sometimes more feasible in practice. Most mass-spring-damper systems are self-adjoint and the mass matrix  $M$  is diagonal, which can be symmetrized to  $M = I$ . This paper aims at constructing the transformation that can either totally or partially decouple a given quadratic system. At the crux of construction is the flow that preserves the so called Lancaster structure which, in turn, preserves the spectrum while the flow follows a projected gradient to diminish the unwanted (coupled) portion of the system. The construction is flexible in that it can be custom-made to handle any kinds of decoupled structure. Error analysis is derived, whose bounds can be used for assessing the quality of the transformation.

**Key words.** self-adjoint quadratic pencil, decoupling, structure preserving transformation, Lancaster structure, isospectrality, gradient flow

**AMS subject classifications.** 15A22, 70H15, 93B10

**1. Introduction.** Coupled systems are ubiquitous. They arise in nature and in almost all areas of disciplines, including economical development, agricultural production, industrial manufacture, environmental evolution, or mechanics applications. The entanglement manifested in the system proves difficult but critical, and is the most basic episode when characterizing a complicate phenomenon that involves many inter-related factors. In general, the dynamics and the associated effects of a coupled system are difficult to analyze or to control. It has been long desired to find simple but equivalent system to represent the very same dynamics. This is the basic concept of decoupling. As simple as a linear transformation, for example, the so called spectral decomposition of representing the original system by a diagonal matrix in terms of the basis of eigenvectors is a classical decoupling process. Depending on the applications, scientists and practitioners have developed different techniques for the task of decoupling of a given system. This paper is concerned about the decoupling of a *second-order linear dynamical system* of the form

$$M_0 \ddot{\mathbf{x}}(t) + C_0 \dot{\mathbf{x}}(t) + K_0 \mathbf{x}(t) = f(t), \quad (1.1)$$

where  $M_0$ ,  $C_0$  and  $K_0$  are given matrix coefficients. Such a system arises frequently in dynamic analysis of applied mechanics, acoustic systems, electrical circuit simulation, fluid mechanics and even microelectronic design [35].

For general discussion on the vibration theory associated with (1.1), see, for example, the books [5], [29], and [30]. Consider the damped mass-spring system depicted in Figure 1.1 as an example. Assuming the Hooke's law for small deformation, the vibration of this particular layout with four

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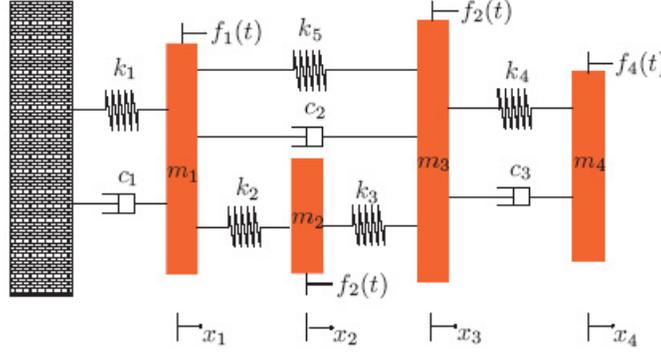


FIG. 1.1. A four-DoF mass-spring-damper system.

degrees of freedom (DoF) is governed by the equation of motion

$$\begin{bmatrix} m_1 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \\ 0 & 0 & m_3 & 0 \\ 0 & 0 & 0 & m_4 \end{bmatrix} \ddot{\mathbf{x}} + \begin{bmatrix} c_1+c_2 & 0 & -c_2 & 0 \\ 0 & 0 & 0 & 0 \\ -c_2 & 0 & c_2+c_3 & -c_3 \\ 0 & 0 & -c_3 & c_3 \end{bmatrix} \dot{\mathbf{x}} + \begin{bmatrix} k_1+k_2+k_5 & -k_2 & -k_5 & 0 \\ -k_2 & k_2+k_3 & -k_3 & 0 \\ -k_5 & -k_3 & k_3+k_4+k_5 & -k_4 \\ 0 & 0 & -k_4 & k_4 \end{bmatrix} \mathbf{x} = \mathbf{f}(t). \quad (1.2)$$

Clearly, the prescribed inter-connectivity results in a special structure of the matrix coefficients. In the context of applied mechanics, the coefficients  $M_0$ ,  $C_0$  and  $K_0$  in (1.1) represent the mass matrix, the damping or gyroscopic matrix and the stiffness or dissipation matrix, respectively, whereas  $\mathbf{x}(t)$  denotes the displacement of the masses from the equilibria and  $f(t)$  is the external excitation in time  $t$ . Different configuration specifics lead to different structures and, hence, different dynamic behaviors. The system is coupled because the movement of each mass is affected by other masses through the linkages.

A dynamical system arising in applications usually consists of multiple subsystems. Coupling is a result of interaction among the subsystems. Mathematically speaking, a second-order linear dynamical system is coupled if the triplets  $(M, C, K)$  are not all diagonal at the same time. Coupling is an inherent nature, but does not give any advantage to vibration analysis. Indeed, engineers often desires to control the vibration of a given system, but the presence of coupling makes the task difficult. It has been long sought in the field for techniques that can break up some or all inner links of subsystems so as to produce a simplified system with the same dynamical behavior.

Decoupling is a classical subject across multiple disciplines. Remarkably many research efforts have been taken to meet the demands from different applications. Articles and books about decoupling are too numerous to list here. We briefly review some basic notions.

**Classical modal analysis.** An *undamped system*, where  $C = 0$ , can be regarded as an generalized eigenvalue problem (GEP), whence can be decoupled by congruence transformation using associated eigenvectors. This technique has been generalized to the so called *classical damping system* [6] where the matrices  $M$ ,  $C$  and  $K$  are required to be symmetric,  $M$  positive definite,  $C$  and  $K$  nonnegative definite, and satisfy the relationship

$$CM^{-1}K = KM^{-1}C. \quad (1.3)$$

Specifically, the technique known as the *classical modal analysis* employs a transformation matrix consisting of eigenvectors of the GEP associated with the pencil  $(M, K)$  only to diagonalize the damping matrix  $C$ . For systems with general damping, the so called *non-proportional damping systems*, the

task of decoupling is much more involved. A variety of techniques can be found in [2], [4], [22], [24], [25], [36], and the many references contained therein. We mention that most of the existing techniques require the complete spectral information, which falls in line with the conventional thinking of using eigenvectors as the basis for transformation. In contrast, as will be explained in the sequel, our approach does the decoupling without employing any spectral information.

**Spectral decomposition.** Specifically, any solution  $(\lambda, \mathbf{u}) \in \mathbb{C} \times \mathbb{C}^n$  to the *quadratic eigenvalue problem* (QEP) [18, 23, 35]

$$Q(\lambda)\mathbf{u} = 0, \quad (1.4)$$

where

$$Q(\lambda) := Q(\lambda; M_0, C_0, K_0) = \lambda^2 M_0 + \lambda C_0 + K_0 \quad (1.5)$$

constitutes a fundamental solution

$$\mathbf{x}(t) = \mathbf{u}e^{\lambda t} \quad (1.6)$$

to the differential system (1.1). The eigeninformation of the quadratic pencil (1.5) entails the dynamical behavior of (1.1). There are effective numerical methods for tackling QEPs. See, for example, `quadeig` in [20] and `polyeig` in MATLAB. The calculation of the entire spectrum and the associated eigenvectors therefore constitutes, in a sense, a process of total decoupling. What is missing and is of significant importance in applications, however, is the single transformation that does the decoupling.

**Phase synchronization.** To find such a transformation, an interesting technique known as phase synchronization has been purposed in [27, 28]. The idea is motivated by the fact that complex conjugate eigenvalues of a classically damping mode must correspond to one real eigenvector [7]. In general, the eigenvalues and corresponding eigenvectors of real-valued QEPs must occur in complex conjugate pairs. Suppose that  $(\mathbf{v}_j, \bar{\mathbf{v}}_j)$  is such a pair of eigenvectors. Write

$$\mathbf{v}_j = [r_{j1}e^{-i\phi_{j1}} \quad r_{j2}e^{-i\phi_{j2}} \quad \dots \quad r_{jn}e^{-i\phi_{jn}}]^\top, \quad (1.7)$$

where  $r_{jk}, \phi_{jk} \in \mathbb{R}$  and  $\phi_{jk}$  is called phase angles. The method of phase synchronization manipulates the algebra to synchronize  $\phi_{jk}$  by phase shift and thus convert the non-classically damped mode to classically damped mode. In this way, the needed transformation can be founded by applying the conventional mode analysis to the resulting classically damped mode. The drawback of such an approach, again, is that it requires the complete eigeninformation. We are interested in decoupling a QEP system, either totally or partially, without invoking the eigeninformation.

**Simultaneous diagonalization.** The question of total decoupling seems to be mathematically related to the problem of simultaneous diagonalization. It is well known that simultaneous diagonalization for even two matrices is not always possible. Necessary or sufficient conditions for simultaneous diagonalization of two matrices can be found in the classic books [14] and [21]. For a QEP system, total decoupling means the simultaneous diagonalization of three coefficient matrices. This is much harder and the theory of existence is hardly available. It was only recently that necessary and sufficient conditions for simultaneous diagonalization of three general symmetric matrices are completely characterized in [31]. It is worth noting that the equation (1.3) for the classical damping systems is only part of the condition for the general systems. These conditions are fairly stringent, so in general no equivalence or congruence coordinate transformations can diagonalize three arbitrary matrices simultaneously.

**Structure preserving isospectral flow.** The truth is that there is a subtle difference between total decoupling and simultaneous diagonalization. Simultaneous diagonalization of  $(M, C, K)$  "directly" is actually a wrong question to ask. What is essential in decoupling is the preservation of

eigeninformation. So long as we can preserve the eigeninformation, we might decouple a system indirectly, which lead to diagonalization in a different context. More specifically, we first linearize the QEP (1.5) the GEP in the Lancaster form

$$L(\lambda) := L(\lambda; M_0, C_0, K_0) = \begin{bmatrix} C_0 & M_0 \\ M_0 & 0 \end{bmatrix} \lambda + \begin{bmatrix} K_0 & 0 \\ 0 & -M_0 \end{bmatrix}. \quad (1.8)$$

It has been shown that there exist two real-valued  $2n \times 2n$  matrices  $\Pi_\ell$  and  $\Pi_r$  such that [15, 16]

$$\Pi_\ell^\top L(\lambda) \Pi_r = L(\lambda; M_D, C_D, K_D) = \begin{bmatrix} C_D & M_D \\ M_D & 0 \end{bmatrix} \lambda + \begin{bmatrix} K_D & 0 \\ 0 & -M_D \end{bmatrix}, \quad (1.9)$$

where  $M_D, C_D, K_D$  are all real-valued  $n \times n$  diagonal matrices. It is in this sense that the original triplets  $(M_0, C_0, K_0)$  are simultaneously diagonalized.

Thus motivated is a differential equation approach that traces three trajectories for matrices  $M(t), C(t), K(t)$ , respectively, until  $M_D, C_D, K_D$  are found [10, 11]. The construction of the flows is to maintain the Lancaster structure and, hence, keep the spectrum invariant, while gradually reducing the off-diagonal entries of  $M(t), C(t), K(t)$ . An alternative approach is to replace the congruent transformation (1.9) by a similarity transformation, which eliminates the need of the left transformation  $\Pi_\ell$  at the cost of decoupling only the products  $M^{-1}C$  and  $M^{-1}K$  [37]. Both of these approaches work directly with the coefficient matrices until the diagonalization is realized at the end. The computation concern only about the evolution of the coefficient matrices. During and at the end of the integration, no information of the transformations  $\Pi_\ell$  and  $\Pi_r$  is generated. In applications, the transformation that results in the diagonalization carries critical information for reorganizing or analyzing the original system [27, 28]. It is also needed for obtaining the responses and transforming external forces after decoupling.

**Smooth transformation.** This paper makes contribution to the field on four points. First, we derive a general framework for computing the transformation needed for the decoupling of a self-adjoint quadratic system. In particular, we reduce the computational overhead by imposing the additional constraint that  $M(t) \equiv I$  for all  $t$ . Such a model is especially applicable to mass-spring-damper systems where the matrix  $M_0$  represents the individual masses and is diagonal. With proper scaling, we may assume  $M_0 = I$ . For a general self-adjoint pencil with positive definite  $M_0$ , we still can symmetrize the system so that  $M_0 = I$ . Our construction maintains this property throughout the calculation. Second, not only we can argue for convergence of this approach by using the Łojasiewicz gradient inequality, but also we develop a bound on the errors due to the numerical integrator and floating-point arithmetic calculation. Third, in contrast to the phase synchronization and many other available techniques that rely on complete eigeninformation, our method does the decoupling without any a priori knowledge of spectral decomposition. Four, our framework is flexible in that it can perform partial decoupling as desired. It is important to point out some advantages of partial decoupling. For large-scale problems, it might be expensive to perform the total decoupling directly. Instead of uncoupling a given system as is, we can first break down the system into several independent subsystems, i.e., modules. Since there are fewer entries to be annihilated, the computation is less stringent and the convergence might be faster. Once these subsystems are attained, they are of smaller DoFs. If necessary, each subsystem can be further decoupled by the same way which is cheaper and can be processed in parallel. This gradual decomposition process might be termed graded decoupling.

We organize our presentation as follows. We begin in Section 2 with a review of conditions that must be imposed on transformations in order to preserve the Lancaster structure for general quadratic pencils. The structure preservation is essential for keeping the spectrum of the original system invariant throughout the transformations. It turns out that there are three full matrix parameters free at our disposal. We explore the symmetry in Section 3 and simplify the differential systems to only two skew-symmetric matrix parameters. In Section 4, we incorporate the isospectral flow into a gradient

flow to carry out the decoupling process through numerical integration. The framework can be applied to any desirable decomposed form. Due to the floating-point arithmetic, inexactness is inevitable. It is important to derive bounds on the errors Section 5. The global convergence is discussed in 6. Some empirical results are presented in Section 7.

**2. Isospectral flows.** So that the presentation is self-contained, we briefly review the basic ideas of structure preserving isospectral flows. Details can be found in [11, 17].

For convenience, denote the Lancaster pair in (1.8) by  $(\mathcal{A}_0, \mathcal{B}_0)$ , that is,

$$\mathcal{A}_0 = \begin{bmatrix} K_0 & 0 \\ 0 & -M_0 \end{bmatrix}, \quad \mathcal{B}_0 = \begin{bmatrix} C_0 & M_0 \\ M_0 & 0 \end{bmatrix}. \quad (2.1)$$

We are interested in finding two one-parameter transformations  $T_L(t), T_R(t) \in \mathbb{R}^{2n \times 2n}$ , starting with  $T_L(0) = T_R(0) = I_{2n}$ , such that matrices  $\mathcal{A}(t)$  and  $\mathcal{B}(t)$  defined by

$$\mathcal{A}(t) := T_L^\top(t) \mathcal{A}_0 T_R(t), \quad \mathcal{B}(t) := T_L^\top(t) \mathcal{B}_0 T_R(t), \quad (2.2)$$

would maintain Lancaster structure for all  $t$ . Note that, regardless of how  $T_L(t)$  and  $T_R(t)$  are defined,  $(\mathcal{A}(t), \mathcal{B}(t))$  is isospectral to  $(\mathcal{A}_0, \mathcal{B}_0)$  for any  $t$ . So, the real focus should be on preserving the Lancaster structure.

Upon differentiating  $(\mathcal{A}(t), \mathcal{B}(t))$  and collecting the terms, we find the relationship

$$\begin{cases} \dot{\mathcal{A}} &= \dot{T}_L^\top \mathcal{A}_0 T_R + T_L \mathcal{A}_0 \dot{T}_R &= \mathcal{L}^\top \mathcal{A} + \mathcal{A} \mathcal{R}, \\ \dot{\mathcal{B}} &= \dot{T}_L^\top \mathcal{B}_0 T_R + T_L \mathcal{B}_0 \dot{T}_R &= \mathcal{L}^\top \mathcal{B} + \mathcal{B} \mathcal{R}, \end{cases} \quad (2.3)$$

or equivalently,

$$\begin{cases} \dot{T}_L(t) &= T_L(t) \mathcal{L}(t) &= T_L(t) \begin{bmatrix} L_{11}(t) & L_{12}(t) \\ L_{21}(t) & L_{22}(t) \end{bmatrix}, \\ \dot{T}_R(t) &= T_R(t) \mathcal{R}(t) &= T_R(t) \begin{bmatrix} R_{11}(t) & R_{12}(t) \\ R_{21}(t) & R_{22}(t) \end{bmatrix}, \end{cases} \quad (2.4)$$

where matrices  $\mathcal{L}(t)$  and  $\mathcal{R}(t)$  are  $2 \times 2$  blocks of  $n \times n$  matrices to be determined so as to maintain the Lancaster structure for all  $t$ . Because  $(\mathcal{A}(t), \mathcal{B}(t))$  must be of the form

$$\mathcal{A}(t) = \begin{bmatrix} K(t) & 0 \\ 0 & -M(t) \end{bmatrix}, \quad \mathcal{B}(t) = \begin{bmatrix} C(t) & M(t) \\ M(t) & 0 \end{bmatrix}, \quad (2.5)$$

we see upon substitution that

$$\begin{bmatrix} \dot{K} & 0 \\ 0 & -\dot{M} \end{bmatrix} = \begin{bmatrix} L_{11}^\top K + K R_{11} & -L_{21}^\top M + K R_{12} \\ L_{12}^\top K - M R_{21} & -L_{22}^\top M - M R_{22} \end{bmatrix}, \quad (2.6)$$

$$\begin{bmatrix} \dot{C} & \dot{M} \\ \dot{M} & 0 \end{bmatrix} = \begin{bmatrix} L_{11}^\top C + C R_{11} + L_{21}^\top M + M R_{21} & L_{11}^\top M + M R_{22} + C R_{12} \\ L_{12}^\top C + L_{22}^\top M + M R_{11} & L_{12}^\top M + M R_{12} \end{bmatrix}. \quad (2.7)$$

To ensure consistency among the block entries in (2.6) and (2.7), the matrices  $L_{ij}$  and  $R_{ij}$ ,  $i, j = 1, 2$ , must satisfy the following system of equations:

$$\begin{cases} KR_{12} - L_{21}^\top M &= 0, \\ L_{12}^\top K - MR_{21} &= 0, \\ L_{12}^\top M + MR_{12} &= 0, \\ L_{11}^\top M - L_{22}^\top M + CR_{12} &= 0, \\ MR_{11} - MR_{22} + L_{12}^\top C &= 0. \end{cases} \quad (2.8)$$

The necessary conditions (2.8) constitute a homogeneous linear system of  $5n^2$  for the  $8n^2$  entries in the unknown matrices  $L_{ij}$  and  $R_{ij}$ ,  $i, j = 1, 2$ . Its solution space contains  $3n^2$  free parameters which we can identify as three  $n \times n$  matrix parameters. The transformations  $T_L(t)$  and  $T_R(t)$  can now be characterized, once these three free matrix parameters are specified.

To solve (2.8), we find first that the quantity  $R_{12}(t)M^{-1}(t)$  is concealed in every equation, provided the matrix  $M(t)$  is invertible. Upon exploiting this hidden structure, we come up with the first matrix parameter  $D(t) \in \mathbb{R}^{n \times n}$  satisfying the relationship

$$R_{12}(t) = -D(t)M(t).$$

It follows after some algebraic manipulations that the solutions to the system (2.8) can now be identified as follows:

$$\left\{ \begin{array}{l} R_{12} := -DM, \\ R_{21} := DK, \\ L_{12} := D^\top M^\top, \\ L_{21} := -D^\top K^\top, \\ L_{11} - L_{22} = D^\top C^\top, \\ R_{11} - R_{22} = -DC. \end{array} \right. \quad (2.9)$$

It is worth noting in retrospect that the assumption that  $M(t)$  is nonsingular is not needed in the list of definitions (2.9). Anything satisfying (2.9) will automatically satisfy the system (2.8). Note also that implicit in the last two equations in (2.9) are the other two free matrix parameters for defining the diagonal blocks of  $\mathcal{L}(t)$  and  $\mathcal{R}(t)$ . There are infinitely many ways to distribute the diagonal blocks of  $\mathcal{L}(t)$  and  $\mathcal{R}(t)$ . The arrangement

$$\left\{ \begin{array}{l} \mathcal{L} = \begin{bmatrix} D^\top & 0 \\ 0 & D^\top \end{bmatrix} \begin{bmatrix} \frac{C^\top}{2} & M^\top \\ -K^\top & -\frac{C^\top}{2} \end{bmatrix} + \begin{bmatrix} N_L^\top & 0 \\ 0 & N_L^\top \end{bmatrix}, \\ \mathcal{R} = \begin{bmatrix} D & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} -\frac{C}{2} & -M \\ K & \frac{C}{2} \end{bmatrix} + \begin{bmatrix} N_R & 0 \\ 0 & N_R \end{bmatrix}, \end{array} \right. \quad (2.10)$$

where the matrices  $D(t)$ ,  $N_L(t)$  and  $N_R(t)$  are free matrix parameters in  $\mathbb{R}^{n \times n}$ , has been proposed in [11, 17]. By substituting (2.10) into the differential system (2.4), the structure preserving flows  $T_\ell$  and  $T_r$  are defined. After a further substitution into (2.6) and (2.7), an isospectral flow for the triplets  $(M(t), C(t), K(t))$  is given by

$$\left\{ \begin{array}{l} \dot{K} = \frac{1}{2}(CDK - KDC) + N_L K + K N_R, \\ \dot{C} = (MDK - KDM) + N_L C + C N_R, \\ \dot{M} = \frac{1}{2}(MDC - CDM) + N_L M + M N_R. \end{array} \right. \quad (2.11)$$

Note that the differential system (2.11) is autonomous in  $(M, C, K)$  and is linear in the parameters  $(D, N_L, N_R)$ .

**3. Maintaining symmetry.** Suppose that the initial triplets  $(M_0, C_0, K_0)$  have some kinds of inborn symmetries to begin with. We certainly want to keep the same nature throughout the flow  $(M(t), C(t), K(t))$  for all  $t$ . For simplicity, assume that the free parameter matrices  $N_L$  and  $N_R$  are limited to the case  $N_L^\top(t) = N_R(t) = N(t)$  (and, hence, only  $2n^2$  free parameters are left). Then the choice of symmetry for the parameter matrix  $D$  alone can maintain both the Lancaster structure and the various symmetries in the triplets  $(M(t), C(t), K(t))$ . Summarized in Table 3.1 are some possible options.

$D(t)$	$M(t)$	$C(t)$	$K(t)$
skew-symmetric	symmetric	symmetric	symmetric
symmetric	symmetric	skew-symmetric	symmetric
symmetric	skew-symmetric	skew-symmetric	skew-symmetric
skew-symmetric	skew-symmetric	symmetric	skew-symmetric

TABLE 3.1  
Preserving symmetries of  $(M(t), C(t), K(t))$  by  $D(t)$ , if  $N_R(t) = N_L^\top(t)$ .

In this paper, we concern ourselves with self-adjoint quadratic pencils only. For mass-spring-damper systems, symmetry is generally expected between any two connected elements because of Newton's third law. The following theorem has been proved by the author in [10, Theorem 7]. See also a discussion in [15].

**THEOREM 3.1.** *Almost all self-adjoint pencil  $Q(\lambda, M_0, C_0, K_0)$  with simple spectrum can be totally decoupled by congruence transformations.*

By taking  $T_r(t) = T_\ell(t) := T(t)$ , the equation (2.2) becomes

$$\mathcal{A}(t) = T^\top(t)\mathcal{A}_0T(t), \quad \mathcal{B}(t) = T^\top(t)\mathcal{B}_0T(t), \quad (3.1)$$

and  $\mathcal{L}(t) = \mathcal{R}(t)$  in (2.4). In this case, without repeating the details, the conditions for maintaining the Lancaster structure for self-adjoint quadratic pencils are reduced to

$$\begin{cases} KR_{12} - R_{21}^\top M = 0, \\ R_{12}^\top M + MR_{12} = 0, \\ MR_{11} - MR_{22} + R_{12}^\top C = 0. \end{cases} \quad (3.2)$$

The second equation in (3.2) is symmetric, so there are  $n(n+1)/2 + 2n^2$  linear equations for the  $4n^2$  unknowns matrices  $R_{ij}$  in the system (3.2). A total of  $n(n-1)/2 + n^2$  parameters can be chosen arbitrarily to preserve the Lancaster structure in the flow (3.1).

To also preserve the symmetry for the triplets  $(M(t), C(t), K(t))$ , we demand that the first matrix parameter  $D(t)$  be skew-symmetric. Together with the second free parameter matrix  $N(t)$  which is subject to no restriction, we find that an analogy to (2.10) is

$$\mathcal{R} = \begin{bmatrix} D & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} -\frac{C}{2} & -M \\ K & \frac{C}{2} \end{bmatrix} + \begin{bmatrix} N & 0 \\ 0 & N \end{bmatrix}. \quad (3.3)$$

The evolution of the congruence transformations  $T(t)$  is governed by the different equation

$$\dot{T} = \begin{bmatrix} -\frac{1}{2}T_{11}DC + T_{12}DK + T_{11}N & -T_{11}DM + \frac{1}{2}T_{12}DC + T_{12}N \\ -\frac{1}{2}T_{21}DC + T_{22}DK + T_{21}N & -T_{21}DM + \frac{1}{2}T_{22}DC + T_{22}N \end{bmatrix}, \quad (3.4)$$

where  $D^\top = -D$  and  $N_L = N_R^\top$  are arbitrary. The corresponding differential system for the triplets  $(M(t), C(t), K(t))$  with symmetry is identical to (2.11).

We now impose one more constraint. If the equation (1.1) represents the motion of a mass-spring-damper system, then the masses are placed along the diagonal of the matrix  $M_0$ . This diagonal structure is mainly a result of the Newton's second law<sup>1</sup>. Even for a general self-adjoint pencil with positive-definite  $M_0$ , if  $M_0 = L_0L_0^\top$  is the Cholesky decomposition, then we may scale the problem

<sup>1</sup>In an RLC circuit, the circuit elements, R, L and C, are connected according to different layouts. The vibration of the current follows Ohm's law and Kirchhoff's law. The resulting equation might not be symmetric. The leading coefficient matrix  $M_0$  reflects the configuration of the inductors L, which might not be diagonal or even invertible.

from the triplets  $(M_0, C_0, K_0)$  to  $(I, L_0^{-1}C_0L_0^{-T}, L_0^{-1}K_0L_0^{-T})$  and still keep the symmetry. The scaling is reversible. Therefore, it suffices to assume the case  $M_0 = I$ . Many discussions in the literature for applied mechanics begin with such a monic assumption [6].

In our decoupling process, we certainly want to maintain  $M(t) \equiv I$  during our transition. By insisting that  $\dot{M}(t) = 0$ , we find from (2.11) that the parameter matrix  $N$  should be of the form

$$N = \frac{1}{4}(CD - DC) + S \quad (3.5)$$

for some skew-symmetric matrix  $S$ . That is, our parameter matrices are now reduced to two skew-symmetric matrices  $D$  and  $S$ . The total dimension of unknown parameter matrices is  $n(n-1)$  which is a significant reduction from the original  $3n^2$ . The structure preserving isospectral transformation  $T(t)$  for monic self-adjoint quadratic pencil is now governed by the much simpler system

$$\dot{T} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} \frac{1}{4}CD - \frac{3}{4}DC + S & -D \\ DK & \frac{1}{4}(CD + DC) + S \end{bmatrix}. \quad (3.6)$$

In case that we want to see the ultimate reduced form of  $(C_0, K_0)$ , say, the diagonal matrices  $(C_D, K_D)$  in the total decoupling, all we need to do is to apply the transformation  $T(t)$  to the original  $(\mathcal{A}_0, \mathcal{B}_0)$  at the end of integration. Alternatively, we could follow the trajectories of the autonomous differential system

$$\begin{cases} \dot{K} &= \frac{1}{4}(KCD - DCK) + \frac{3}{4}(CDK - KDC) + (KS - SK), \\ \dot{C} &= \frac{1}{4}(CCD - DCC) + (DK - KD) + (CS - SC), \end{cases} \quad (3.7)$$

to the end without referring to the transformation  $T(t)$  at all.

It only remains to choose the control parameters  $D(t)$  and  $S(t)$  to steel the flow  $T(t)$  so that it does decouple  $C(t)$  and  $K(t)$ . Toward this end, we incorporate the structure preserving isospectral  $T(t)$  into a gradient flow in the next section.

**4. Gradient flow.** We begin our discussion with the motivation to totally decouple a given  $n$ -DoF mass-spring-damper system into  $n$  independent 1-DoF single systems. In practice, sometimes it is more sensible that the underlying system is made of separate modules each of which is itself a vibrating system but with lower DoF. There are also situations when the original connectivity is to be modified to a newly specified configuration without alternating the spectral information. That is, instead of total decoupling, we might want to transform the triplets  $(M_0, C_0, K_0)$  into block forms, each block representing one module of a specified size, or into a particular structure resulting from a specifically configured inter-connectivity (say, that depicted in Figure 1.1). We refer to such a goal as partial decoupling. Total decoupling is a special case of partial decoupling. We now describe a general framework to accomplish such a task.

By construction, we know

$$\begin{cases} C(t) &= C(T(t)) = T_{11}^\top(t)C_0T_{11}(t) + T_{21}^\top(t)T_{11}(t) + T_{11}^\top(t)T_{21}(t), \\ K(t) &= K(T(t)) = T_{11}^\top(t)K_0T_{11}(t) - T_{21}^\top(t)T_{21}(t). \end{cases} \quad (4.1)$$

At first glance, it appears that  $C(t)$  and  $K(t)$  depends on  $T_{11}(t)$  and  $T_{21}(t)$  only. By (3.6), however,  $T_{11}(t)$  and  $T_{21}(t)$  do depend on  $T_{12}(t)$  and  $T_{22}(t)$ , which will come into play in our gradient flow. Suppose that certain desired patterns of  $C$  and  $K$  have been specified. In the total decoupling, for instance, the desired pattern is that both  $C$  and  $K$  are diagonal. Let  $\mathcal{P}_C$  and  $\mathcal{P}_K$  denote the projector so that  $\mathcal{P}_C(C)$  and  $\mathcal{P}_K(K)$  represent, respectively, the portions of  $C$  and  $K$  not agreeing with the desired patterns. Introduce the objective function

$$f(T) := \frac{1}{2} \{ \alpha \|\mathcal{P}_C(C(T))\|_F^2 + \beta \|\mathcal{P}_K(K(T))\|_F^2 \}, \quad (4.2)$$

where  $\alpha$  and  $\beta$  are adjustable weights. Our goal is to choose the control parameters  $D(t)$  and  $S(t)$  so that the resulting transformation flow  $T(t)$  minimizes the objective value  $f(T(t))$ .

To achieve our goal, we first learn from the open-loop optimal control problem:

$$\begin{aligned} \min_{\mathbf{x} \in \mathbb{R}^n} \quad & h(\mathbf{x}), \\ \text{subject to} \quad & \dot{\mathbf{x}} = g(\mathbf{x})\mathbf{u}, \quad \mathbf{x}(0) = \mathbf{x}_0, \end{aligned} \quad (4.3)$$

where the objective function  $h : \mathbb{R}^n \rightarrow \mathbb{R}$  is sufficient smooth,  $g : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times p}$  is piecewise continuous with  $\text{rank}(g(\mathbf{x})) = p$ , and  $\mathbf{u} = \mathbf{u}(t) \in \mathbb{R}^p$  is the control. A natural choice for the control  $\mathbf{u}$  is to make the resulting vector  $\dot{\mathbf{x}}$  be as close to  $-\nabla f(\mathbf{x})$  as possible. This amounts to the selection of the least squares solution  $\mathbf{u}$  to the problem

$$\min_{\mathbf{x} \in \mathbb{R}^n} \quad \|g(\mathbf{x})\mathbf{u} + \nabla h(\mathbf{x})\|_2.$$

For convenience, we write explicitly

$$\mathbf{u} := -g(\mathbf{x})^\dagger \nabla h(\mathbf{x}), \quad (4.4)$$

where  $g(\mathbf{x})^\dagger$  stands for the Moore-Penrose generalized inverse of  $g(\mathbf{x})$ . In this way, the closed-loop dynamical system,

$$\dot{\mathbf{x}} = -g(\mathbf{x})g(\mathbf{x})^\dagger \nabla h(\mathbf{x}), \quad (4.5)$$

defines a descent flow  $\mathbf{x}(t)$  for the objective function  $f(\mathbf{x})$ .

The model (4.3) fits our application because the dynamical system (3.6) is linear in the control parameters  $S$  and  $D$ . Furthermore, since  $S$  and  $D$  are skew-symmetric, it suffices to consider only the strictly triangular parts of  $S$  and  $D$ . It need not be formulated explicitly in the code, but let  $Z \in \mathbb{R}^{2n^2 \times n(n-1)}$  denote the linear transformation that does

$$\begin{bmatrix} \text{vec}(S) \\ \text{vec}(D) \end{bmatrix} = Z \begin{bmatrix} \mathbf{s} \\ \mathbf{d} \end{bmatrix}, \quad (4.6)$$

where  $\mathbf{d}$  and  $\mathbf{s}$  are vectors in  $\mathbb{R}^{\frac{n(n-1)}{2}}$  representing the strictly upper triangular parts of  $S$  and  $D$ , respectively. Define

$$g(T) := \begin{bmatrix} I \otimes T_{11} & \frac{1}{4}(I \otimes (T_{11}C)) - \frac{3}{4}(C \otimes T_{11}) + K \otimes T_{12} \\ I \otimes T_{21} & \frac{1}{4}(I \otimes (T_{21}C)) - \frac{3}{4}(C \otimes T_{21}) + K \otimes T_{22} \\ I \otimes T_{12} & \frac{1}{4}(I \otimes (T_{12}C)) + \frac{1}{4}(C \otimes T_{12}) - I \otimes T_{11} \\ I \otimes T_{22} & \frac{1}{4}(I \otimes (T_{22}C)) + \frac{1}{4}(C \otimes T_{22}) - I \otimes T_{21} \end{bmatrix} Z \in \mathbb{R}^{4n^2 \times n(n-1)}. \quad (4.7)$$

Then the controls  $\mathbf{s}$  and  $\mathbf{d}$  can be obtained as the least squares solution to the equation

$$g(T) \begin{bmatrix} \mathbf{s} \\ \mathbf{d} \end{bmatrix} = -\nabla f(T), \quad (4.8)$$

where, because the maps  $\mathcal{P}_C$  and  $\mathcal{P}_K$  are linear, we can easily calculate that  $-\nabla f(T)$  is given by

$$-\nabla f(T) = \begin{bmatrix} \text{vec}(-2\alpha(C_0T_{11} + T_{21})(\mathcal{P}_C(C)) - 2\beta K_0T_{11}(\mathcal{P}_K(K))) \\ \text{vec}(-2\alpha T_{11}(\mathcal{P}_C(C)) + 2\beta T_{21}(\mathcal{P}_K(K))) \\ 0 \\ 0 \end{bmatrix}. \quad (4.9)$$

Once these controls are calculated, we may define the descent flow  $\mathbf{T}(t)$  for the objective function  $f(T)$  by solving the initial value problem (IVP)

$$\dot{T} = -g(T)g(T)^\dagger \nabla f(T), \quad T(0) = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}. \quad (4.10)$$

Alternatively, they can be fed to (3.7) to define the trajectories of  $(C(t), K(t))$ . Note that the vector field in (4.10) is precisely the projection of  $-\nabla f(T)$  onto the range space of  $g(T)$ , whereas in exact arithmetic  $g(T)$  is capable of preserving the Lancaster structure and, hence, the spectrum.

**5. Error analysis.** The Lancaster structure is preserved if and only if the four matrices defined by

$$\begin{cases} F_1(t) &= T_{22}^\top(t)T_{22}(t) - T_{12}^\top(t)K_0T_{12}(t) - I, \\ F_2(t) &= T_{11}^\top(t)K_0T_{12}(t) - T_{21}^\top(t)T_{22}(t), \\ F_3(t) &= T_{11}^\top(t)C_0T_{12}(t) + T_{21}^\top(t)T_{12}(t) + T_{11}^\top(t)T_{22} - I, \\ F_4(t) &= T_{12}^\top(t)C_0T_{12}(t) + T_{22}^\top(t)T_{12} + T_{12}^\top(t)T_{22}(t), \end{cases} \quad (5.1)$$

are identically zero for all  $t$ . Because the differential system (4.10) is to be integrated numerically, the computed solution, denoted by  $\tilde{T}(t)$ , is not exact. Inevitably the condition imposed by (5.1) will not be satisfied. Once the Lancaster structure is lost, so is the spectrum. In this section, we analyze the errors and show that there is an upper bound.

Suppose

$$\tilde{T}(t) = T(t) + E(t), \quad (5.2)$$

where  $E(t)$  represents the global error of  $T(t)$  due to the numerical integration. Using  $\tilde{T}$  instead of  $T$  in (3.1), we obtained the pencil  $(\tilde{A}(t), \tilde{B}(t))$  with

$$\tilde{A}(t) := \tilde{T}^\top \mathcal{A}_0 \tilde{T} = \mathcal{A}(t) + \mathcal{E}(\mathcal{A}(t)), \quad \tilde{B}(t) := \tilde{T}^\top \mathcal{B}_0 \tilde{T} = \mathcal{B}(t) + \mathcal{E}(\mathcal{B}(t)), \quad (5.3)$$

where  $\mathcal{E}(\mathcal{A}(t))$  and  $\mathcal{E}(\mathcal{B}(t))$  represent the deviation from the true Lancaster structure. It can easily be checked that

$$\mathcal{E}(\mathcal{A}(t)) = O(\|E(t)\|), \quad \mathcal{E}(\mathcal{B}(t)) = O(\|E(t)\|).$$

We claim that the loss of isospectrality is of the same order  $O(\|E(t)\|)$  of the integration error.

Given a matrix pencil  $W = (A, B) \in \mathbb{C}^{m \times m} \times \mathbb{C}^{m \times m}$  let  $(\lambda_i, \omega_i)$ ,  $i = 1, 2, \dots, m$ , denote an generalized eigenpair in the sense that

$$A\omega_i = \lambda_i B\omega_i. \quad (5.4)$$

It is well known in the literature that if  $X := [\omega_1, \dots, \omega_m]$  denotes the matrix of eigenvectors, then there exists a nonsingular matrix  $Y$  such that [18]

$$YAX = \text{diag}(\lambda_i), \quad YBX = I. \quad (5.5)$$

It is in this sense that this pencil  $W = (A, B)$  is said to be diagonalizable [13]. Suppose  $\tilde{W}$  is a perturbed matrix pencil of  $W$  with spectrum  $\{\tilde{\lambda}_i\}$ . We measure the distance between two eigenvalues by the so called chordal metric<sup>2</sup> [13]

$$\rho(\lambda_i, \tilde{\lambda}_i) := \frac{|\lambda_i - \tilde{\lambda}_i|}{\sqrt{|\lambda_i|^2 + 1} \sqrt{|\tilde{\lambda}_i|^2 + 1}}. \quad (5.6)$$

<sup>2</sup>This metric is meant to be more general than the one we used here. Basically, to take in account degenerate pencils, (5.4) is expressed as  $a_i A \omega_i = b_i B \omega_i$  and the pair  $(a_i, b_i)$  is called an eigenvalue. The chordal metric ensures that the distance in the projective space (of  $(a_i, b_i)$ ) is the same regardless of the representatives selected.

Since complex numbers cannot be ordered, we measure the sensitivity of the eigenvalues of  $W$  subject to perturbation by taking into account the worst change of eigenvalues. That is, we define the *generalized spectral variation* of  $\widetilde{W}$  respect to  $W$  by [13]

$$S_W(\widetilde{W}) := \max_i \min_j \rho(\lambda_j, \widetilde{\lambda}_i). \quad (5.7)$$

The following result is a generalization of the well known Bauer-Fike theorem to the generalized eigenvalue problems [13, Theorem 2.1].

LEMMA 5.1. *Suppose that  $W = (A, B)$  is diagonalizable by the matrices  $X, Y$  in the sense of (5.5). Let  $\widetilde{W}$  be a regular pencil. Then*

$$S_W(\widetilde{W}) \leq \|X^{-1}\|_2 \|X\|_2 \|P_W - P_{\widetilde{W}}\|_2, \quad (5.8)$$

where  $P_W = WW^\dagger$  and  $P_{\widetilde{W}} = \widetilde{W}\widetilde{W}^\dagger$  are the orthogonal projectors of  $W$  and  $\widetilde{W}$ , respectively.

To put Lemma 5.1 in good use, we also need to measure the difference between the orthogonal projectors, which can be estimated as follows [8, 33, 34].

LEMMA 5.2. *Let  $W$  and  $\widetilde{W}$  be two matrix pencils. Then*

$$\|P_{\widetilde{W}} - P_W\|_2 \leq \max\{\|W^\dagger\|_2, \|\widetilde{W}^\dagger\|_2\} \|\mathcal{E}(W)\|_2, \quad (5.9)$$

where  $\mathcal{E}(W) := \widetilde{W} - W$ .

Lemma 5.2 is still not fine enough because it involves the perturbation of the generalized inverse. If the perturbation does not alter the rank, then we have the following classical result [19, Page 650].

LEMMA 5.3. *Suppose that  $\widetilde{W} = W + \mathcal{E}(W)$  and  $\text{rank}(W) = \text{rank}(\widetilde{W})$ . Then*

$$\|\widetilde{W}^\dagger\|_2 = \|W^\dagger\|_2 + O(\|\mathcal{E}(W)\|_2). \quad (5.10)$$

For our application, the Lancaster structure (2.5) that is supposed to be inherited in the pencil  $W(t) = (A(t), B(t))$  is lost due to the global error accumulated in the numerical integration of (4.10). Instead, what we have in hand is the perturbed pencil  $\widetilde{W} = (\widetilde{A}(t), \widetilde{B}(t))$  which is estimated at (5.3). The deviation from the desired isospectrality is summarized in the following theorem.

THEOREM 5.4. *Suppose  $\widetilde{T}(t)$  represents the transformation obtained by a numerical integrator applied to the differential system (4.10). Assume that  $\widetilde{T}(t) = T(t) + E(t)$ . Suppose also that the Lancaster pair  $(\mathcal{A}(t), \mathcal{B}(t))$  is diagonalizable with eigenvectors  $X(t) := [\omega_1(t), \dots, \omega_n(t)]$ . Then the generalized spectral variation is of the order*

$$S_{W(t)}(\widetilde{W}(t)) \leq O(\|E(t)\|_2). \quad (5.11)$$

*Proof.* By substituting (5.10) and (5.9) into (5.8), we see that

$$S_{W(t)}(\widetilde{W}(t)) \leq \|X(t)^{-1}\|_2 \|X(t)\|_2 (\|W^\dagger\|_2 + O(\|\mathcal{E}(W)\|_2)) \|\mathcal{E}(t)\|_2 = O(\|E(t)\|_2). \quad (5.12)$$

But we know from (5.3),  $O(\|\mathcal{E}(t)\|_2) = O(\|E(t)\|_2)$ .  $\square$

Theorem 5.4 is significant in that if there exists a perturbation in Lancaster structure, then so is there a deviation from the original spectrum. But the deviation will not drift too far away. The theorem asserts that the error of eigenvalue has an upper bound of the size indicated in (5.11). It might be of interest to develop a way, similar to the penalty method or the regularization method used in optimization, to reduce the drift of  $T(t)$ , but that will be another research topic in the future.

For our numerical experiments, we use the ODE solvers available in `Matlab` [32]. These methods indirectly control the global error by attempting to bound the local error at each step. More specifically, the methods adjust step sizes by estimating the local error  $e_i$  for each entry  $y_i$  in the solution vector and choosing step sizes to ascertain that the criterion

$$|e_i| \leq \max\{|y_i|\text{RelTol}, \text{AbsTol}\}, \quad (5.13)$$

where `RelTol` and `AbsTol` are user-specified local error tolerance, is satisfied. This technique has proved to be efficient in practice, but there is no rigorous theory characterizing the relationship between the final accuracy and the prescribed tolerance. Applied to our decoupling problems, our experiences suggest that the integration is fast and that the global error is approximately about 1 to 3 orders higher than the prescribed `RelTol` and `AbsTol`. We shall demonstrate numerically that the magnitude of the deviation of the spectrum is of the same order as the loss of Lancaster structure.

**6. Global convergence.** Consider the gradient flow

$$\dot{\mathbf{x}} = -\nabla\xi(\mathbf{x}(t)) \quad (6.1)$$

for a general differentiable objective function  $\xi : \mathbb{R}^n \rightarrow \mathbb{R}$ . It is clear that  $\xi(\mathbf{x}(t))$  is a decreasing function in  $t$ . It is further known that if the semi-orbit  $\mathbf{x}(t)$  is bounded for  $t \geq 0$ , then the set of accumulation points

$$\omega(\mathbf{x}(0)) := \{\mathbf{x}^* \in \mathbb{R}^n | \mathbf{x}(t_\nu) \rightarrow \mathbf{x}^* \text{ for some sequence } t_\nu \rightarrow \infty\} \quad (6.2)$$

is a nonempty, compact, and connected subset of stationary points

$$C := \{\mathbf{x} \in \mathbb{R}^n | \nabla\xi(\mathbf{x}) = 0\}. \quad (6.3)$$

So, if  $\xi(\mathbf{x})$  is bounded below,  $\xi(\mathbf{x}(t))$  must converge to a local minimum. For general objective function  $\xi(\mathbf{x})$ , however, there is no guarantee about the limiting behavior of the flow  $\mathbf{x}(t)$  itself. On the other hand, if  $\xi(\mathbf{x})$  is analytic, then the Łojasiewicz gradient inequality [9, 26] can be used to argue that the semi-orbit  $\mathbf{x}(t)$  of an analytic gradient flow must be of finite length. It follows that the set  $\omega(\mathbf{x}(0))$  of any analytic gradient flow  $\mathbf{x}(t)$  is necessarily a singleton [1, Theorem 2.2].

**THEOREM 6.1.** *Suppose that  $\xi : U \rightarrow \mathbb{R}$  is real analytic in an open set  $U \subset \mathbb{R}^n$ . Then for any bounded semi-orbit of (6.1), there exists a point  $\mathbf{x}^* \in S$  such that  $\mathbf{x}(t) \rightarrow \mathbf{x}^*$  as  $t \rightarrow \infty$ .*

For our application, the vector field in (4.10) is a polynomial system which certainly is analytic. Thus we know by Theorem 6.1 that the flow  $T(t)$  must converge to a single point  $T^*$  which is necessarily a local minimizer of  $f(T)$ . Being nonlinear in  $T$ , the objective function (4.2) might have many local minima. If  $f(T^*)$  is nearly zero up to the prescribed tolerance, then we may proclaim that the system is decoupled. However, there are several possible scenarios causing  $f(T^*)$  to be far away from zero.

1. The flow  $T(t)$  is trapped in the basin of attraction of a local minimum.
2. The flow  $T(t)$  converges to a point  $\hat{T}$  at which  $g^\dagger(\hat{T})\nabla f(\hat{T}) = 0$ , but  $\nabla f(\hat{T}) \neq 0$ . That is,  $\hat{T}$  is not a local minimum, but the flow stagnates because  $\nabla f(\hat{T})$  is perpendicular to the range space of  $g(\hat{T})$ .
3. The flow  $T(t)$  as it is cannot connect the initial point  $T(0)$  to the transformation that decouples the original pencil.

There are several possible tactics to remedy the non-desirable convergence.

- We might monitor the evolution by modifying the weights  $\alpha$  and  $\beta$  adaptively with the hope of navigating the  $(\alpha, \beta)$ -dependent flow  $T(t)$  into a more desired region.
- We need not follow the rule (4.8) for all  $t$ . We might modify the controls  $\mathbf{s}$  and  $\mathbf{d}$  intermittently to force changes of directions for the flow  $T(t)$ . See Example 3 in the next section for a mimic of the simulated annealing scheme.

- We might modify  $(C_0, K_0)$  to an different but equivalent pair.

So as to focus on the main ideas of this discussion, we have not implemented all the above-mentioned tactics in our computer codes. Our primitive testing in the next section seems to support that our approach is very promising.

Finally, we mention that our discussion thus far is only about the forward problem, namely, decoupling a given quadratic system into subsystems. It is sensible to ask the inverse problem, that is, building a fully structured system from prescribed subsystems. The so called *quadratic inverse eigenvalue problem* of constructing a quadratic model with a prescribed spectrum and a desired connectivity structure is one such an application [12]. Our framework can be exploited to answer this question. See Example 4 in Section 7.

**7. Numerical experiments.** In this section we report four empirical results with the transformation flow  $T(t)$  by numerically integrating the IVP (4.10). Each experiment demonstrates one particular setting. Our primary concern at the moment is not so much on the efficiency of this method. Rather, our goal is to demonstrate that the flow approach can indeed be employed to find structure preserving isospectral transformation for the purpose of decoupling. Understandably, to make this method computationally more effective requires additional ruminations, such as employing specially designed geometric integrators or streamlining memory management, which are not investigated in this paper. Also, to make this method universally robust in dealing with undesirable limit points, tactics such as those mentioned in the preceding section should be implemented. At present, we carry out the integration by standard ODE solvers available in `Matlab`. In Example 3 we take advantage of the event capability in the ODE suite to detect the existence of undesirable limit point. In all our experiments, we set the local tolerances `AbsTol` and `RelTol` at  $10^{-10}$ . For the ease of displaying the data in the running text, we report numerical results on small-scale problems and show 4 significant digits only.

**Example 1.** Consider the total decoupling of a 4-DoF system depicted in Figure 1.1 with  $M_0 = I$  and

$$C_0 = \begin{bmatrix} 0.4108 & 0.0000 & -0.3529 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ -0.3529 & 0.0000 & 2.1661 & -1.8132 \\ 0.0000 & 0.0000 & -1.8132 & 1.8132 \end{bmatrix}, \quad K_0 = \begin{bmatrix} 22.3480 & -9.3547 & -8.9365 & 0.0000 \\ -9.3547 & 18.5240 & -9.1690 & 0.0000 \\ -8.9365 & -9.1690 & 22.2080 & -4.1027 \\ 0.0000 & 0.0000 & -4.1027 & 4.1027 \end{bmatrix}.$$

Upon integrating (4.10) to  $t \approx 400$ , we obtain a near-convergence transformation

$$T = \begin{bmatrix} 0.7059 & -0.4465 & -0.4965 & -1.1729 & | & 0.0168 & 0.0426 & 0.7025 & -0.5292 \\ -0.7082 & -0.3461 & -0.4115 & -0.2544 & | & 0.0139 & 0.0092 & -0.7111 & -0.3641 \\ 0.1405 & 0.8502 & 0.5264 & -0.6789 & | & -0.0178 & 0.0246 & 0.1442 & 0.8023 \\ -0.0417 & -0.1065 & 0.2927 & 1.8316 & | & -0.0099 & -0.0665 & -0.0396 & 0.0228 \\ \hline 0.3599 & 0.4143 & -0.0606 & -0.0111 & | & 0.0099 & 0.0132 & 0.3380 & 0.4138 \\ 0.4131 & 0.4772 & -0.0918 & -0.0127 & | & 0.0151 & 0.0151 & 0.3799 & 0.4766 \\ 0.2150 & 0.4975 & 0.4935 & -0.0116 & | & -0.0810 & 0.0137 & 0.3934 & 0.4970 \\ -0.9177 & 0.6033 & 0.1258 & 0.0303 & | & -0.0207 & -0.0359 & -0.8722 & 0.6046 \end{bmatrix}.$$

which transforms the damping matrix and stiffness matrix into

$$C = \begin{bmatrix} 0.2073 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 2.2029 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 1.9436 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0363 \end{bmatrix}, \quad K = \begin{bmatrix} 29.5482 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 6.0921 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 27.5510 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.8435 \end{bmatrix},$$

representing four 1-DoF subsystems.

To demonstrate the effect of the flow approach, we plot in Figure 7.1 the evolution history of the off-diagonal parts of  $C$  and  $K$  which are successfully annihilated. Also plotted on the left side in Figure 7.2 is the history of deviation from the Lancaster structure because  $T(t)$  is obtained by numerical integration. Recall that our local tolerance is set at  $10^{-10}$ , the global error at around  $10^{-8}$

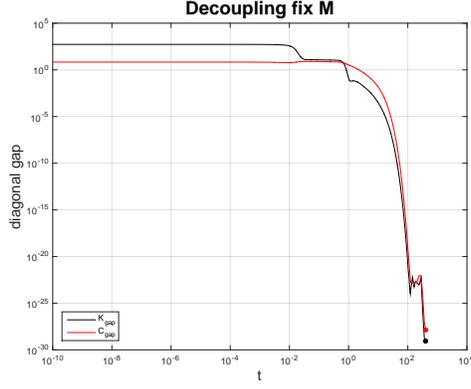


FIG. 7.1. The evolution of off-diagonal portions of matrices  $C$  and  $K$  for Example 1.

should be deemed acceptable. We also calculate the spectra of the triplets  $(I, C(t), K(t))$  and compare them with the true spectrum of  $(I, C_0, K_0)$ . The loss of isospectrality is plotted in the graph on the right side of Figure 7.2. Our theory asserts that the loss of isospectrality should be of the same order as the deviation from the Lancaster structure. The two graphs in Figure 7.2 side by side confirm the theory.

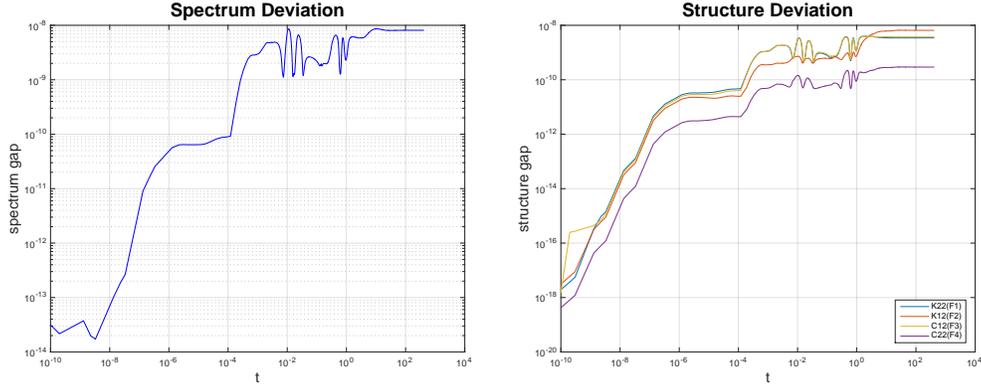


FIG. 7.2. The deviations of spectrum and Lancaster structure for Example 1.

**Example 2.** For a serially linked mass-spring-damper system, the coefficient matrices are necessarily tridiagonal. We artificially generate a 4-DoF test data

$$C_0 = \begin{bmatrix} 0.1000 & -0.1000 & 0.0000 & 0.0000 \\ -0.1000 & 0.2000 & -0.1000 & 0.0000 \\ 0.0000 & -0.1000 & 0.2000 & -0.1000 \\ 0.0000 & 0.0000 & -0.1000 & 1.3500 \end{bmatrix}, \quad K_0 = \begin{bmatrix} 1.0000 & -1.0000 & 0.0000 & 0.0000 \\ -1.0000 & 2.0000 & -1.0000 & 0.0000 \\ 0.0000 & -1.0000 & 2.0000 & -1.0000 \\ 0.0000 & 0.0000 & -1.0000 & 1.1000 \end{bmatrix}.$$

Note the first three elements have identical physical parameters. It will be interesting to see if similar elements linked together can be separated. We find a decoupled system

$$C = \begin{bmatrix} 0.4123 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.3865 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.3966 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.6547 \end{bmatrix}, \quad K = \begin{bmatrix} 0.0374 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 3.3190 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 1.7488 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.4613 \end{bmatrix}$$

via the transformation

$$T = \left[ \begin{array}{cccc|cccc} 0.6974 & -0.5321 & -0.0198 & -0.1232 & 0.5297 & 0.0704 & 0.4790 & -0.5600 \\ 0.6787 & 0.4475 & -0.0114 & -0.1250 & 0.3050 & 0.0714 & 0.5529 & 0.4191 \\ 0.6384 & 0.6551 & 0.0062 & 0.2629 & -0.1671 & -0.1503 & 0.7073 & 0.7147 \\ 0.5706 & -0.3685 & 0.0349 & 0.4640 & -0.9339 & -0.2653 & 0.9556 & -0.2633 \\ \hline -0.2971 & -0.3824 & -0.0677 & -0.2267 & 0.0204 & 0.4914 & -0.3050 & -0.7041 \\ 0.6980 & -0.0477 & 0.1027 & -0.3258 & -0.0310 & 0.7063 & 0.7100 & -0.5101 \\ -0.6419 & 0.5215 & 0.0597 & -0.4174 & -0.0180 & 0.9048 & -0.6350 & -0.0709 \\ 0.1571 & 1.1050 & -0.2983 & -0.3251 & 0.0899 & 0.7049 & 0.1224 & 0.6436 \end{array} \right].$$

**Example 3.** In this example, we experiment with partial decoupling. We also demonstrate how the event of undesirable limit point can be handled. Suppose that we wish to transform the following randomly generated system in  $\mathbb{R}^5$ , which does not even represent a mechanical system, into 4 subsystems in  $\mathbb{R}^2 \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ .

$$C_0 = \begin{bmatrix} 2.4692 & -1.2722 & -0.5179 & -0.6703 & 2.4752 \\ -1.2722 & 3.7048 & 0.3883 & 1.7535 & -2.0158 \\ -0.5179 & 0.3883 & 2.3288 & -0.2062 & -0.1141 \\ -0.6703 & 1.7535 & -0.2062 & 8.7696 & 0.2896 \\ 2.4752 & -2.0158 & -0.1141 & 0.2896 & 2.9922 \end{bmatrix}, K_0 = \begin{bmatrix} 5.9325 & -3.4215 & 1.9520 & -0.6521 & 0.4059 \\ -3.4215 & 6.3475 & 1.1815 & 4.2934 & -2.0081 \\ 1.9520 & 1.1815 & 5.0092 & 2.3673 & 1.4432 \\ -0.6521 & 4.2934 & 2.3673 & 5.7336 & -0.7962 \\ 0.4059 & -2.0081 & 1.4432 & -0.7962 & 4.4014 \end{bmatrix}.$$

Our framework works by defining the projectors  $\mathcal{P}_C$  and  $\mathcal{P}_K$  accordingly in (4.2) and finds the partially decoupled isospectral subsystems

$$C = \begin{bmatrix} 4.5820 & -2.5243 & 0.0000 & 0.0000 & 0.0000 \\ -2.5243 & 4.3581 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 2.3999 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 8.7908 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.1340 \end{bmatrix}, K = \begin{bmatrix} 3.3456 & -4.1547 & 0.0000 & 0.0000 & 0.0000 \\ -4.1547 & 6.3737 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 6.7071 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 2.3894 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 5.0578 \end{bmatrix}.$$

via the transformation

$$T = \left[ \begin{array}{ccccc|ccccc} 0.1977 & 0.1447 & -0.6904 & -0.2161 & 0.1960 & -0.1737 & -0.0475 & -0.0402 & -0.4077 & -0.1208 \\ 0.0866 & 0.2271 & 0.1652 & 0.0617 & 0.3941 & -0.2952 & -0.0125 & -0.0127 & 0.6491 & 0.3148 \\ -0.3908 & 0.8533 & -0.0981 & 0.0921 & -0.1433 & 0.0146 & -0.0192 & -0.0417 & 0.1308 & -0.0596 \\ 0.3803 & 0.1230 & -0.1036 & -0.9533 & 0.0417 & 0.2955 & 0.0521 & 0.0069 & -0.3373 & 1.0046 \\ 0.2275 & 0.4381 & 0.7174 & 0.0890 & 0.1376 & -0.0446 & 0.0576 & -0.0555 & -0.5754 & -0.0554 \\ \hline -0.0653 & -1.0116 & 0.0862 & 0.3186 & 0.2035 & 0.0313 & -0.1020 & 0.5899 & 0.2587 & -0.6850 \\ 0.9094 & -1.4765 & 0.1655 & 0.0841 & 0.0643 & 0.0063 & -0.2051 & 0.7188 & 0.2572 & 0.1669 \\ 0.0466 & 0.5918 & -0.0040 & 0.1291 & 0.2109 & -0.0107 & 0.0746 & -0.4298 & 0.8995 & -0.0926 \\ 0.0707 & 0.8149 & -0.2051 & -0.3497 & -0.0347 & 0.0910 & -0.0217 & -0.1810 & -0.0022 & -0.1045 \\ -0.5957 & -0.6809 & 0.0285 & -0.3865 & 0.2806 & -0.0077 & -0.0716 & 0.3167 & 0.2998 & 0.7248 \end{array} \right].$$

If further decoupling is desired, we only need to work on the upper left  $2 \times 2$  blocks of  $C$  and  $K$ , i.e.,

$$\widehat{C}_0 = \begin{bmatrix} 4.5820 & -2.5243 \\ -2.5243 & 4.3581 \end{bmatrix}, \widehat{K}_0 = \begin{bmatrix} 3.3456 & -4.1547 \\ -4.1547 & 6.3737 \end{bmatrix},$$

which at present stands as a coupled subsystem. Surprisingly, our gradient flow approach by using the control strategy (4.8) alone fails to break down such a simple subsystem. What has happened is that at a certain point of the integration the product  $g^\top(T(t))\nabla f(T(t))$  becomes nearly zero, which causes the controls  $\mathbf{s}$  and  $\mathbf{d}$  defined via (4.8) also becomes nearly zero. Consequently, the vector field  $\dot{T}$  defined by (4.10) is nearly zero and the flow  $T(t)$  stagnates.

To fix such an undesirable phenomenon, we build in our code a mechanism to automatically detect the "event" when  $\|\nabla f(T(t))\|$  is greater than a specified lower bound and  $\|g^\top(T(t))\nabla f(T(t))\|$  is less

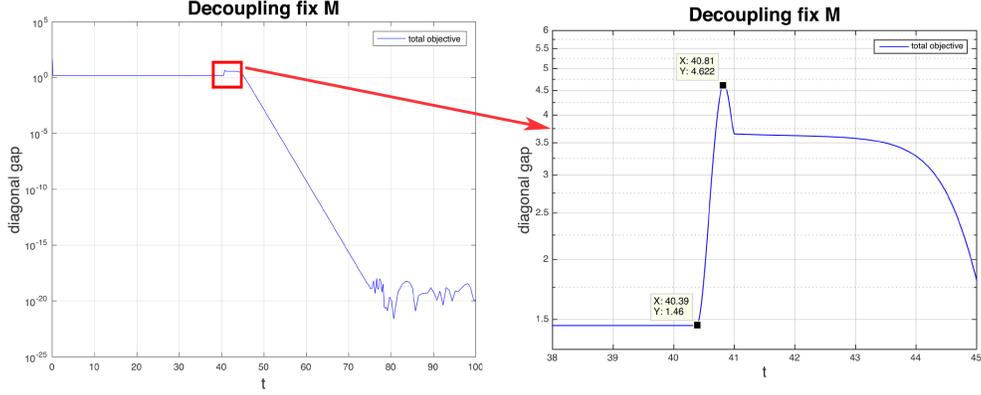


FIG. 7.3. The objective function  $f_u(T)$ .

than a specified upper bound. When such an event occurs, we momentarily modify the vector field by providing an excitement. The idea is to mimic the simulated annealing where the descent flow is excited to become an ascent flow briefly and, hence, jumps out of the basin of the local minimum. The resulting dynamics of such a control strategy is demonstrated in the left graph of Figure 7.3. At approximately  $t = 40.39$  the event occurs. We zoom in the behavior near this point in the right graph of Figure 7.3. We exert the excitement for a brief moment, say, until  $t = 40.81$ . During that interval, the objective value is increased. We then return to the ordinary descent flow (4.10). We are able to decouple the subsystem into

$$C_u = \begin{bmatrix} 3.3593 & 0.0000 \\ 0.0000 & 5.5808 \end{bmatrix}, \quad K_u = \begin{bmatrix} 3.3591 & 0.0000 \\ 0.0000 & 1.2095 \end{bmatrix},$$

with the corresponding transformation

$$T_u = \begin{bmatrix} -0.8724 & 2.1208 & -0.6314 & 1.2485 \\ 0.5095 & 0.1933 & -0.0576 & 0.7028 \\ -1.4553 & 0.1225 & -0.1012 & -0.8903 \\ -1.0175 & 0.3947 & -0.3263 & 0.8037 \end{bmatrix}.$$

**Example 4.** In this example, we experiment with the ability of our flow approach for the inverse eigenvalue problem. We first generate a random self-adjoint quadratic system

$$C_0 = \begin{bmatrix} 3.8604 & -1.0560 & -0.1034 & 1.1005 \\ -1.0560 & 2.4316 & 0.1350 & 1.4677 \\ -0.1034 & 0.1350 & 0.9208 & -0.9336 \\ 1.1005 & 1.4677 & -0.9336 & 3.0482 \end{bmatrix}, \quad K_0 = \begin{bmatrix} 3.3899 & 2.2202 & 5.7810 & 1.0527 \\ 1.0527 & 2.8559 & 4.1710 & -0.7054 \\ 5.7810 & 5.7810 & 12.8220 & 3.4286 \\ 3.4286 & 0.34286 & 3.4286 & 3.4286 \end{bmatrix},$$

to obtain their spectrum

$$\{-0.0274, -0.1492, -0.1812 \pm 1.8714i, -0.6176 \pm 4.0761i, -0.61761 + 4.0761i, -4.1340, -4.3528\}.$$

We specify the desired structure such as that in (1.2). Upon defining the projectors  $\mathcal{P}_C$  and  $\mathcal{P}_K$  properly and following the gradient flow (4.10), we find

$$C = \begin{bmatrix} 4.6134 & 0.0000 & 0.0948 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0948 & 0.0000 & 1.8592 & -1.2274 \\ 0.0000 & 0.0000 & -1.2274 & 3.7885 \end{bmatrix}, \quad K = \begin{bmatrix} 1.7895 & 2.2868 & 2.0202 & 0.0000 \\ 2.2868 & 4.6182 & 4.3250 & 0.0000 \\ 2.0202 & 4.3250 & 1.4366 & 4.3274 \\ 0.0000 & 0.0000 & 4.3274 & 1.9609 \end{bmatrix}.$$

with the corresponding transformation

$$T = \left[ \begin{array}{cccc|cccc} 1.1571 & 0.1970 & 0.1171 & -0.4775 & 0.0450 & 0.0656 & 0.9433 & 0.0905 \\ 0.0730 & -0.0433 & -0.3096 & -0.8089 & 0.0995 & 0.0357 & -0.3895 & -0.1241 \\ -0.3293 & 1.0243 & 0.1064 & 0.4262 & -0.0444 & -0.0103 & -0.1232 & 0.9978 \\ -0.2562 & -0.0111 & 0.1033 & -0.0424 & -0.0856 & 0.0109 & 0.1375 & -0.0213 \\ \hline 0.4031 & 0.0846 & 0.2801 & -0.3150 & -0.1443 & 0.0160 & 0.4031 & 0.1046 \\ 0.7965 & 0.4612 & -0.5020 & -0.1466 & 0.0260 & -0.0040 & 0.7965 & 0.5202 \\ 0.0197 & -0.0825 & 0.1584 & 0.1248 & -0.0026 & -0.0407 & 0.0197 & 0.0591 \\ -0.5085 & 0.8471 & 0.0924 & -0.0499 & 0.0122 & 0.0015 & -0.5085 & 0.8546 \end{array} \right].$$

It is worth noting that while we do find a quadratic system which carries the prescribed spectrum and satisfies the desired structure in mathematics, the constructed system  $(C, K)$  does not necessarily meet the physical feasibility, e.g., the hereditary property of zero row sums for internally linked masses. Solving a mathematical problem is the first step. To meet physical feasibilities requires additional improvement of our framework.

**8. Conclusion.** In this paper we propose a framework for computing structure preserving isospectral transformations for the decoupling of a monic self-adjoint quadratic pencil. The approach is an echo to the general but interesting question raised by V.I. Arnol'd in the seminal book [3]:

What is the simplest form to which a family of matrices depending smoothly on the parameters can be reduced by a change of coordinates depending smoothly on the parameters?

We explain conditions for preserving the Lancaster structure throughout the smooth change of coordinates, which thus preserves the spectrum, and incorporate such a flow into a gradient flow to decouple the quadratic pencils. The framework is capable of handling both total and partial decoupling. A theory on the bounds of error is established. Fine tuning of the framework as well as the codes is yet to be further studied. However, preliminary experiments seem to evidence the working of the framework.

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