ON THE INVERSE EIGENVALUE PROBLEM FOR REAL CIRCULANT MATRICES

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Abstract.
The necessary condition for eigenvalue values of a circulant matrix is studied. It is then proved that the necessary condition also suffices the existence of a circulant matrix with the prescribed eigenvalue values.

1. Introduction.

An $n \times n$ matrix C of the form

$$C = \begin{bmatrix} c_0 & c_1 & & \dots & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \dots & c_{n-2} \\ c_{n-2} & c_{n-1} & c_0 & \dots & c_{n-3} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ c_1 & c_2 & & c_{n-1} & c_0 \end{bmatrix}$$

is called a circulant matrix. As each row of a circulant matrix is just the previous row cycled forward one step, a circulant matrix is uniquely determined by the entries of its first row. We shall denote a circulant matrix by $C(\gamma)$ if its first row is γ. In this paper, we are mainly concerned with the case when $\gamma \in R^n$.

Let Π $(= \Pi_n)$ denote the permutation matrix of order n

$$\Pi := \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & & & & 1 \\ 1 & 0 & & \dots & 0 \end{bmatrix}. \tag{1}$$

It is easy to see that

$$C = \sum_{k=0}^{n-1} c_k \Pi^k \tag{2}$$

if and only if $C = C(\gamma)$ with $\gamma := [c_0, \dots, c_{n-1}]$. It is convenient to represent this relationship as

$$C(\gamma) = P_\gamma(\Pi) \tag{3}$$

where

$$P_\gamma(x) = \sum_{k=0}^{n-1} c_k x^k \tag{4}$$

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is called the characteristic polynomial of $C(\gamma)$. Because of this representation, it follows that circulant matrices are closed under multiplication. It is also clear that circulant matrices commute under multiplication.

Circulant matrices have important applications to diverse areas of disciplines including physics, image processing, probability and statistics, numerical analysis, number theory and geometry. The built-in periodicity also means that circulants is closely related to Fourier analysis and group theory. On the other hand, circulant matrices have very nice structure, mainly due to the matrix Π, which enable us to resolve many matrix-theoretic questions in "closed form". A most complete general reference on circulant matrices is [1].

We now briefly review some of the basic spectral properties that are relevant to out study. Most of these results are not new and the proofs can be found in [1].

Let $i := \sqrt{-1}$. For a fixed integer $n \geq 1$, let ω $(= \omega_n)$ denote the primitive n^{th} root of unity

$$\omega := \exp(\frac{2\pi i}{n}). \tag{5}$$

Let Ω $(= \Omega_n)$ denote the diagonal matrix

$$\Omega := \mathrm{diag}(1, \omega, \omega^2, \ldots, \omega^{n-1}), \tag{6}$$

and let F $(= F_n)$ denote the so called Fourier matrix whose Hermitian adjoint F^* is defined by

$$F^* := \frac{1}{\sqrt{n}} \begin{bmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & \omega & \omega^2 & \ldots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \ldots & \omega^{n-2} \\ \vdots & & & & \vdots \\ 1 & \omega^{n-1} & \omega^{n-2} & \ldots & \omega \end{bmatrix}, \tag{7}$$

that is, $\sqrt{n}F^*$ is the Vandermonde matrix generated by the row vector $[1, \omega, \omega^2, \ldots, \omega^{n-1}]$. Observe that F is a unitary matrix.

The following spectral decomposition of the forward shift matrix Π is key to our discussion.

THEOREM 1.1.

$$\Pi = F^* \Omega F. \tag{8}$$

It follows immediately that

THEOREM 1.2.

$$C(\gamma) = F^* P_\gamma(\Omega) F. \tag{9}$$

From (9), clearly the eigenvalues of $C(\gamma)$ are $P_\gamma(\omega^k)$, $k = 0, 1, \ldots, n-1$. The inverse problem of finding a circulant matrix with a prescribed spectrum is also easy to answer — Given any row vector $\ell := [\lambda_1, \ldots, \lambda_n]$, the circulant matrix $C(\gamma)$ with γ defined by

$$\gamma^T = \frac{1}{\sqrt{n}} F \ell^T \tag{10}$$

will have eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$. It is interestng to observe that if all the eigenvalues are distinct then there are precisely $n!$ many distinct circulant matrices with the prescribed spectrum.

For real circulant matrices, the eigenvalues appear in complex conjugate pairs. We claim that

THEOREM 1.3. *If the eigenvalues are arranged in the order*

1. $\lambda_1, \lambda_2, \ldots, \lambda_m, , \lambda_{m+1}, \overline{\lambda_m}, \ldots, \overline{\lambda_2}$, *if* $n = 2m$; *or*

2. $\lambda_1, \lambda_2, \ldots, \lambda_m, \overline{\lambda_m}, \ldots, \overline{\lambda_2}$, *if* $n = 2m+1$,

then there exist a unique real circulant matrix $C(\gamma)$ *with the prescribed spectrum.*

By rewriting (9) as

$$C(\gamma) = (F^* P_\gamma(\Omega)|P_\gamma(\Omega)|^{-1})|P_\gamma(\Omega)|F \tag{11}$$

where $|X|$ means the absolute value of the elements of X, we also realize that (11) is the singular value decomposition of $C(\gamma)$. The singular values of $C(\gamma)$ are $|P_\gamma(\omega^k)|$, $k = 0, 1, \ldots, n-1$. Since all the roots of unity are uniformly distributed on the unit circle, we obtain the necessary condition for singular values of a circulant matrix:

THEOREM 1.4. *The singular values of any real circulant matrix must appear in pairs as follows:*

1. $\sigma_1, \sigma_1, \ldots, \sigma_{m-1}, \sigma_{m-1}, \sigma_{2m-1}, \sigma_{2m}$, *if* $n = 2m$; *or*

2. $\sigma_1, \sigma_1, \ldots, \sigma_m, \sigma_m, \sigma_{2m+1}$, *if* $n = 2m+1$,

where the indices not necessarily reflecting the magnitude of the singular values. In any case, there can be at most $m+1$ *distinct singular values.*

REFERENCES

[1] P. J. Davis, Circulant Matrices, John Wiley and Sons, New York, 1979.