The Stability Group of Symmetric Toeplitz Matrices

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Abstract

In regard to the linear subspace $\mathcal{T}(n)$ of $n \times n$ symmetric Toeplitz matrices over the real field, the collection $\mathcal{S}(n)$ of all real and orthogonal matrices Q such that $QTQ^T \in \mathcal{T}(n)$ whenever $T \in \mathcal{T}(n)$ forms a group, called the stability group of $\mathcal{T}(n)$. This paper shows that $\mathcal{S}(n)$ is finite. In fact, $\mathcal{S}(n)$ has exactly eight elements regardless of the dimension n. Group elements in $\mathcal{S}(n)$ are completely characterized.

1. Introduction.

In all discussion that follows, we shall consider only matrices over the real field.

An $n \times n$ matrix $T = (t_{ij})$ is symmetric and Toeplitz if there exist real scalars $r_1, \ldots, r_n$ such that

$$t_{ij} = r_{|i-j+1|}$$

for all i and j. Clearly a symmetric Toeplitz matrix is uniquely determined by the entries of its first column. Thus a symmetric Toeplitz matrix will be denoted by $T(r)$ if its first column is the vector $r \in R^n$. It is also clear that the linear subspace $\mathcal{T}(n)$ of all symmetric Toeplitz matrices is spanned by the matrices

$$E^{(i)} := T(e^{(i)}) \tag{1}$$

where $e^{(i)}$, $i = 1, \ldots, n$, is the i^{th} standard basis in R^n. With respect to the Frobenius norm, elements in the basis $\{E^{(i)}\}$ are mutually orthogonal.

Due to their role in important applications like the trigonometric moment problem, the Szegö theory and the signal processing, many properties of Toeplitz matrices have been studied over the years. See, for example, [1, 4, 5, 8] and the many references contained therein.

One interesting problem that remains unsolved is the so called inverse Toeplitz eigenvalue problem where a vector r is to be found so that $T(r)$ has a prescribed set of eigenvalues. Equally difficult is the problem of identifying an orthogonal matrix so that its columns are eigenvectors of a certain Toeplitz matrix. It is toward the second problem that we find ourselves faced the notion of stability subgroup of $\mathcal{T}(n)$ in the orthogonal group $\mathcal{O}(n)$.

The stability group $\mathcal{S}(n)$ of $\mathcal{T}(n)$ is defined to be

$$\mathcal{S}(n) := \{Q \in \mathcal{O}(n) |\, QTQ^T \in \mathcal{T}(n) \text{ if } T \in \mathcal{T}(n)\}. \tag{2}$$

The name "group" is well justified because for each fixed $Q \in \mathcal{S}(n)$ the action QTQ^T is a bijection from $\mathcal{T}(n)$ to itself. Definition (2) is a generalization of the usual stability subgroup defined at a single point in a topological space [7, p74].

Obviously, the identity matrix I and the backward identity matrix J where

$$J := \begin{bmatrix} 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & & 1 & 0 \\ \vdots & & & & \vdots \\ 0 & & & & 0 \\ 1 & 0 & \ldots & 0 & 0 \end{bmatrix} \tag{3}$$

are in $\mathcal{S}(n)$. In this note, we show that $\mathcal{S}(n)$ has exactly eight elements regardless of the dimension n. We characterize all elements in $\mathcal{S}(n)$ explicitly.

2. Main Result.

For each fixed $Q \in \mathcal{O}(n)$, the action $Q^T T Q$ is linear on $\mathcal{T}(n)$. So it suffices to consider the actions of Q on the basis $\{E^{(1)}, \ldots, E^{(n)}\}$.

For $k = 1, \ldots, n$, let

$$\tilde{E}^{(k)} := Q E^{(k)} Q^T. \tag{4}$$

Obviously $\tilde{E}^{(k)}$ are still mutually orthogonal with respect to the Frobenius norm. Since $\tilde{E}^{(1)} = E^{(1)}$, we see that if $Q \in \mathcal{S}(n)$ then the diagonal elements of $\tilde{E}^{(k)}$ must be identically zero for all $k \geq 2$. This observation alone suffices to show that $\mathcal{S}(n)$ is finite. The reason is given below.

Lemma 2.1. *The vector $x = [x_1, \ldots, x_n]^T$ solves the system of polynomials*

$$\sum_{j=1}^{n} x_j^2 = 1 \tag{5}$$

$$\sum_{j=1}^{n-k+1} x_j x_{j+k-1} = 0, \quad \text{for } k = 2, \ldots n \tag{6}$$

if and only if $x = \pm e^{(j)}$ for $j = 1, \ldots n$.

Proof. The lemma can be proved easily by induction. The last equation (i.e., when $k = n$) implies that $x_1 = 0$ or $x_n = 0$. In either case we can apply the induction hypothesis to the reduced equations.

The result also follows from a tedious but straightforward computation by using the theory of Gröbner bases [6]. $\square$

Denote the matrix Q by columns $Q := [q^{(1)}, \ldots, q^{(n)}]$ where $q^{(j)} = [q_1^{(j)}, \ldots, q_n^{(j)}]^T$. Then for all $k \geq 2$,

$$\tilde{E}^{(k)} = \sum_{j=1}^{n-k+1} \left(q^{(j)} q^{(j+k-1)^T} + q^{(j+k-1)} q^{(j)^T} \right). \tag{7}$$

As mentioned earlier, if $Q \in \mathcal{S}(n)$ then the (i, i) diagonal element

$$\tilde{E}_{ii}^{(k)} = 2 \sum_{j=1}^{n-k+1} q_i^{(j)} q_i^{(j+k-1)^T} \tag{8}$$

of $\tilde{E}^{(k)}$ must be zero for $k = 2, \ldots, n$. Applying Lemma 2.1 to (8) for each fixed i, together with the fact that all rows of Q are mutually orthogonal, we conclude that

Lemma 2.2. *If $Q \in \mathcal{S}(n)$, then Q must be of the form of a permutation matrix where the value 1 may possibly be replaced by -1. In particular, the stability group $\mathcal{S}(n)$ of $\mathcal{T}(n)$ has at most $2^n n!$ elements.*

We can deduce the structure of $\mathcal{S}(n)$ further from Lemma 2.2. Indeed, since $Q \in \mathcal{S}(n)$ is a permutation-type matrix, it must be that either $Q E^{(n)} Q^T = E^{(n)}$ or $Q E^{(n)} Q^T = -E^{(n)}$. This implies that

LEMMA 2.3. *If $Q \in \mathcal{S}(n)$, then Q must be of the structure that either*

$$Q = \begin{bmatrix} 0 & 0 & \dots & 0 & \pm 1 \\ 0 & & & & 0 \\ \vdots & & P & & \vdots \\ 0 & & & & 0 \\ \pm 1 & 0 & \dots & 0 & 0 \end{bmatrix} \tag{9}$$

or

$$Q = \begin{bmatrix} \pm 1 & 0 & \dots & 0 & 0 \\ 0 & & & & 0 \\ \vdots & & P & & \vdots \\ 0 & & & & 0 \\ 0 & 0 & \dots & 0 & \pm 1 \end{bmatrix} \tag{10}$$

where P is a permutation-type matrix of dimension $n-2$.

We observe a Toeplitz matrix $T(r)$ with $r = [r_1, \dots, r_n]^T$ can be written as

$$T(r) = \begin{bmatrix} r_1 & r_2 & \dots & r_{n-1} & r_n \\ r_2 & & & & r_{n-1} \\ \vdots & & T(r') & & \vdots \\ r_{n-1} & & & & r_2 \\ r_n & r_{n-1} & \dots & r_2 & r_1 \end{bmatrix}$$

where $T(r')$ is the Toeplitz matrix with $r' = [r_1, \dots, r_{n-2}]^T$. It follows that if $QT(r)Q^T \in \mathcal{T}(n)$, then $PT(r')P^T \in \mathcal{T}(n-2)$. Applying Lemma 2.3, we conclude that P will have the same structure as in (9) or (10). In fact, we claim

LEMMA 2.4. *Suppose $Q \in \mathcal{S}(n)$. If Q has structure (9), the P cannot have structure (10). If Q has structure (10), then P cannot have structure (9).*

Proof. It suffices to consider one case. Suppose Q is of the structure

$$Q = \begin{bmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & & 1 & 0 \\ \vdots & & & & \vdots \\ 0 & 1 & & 0 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{bmatrix}.$$

Then

$$QE^{(2)}Q^T = \begin{bmatrix} 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & & & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix}$$

which certainly is not Toeplitz. $\square$

Repeatedly applying Lemmas 2.3 and 2.4, we conclude that

LEMMA 2.5. *If $Q \in \mathcal{S}(n)$, then Q must be of the form of the identity matrix I or the backward identity matrix J where the value 1 may possibly be replaced by the value -1. In particular, $\mathcal{S}(n)$ has at most 2^{n+1} elements.*

We now determine the sign pattern for elements in $\mathcal{S}(n)$. Again, we observe that any 3×3 principal submatrix, indexed by $\{i-1, i, i+1\}$, of a Toeplitz matrix $T(r)$ is a Toeplitz matrix $T(r'')$ with $r'' = [r_1, r_2, r_3]$. Consider the submatrix R of $Q \in \mathcal{S}(n)$ that corresponds to the same index set $\{i-1, i, i+1\}$. From Lemma 2.5, R is either a diagonal or a backward diagonal matrix with 1 or -1 as the non-zero elements. We now claim

LEMMA 2.6. *The non-zero elements of R must be either that all are of the same sign or that they alternate signs.*

Proof. The otherwise case is that two consecutive elements of R have the same sign and the third element has the opposite sign. It suffices to consider

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

But then

$$RT(r'')R^T = \begin{bmatrix} r_1 & -r_2 & -r_3 \\ -r_2 & r_1 & r_2 \\ -r_3 & r_2 & r_1 \end{bmatrix}.$$

which in general cannot be Toeplitz. $\square$

Since the matrix R proved in Lemma 2.6 represents a typical block of any $Q \in \mathcal{S}(n)$, we finally conclude that

THEOREM 2.7. *Suppose $n > 1$. The stability group $\mathcal{S}(n)$ of $\mathcal{T}(n)$ has exactly eight elements, regardless of the dimension n. The elements are $\pm I$, $\pm J$, $\pm I'$ and $\pm J'$ where*

$$I' := \begin{bmatrix} -1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & & & \\ 0 & 0 & -1 & & & \\ \vdots & & & \ddots & & \vdots \\ 0 & & & & (-1)^{n-1} & 0 \\ 0 & & & \dots & 0 & (-1)^n \end{bmatrix} \tag{11}$$

and

$$J' := \begin{bmatrix} 0 & 0 & \dots & 0 & 0 & -1 \\ 0 & 0 & & 0 & 1 & 0 \\ 0 & 0 & & -1 & 0 & 0 \\ \vdots & & & & & \vdots \\ 0 & (-1)^{n-1} & & & 0 & 0 \\ (-1)^n & 0 & \dots & & 0 & 0 \end{bmatrix}. \tag{12}$$

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