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A NUMERICAL METHOD FOR SOLVING INTERFACE PROBLEMS ARISING IN TWO-POINT BOUNDARY VALUE PROBLEMS

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A numerical approach iterating on the position of interface points is suggested for solving interface problems arising in two-point boundary value problems. Given an interface problem, it is decomposed into several standard local boundary value problems which are coupled at the interface points. Meanwhile, a nonlinear equation involving the interface points are formulated from the interface conditions. The advantages of this approach are that the boundary conditions for each local problem can easily be selected by considering the natural physical requirements, and that each of the local problems can be solved independently by standard numerical BVP techniques.

1. Introduction

In many physical systems modeled by differential equations, the parameters of the system may not be continuous functions of time and space. Often these discontinuities are reflected in interface conditions for the differential system whereas the location of the interfaces may or may not be known a priori. This paper deals especially with interface problems arising in two-point boundary value problems. Such a problem typically can be described in the following form:

Given an ODE for a function y together with separated boundary conditions at fixed points $x = a$ and $x = b$ $(a < b)$, trigger conditions

$$f(c, y(c^-), y'(c^-), y(c^+), y'(c^+)) = 0 . \tag{1}$$

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and interface conditions

$$g(c, y(c^-), y'(c^-), y(c^+), y'(c^+)) = 0, \tag{2}$$

find the unknown function y satisfying the ODE and BCs, and such that the interface conditions hold at any point at which the trigger conditions hold. (Such a point $x = c$ is called an interface point.)

In recent years the development of numerical techniques for solving standard BVPs has reached some level of sophistication. Extensive references in this aspect may be found in [1–5]. All these various approaches are generally applicable to fixed interface problems as well. For instance, in the shooting method one-sided limits at the interface are known so that the solution can be continued past the interface by computing new initial values from the interface condition. In finite difference techniques the algebraic equation at the interface mesh point is derived from the interface condition rather than the differential equations [6]. We find, however, that free interface problems where the location of the interface has to be determined as part of the problem, generally are not amenable to the standard solution techniques mentioned above. The breakdown could be caused by several reasons depending upon the nature of the underlying problem. This observation will be better demonstrated when we discuss our examples in the later part of this paper.

If the number of interface points is known ahead of time, then a procedure [7] that transforms the intervals into ones with known endpoints and introduces trivial ODEs to determine the unknown point values, may be employed. While this procedure would result in a standard BVP and hence many standard codes could be used from then on, the penalty is a significant increase in the size of the system. Alternatively, the invariant imbedding method [8] seems to be a more flexible and mechanically applicable approach for solving free interface problems. The basic idea of this method is to interpret the ordinary differential equations of the boundary value problem as the characteristic equations of an associated partial differential equation of an initial value problem. As a consequence, numerical PDE techniques are then required to solve the corresponding initial value problem.

In this paper we propose a direct approach to solving interface problems for two-point BVPs. The idea is to utilize the natural structure in the interface condition to construct what we call the target function of the interface points. We then apply the Newton method to iterate on the position of interface points. Since the derivative of the target function with respect to the interface points is generally not available, we use the secant method as a matter of course. Although the mathematical facts of interface problems could be very complicated and difficult, the algorithm we propose herein is quite straightforward and is explained in the next section. The main purpose of this paper is to demonstrate how the idea works. Towards this end, we apply the method to four different types of interface problems. We show how the target function could be formed and how the algorithm performs.

2. Algorithm

One powerful technique used for solving standard two-point BVPs is the parallel shooting method. Our idea is analogous to that method. See also [9]. The main difference is that in the

parallel shooting method one attempts to satisfy the continuity and boundary conditions by adjusting the solution values at certain prescribed and fixed nodal points whereas in our method we attempt to solve certain target equations by adjusting the location of interface points. In the parallel shooting method the interface conditions remain to cause difficulty in the computation, but in our method the interface conditions are converted into standard boundary conditions.

To illustrate the idea, let us consider the following example. Let the BVP be defined by

$$y'' = f(x, y, y', c), \quad 0 < x < 1, \tag{3}$$

with boundary conditions

$$y(0) = 0, \quad y(1) = 0, \tag{4}$$

where the solution y is also required to satisfy the interface condition

$$g(y'(c^-), y'(c^+)) = 0, \tag{5}$$

whenever the trigger condition

$$y(c) = 0 \tag{6}$$

holds. For simplicity of exposition, we shall assume further that somehow we know there is only one interface point. Then solving the above problem is equivalent to finding the scalar c and two functions y_1 and y_2, such that

$$P_1: y_1'' = f(x, y_1, y_1'), \quad 0 < x < c, \qquad y_1(0) = 0, \qquad y_1(c) = 0, \tag{7}$$

$$P_2: y_2'' = f(x, y_2, y_2'), \quad c < x < 1, \qquad y_2(c) = 0, \qquad y_2(1) = 0, \tag{8}$$

and that

$$g(y_1'(c), y_2'(c)) = 0, \tag{9}$$

where $y_1'(c)$ and $y_2'(c)$ are understood in the sense of a one-sided limit. By using the Green's functions

$$\Phi_1(x, \xi; c) = \begin{cases} (1 - \xi/c)x, & 0 \leq x \leq \xi, \\ (1 - x/c)\xi, & \xi \leq x \leq c, \end{cases} \tag{10}$$

$$\Phi_2(x, \xi; c) = \begin{cases} (1 - \xi)(x - c)/(1 - c), & c \leq x \leq \xi, \\ (1 - x)(\xi - c)/(1 - c), & \xi \leq x \leq 1, \end{cases} \tag{11}$$

for problems P_1 and P_2, respectively, we can see the sensitivity dependence of $y_1'(c)$ and $y_2'(c)$ upon the change of c:

$$\frac{\partial y_1'(x; c)}{\partial c} = \frac{1}{c^2} \int_0^c \xi f(x, y_1(\xi), y_1'(\xi))\, d\xi, \tag{12}$$

$$\frac{\partial y_2'(x;c)}{\partial c} = \frac{1}{(1-c)^2}\int_c^1 (1-\xi) f(x, y_2(\xi), y_2'(\xi))\,d\xi\,. \tag{13}$$

Thus the location of the interface point c may be approximated iteratively by applying the secant method to the target equation (9) in which functions y_1 and y_2 corresponding to problems P_1 and P_2 with c equal to the current approximate value of the interface point are obtained by using any standard numerical BVP technique.

The above idea obviously can be extended to the multi-interface case. Suppose it is conjectured that there are n interface points. Let $0 = c_0 < c_1 < \cdots < c_n < c_{n+1} = 1$. Then the target equation becomes

$$G(c_1, \ldots, c_n) = \begin{bmatrix} g(y_1'(c_1), y_2'(c_1)) \\ \vdots \\ g(y_n'(c_n), y_{n+1}'(c_n)) \end{bmatrix} = 0\,, \tag{14}$$

where for $i = 1, \ldots, n+1$, the function y_i is the solution to the BVP

$$y_i'' = f(x, y_i, y_i')\,, \quad c_{i-1} < x < c_i\,, \qquad y_i(c_{i-1}) = 0\,, \qquad y_i(c_i) = 0\,. \tag{15}$$

It is easy to see that the Jacobian of the target function has a tridiagonal stucture. One should note, however, that there is no guarantee that the target equation will always have a solution. One should also note that in certain cases the target equation may have more than one solution. Some examples are observed in the next section. For the time being, we do not know of any general theory that can predict the number of interface points before actually solving the problem. Such a theory certainly would be advantageous. In practice, the underlying physical background sometimes can provide information in estimating the number of interface points.

For general BVPs the continuous dependence of the solution on the location of interface points may not be addressed as easily as before [10]. In fact, even the theoretical existing question of a solution itself is difficult to answer in general [11]. Nevertheless, this mathematical difficulty should not deter us from seeking a numerical solution. A numerical algorithm based upon the above idea can still be formulated, provided each of the local problems with the trigger conditions as its boundary conditions at any two consecutive interface points is solvable.

3. Applications

The idea explained in the preceding section is now applied to four nontrivial free interface problems. We demonstrate how the target equation could be constructed and we present some empirical results. From case to case, we analyze the theoretical solution to a certain extent just to expose some intrinsically important properties. Readers should not mistake that all problems can be analyzed in a similar way. In fact, even if one of the problem parameters is changed a little bit, the analytic solution may become too complicated to be tracked.

All the local boundary value problems hereinafter are solved by a standard finite element

method with linear elements [12]. Other numerical BVP techniques certainly can be used as well. In calculating the finite difference for the secant method, the perturbation ε on the values of interface points is chosen to be $\varepsilon = O(h^{p+1})$, if the ODE discretization is $O(h^p)$. With this choice of ε, the computation of the perturbed problem can be obtained inexpensively by applying an iterative refinement technique to the solution of the unperturbed problem. Indeed, the solution obtained for the unperturbed problem is used as the initial guess for the perturbed problem.

We note also that all the local problems are totally uncoupled. Therefore, it is feasible to carry out the computation concurrently on a parallel computer. The following results are done on an IBM 4361 machine.

EXAMPLE 1. A one-dimensional scattering problem.

Consider the motion of a particle under the influence of a known force depending on its speed and position. Suppose that whenever this particle collides with a known barrier it is scattered back with a fraction of its momentum. Assume that the particle starts from the origin and that at a known future time T it passes again through the origin. We are interested in its initial momentum. The corresponding free interface problem is described as:

$$\begin{aligned} &u'' = G(t, u, u'), \\ &u(0) = 0, \quad u(T) = 0, \\ &u'(L^+) = -\alpha u'(L^-), \quad \text{whenever } u(L) = \beta, \end{aligned} \tag{16}$$

where u is the displacement, u' the speed of the particle, β the location of the barrier and $0 < \alpha < 1$. The above information in general is not adequate in determining a unique solution because corresponding to different initial momentum values the particle could have been scattered more than once during the time interval $[0, T]$ and still satisfy the boundary conditions. So the number of scatterings should also be specified as an additional condition for this problem.

For illustration purpose, let us consider a rather simple but interesting bouncing ball problem. We shall assume $G(t, u, u') \equiv -g$ where g is the gravitational constant. The parameters are chosen to be $u(0) = u_0$, $T = 1$ and $\beta = 0$. The solution to this particular problem can be derived explicitly, which is done below to show some theoretical facts.

Suppose we are looking for the solution which has n bounces in the interval $(0, 1)$. Let the time of the bounces (the interface points) be denoted as $t_1, \ldots, t_n$. By requiring the analytical solutions of the BVP (16) to satisfy the interface conditions at every interface points, we obtain the following system:

$$t_1 - t_2 = -\alpha\left(t_1 + \frac{2u_0}{gt_1}\right),$$

$$t_k - t_{k+1} = -\alpha(t_k - t_{k-1}), \quad k = 2, \ldots, n-1, \tag{17}$$

$$t_n - 1 = -\alpha(t_n - t_{n-1}). \tag{18}$$

It is easy to see that the solution of (17) satisfies the recurrence relation:

$$t_n = \frac{1+\alpha t_{n-1}}{1+\alpha},$$

$$t_k = \frac{1+\alpha(1+\alpha+\cdots+\alpha^{n-k})t_{n-k}}{1+\alpha+\cdots+\alpha^{n-k+1}}, \tag{19}$$

$$gt_1^2(1+\alpha+\cdots+\alpha^n) - gt_1 + 2\alpha(1+\alpha+\cdots+\alpha^{n-1})u_0 = 0. \tag{20}$$

Thus, given α and n, the necessary condition for having a solution of (16) is that the initial height of the ball does not go beyond the upper bound

$$u_0 \leq \frac{g}{8\alpha(1+\alpha+\cdots+\alpha^{n-1})(1+\alpha+\cdots+\alpha^n)}. \tag{21}$$

In the strict inequality case, it is interesting to note that there should always be two sets of bounces $\{t_k\}$ corresponding to two different values of initial momentum values.

We now solve the problem numerically without referring to any of the theoretical facts mentioned above. Let $t_0 = 0$ and $t_{n+1} = 1$. It seems nature to use the jump conditions of the first derivatives at the interfaces of the solution as the target equation. That is, we want to find $t_1, \ldots, t_n$, so that

$$G(t_1, \ldots, t_n) = \begin{bmatrix} u_2'(t_1) + \alpha u_1'(t_1) \\ \vdots \\ u_{n+1}'(t_n) + \alpha u_n'(t_n) \end{bmatrix} = 0, \tag{22}$$

where $u_k(t)$ is the solution to the BVP $u'' + g = 0$ over the interval $[t_{k-1}, t_k]$ with boundary conditions $u_k(t_{k-1}) = u_k(t_k) = 0$ for $k = 2, \ldots, n$, and $u_1(t_1) = 0$, $u_1(t_0) = u_0$ and $u_{n+1} = 0$.

We test our algorithm for the case where $n = 3$, $\alpha = 1$. The mesh size in the finite element method for each local problem is chosen to be $h = 10^{-2}$. The perturbation value used in the secant method is set to be $\varepsilon = 10^{-6}$. We count that convergence occurs whenever the 2-norm of updates on the location of interface points is less than 10^{-7}. Table 1 shows the approximate locations of the three interface points.

Figure 1 contains the graphs of the coupled solutions corresponding to the first and final iterates.

Table 1
Iterates of three interface points for the bouncing ball problem

Number of iterations	t_1	t_2	t_3
0	0.40000000	0.60000000	0.70000000
1	0.41143894	0.50884664	0.60625434
2	0.40837241	0.50011439	0.59185637
3	0.40833756	0.50000002	0.59166247

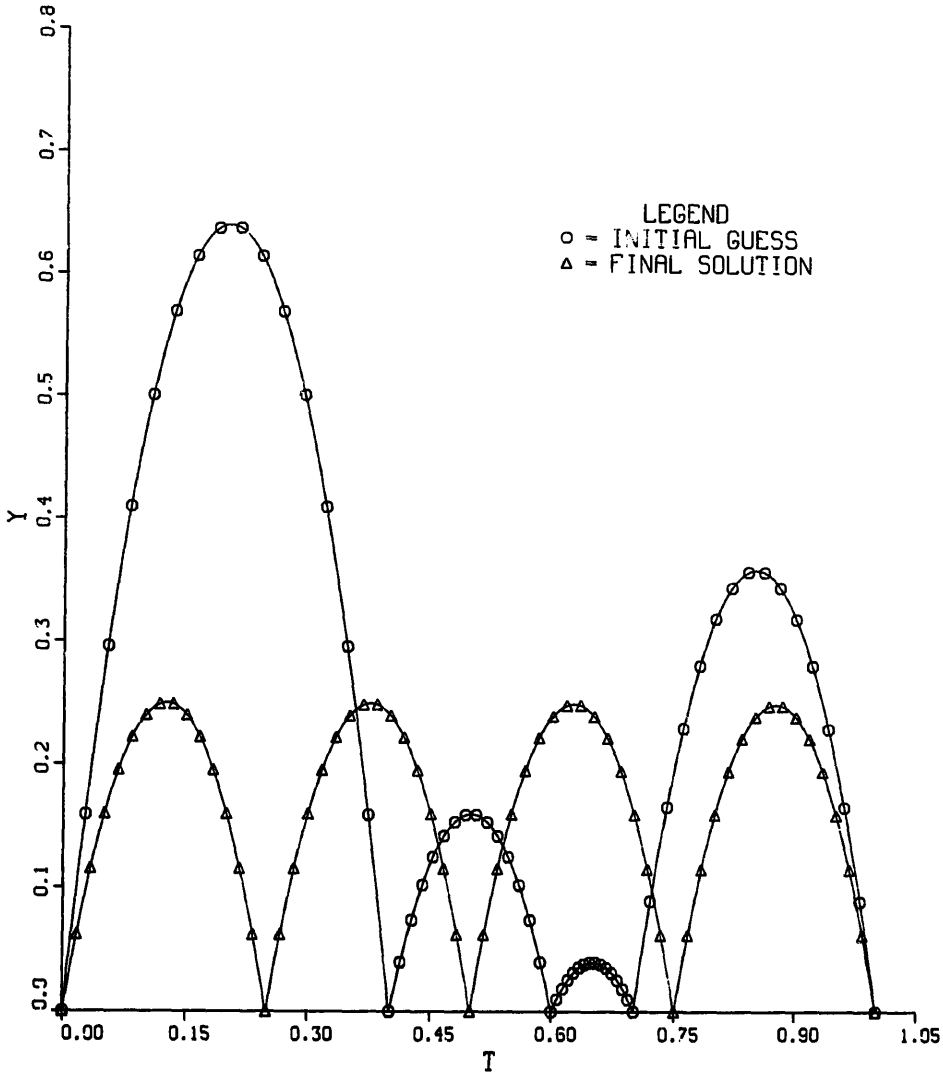

Fig. 1. Bouncing ball problem.

EXAMPLE 2. An undamped vibration with constant restoring force:

Another important class of BVP's in application are those with discontinuous nonlinearities. Quite often this kind of problem has multiple solutions. Here we consider a simple vibration problem subject to the signum nonlinearity of the restoring force:

$$u'' = -\mathrm{sgn}(u), \qquad u(0) = u(1) = 0. \tag{23}$$

This problem is different from Example 1 in that a solution to (23) is required to be in $C^1[0, 1]$ and to have piecewise continuous second derivative only. The interface condition, therefore, is the continuity of the first derivative u' at the interface point where u vanishes.

Again, problem (23) can be solved explicitly. Indeed, it is not difficult to see that the totality of solutions of (23) is composed of piecewise parabolas denoted by $\pm\varphi_n$, $n =$

0, 1, 2, . . . , where

$$\varphi_n(t) = \frac{1}{2}(-1)^k\left(t - \frac{k}{n}\right)\left(\frac{k+1}{n} - t\right) \tag{24}$$

for $k/n \leq t \leq (k+1)/n$, $k = 0, 1, \ldots, n-1$. Note that for each n there are $n-1$ interface points uniformly distributed throughout the interval (0, 1).

Suppose now that we want to solve (23) numerically. We may choose the target equation for $t_1, \ldots, t_{n-1}$ to be

$$G(t_1, \ldots, t_n) = \begin{bmatrix} u_2'(t_1) - u_1'(t_1) \\ \vdots \\ u_n'(t_{n-1}) - u_{n-1}'(t_{n-1}) \end{bmatrix} = 0, \tag{25}$$

where $u_k(t)$ solves the trivial problem $u_k'' = (-1)^{k+1}$ over the interval $[t_{k-1}, t_k]$ with boundary conditions $u_k(t_{k-1}) = u_k(t_k) = 0$ for $k = 1, \ldots, n$ (Here $t_0 = 0$ and $t_n = 1$). We test our algorithm for the case $n = 3$. All the control parameters in the numerical computation are set to be the same as those in Example 1. Table 2 shows the effective convergence of our scheme on locating the interface points. The coupled solutions corresponding to the initial and the final locations of the interface points are sketched in Fig. 2.

REMARKS. When reviewing this work, we have found several other approaches suggested in the literature. Since the results are interesting, we mention them below for comparison.

(1) It has been conjectured [13] that the Picard iteration,

$$u_n'' = -\text{sgn}(u_{n-1}), \qquad u_n(0) = u_n(1) = 0, \tag{26}$$

even starting close to nonextreme solutions, would always converge in a finite number of steps to the extreme solutions $\pm\varphi_1$.

(2) It also has been suggested to solve the weak formulation,

$$F(u; \phi) = \langle u', \phi' \rangle - \langle \text{sgn}(u), \phi \rangle = 0 \quad \forall \phi \in C_0^\infty(0, 1), \tag{27}$$

of (23) by the Newton method in which the derivative of F is understood in the weak sense

$$\frac{\partial F(u; \phi)}{\partial u} s = \langle s', \phi \rangle - 2 \sum_i \frac{s(x_i)\phi(x_i)}{|u'(x_i)|}, \tag{28}$$

Table 2
Iterates of three interface points for the constant restoring force vibration problem

Number of iterations	t_1	t_2	t_3
0	0.40000000	0.60000000	0.70000000
1	0.24999999	0.49999999	0.75000000

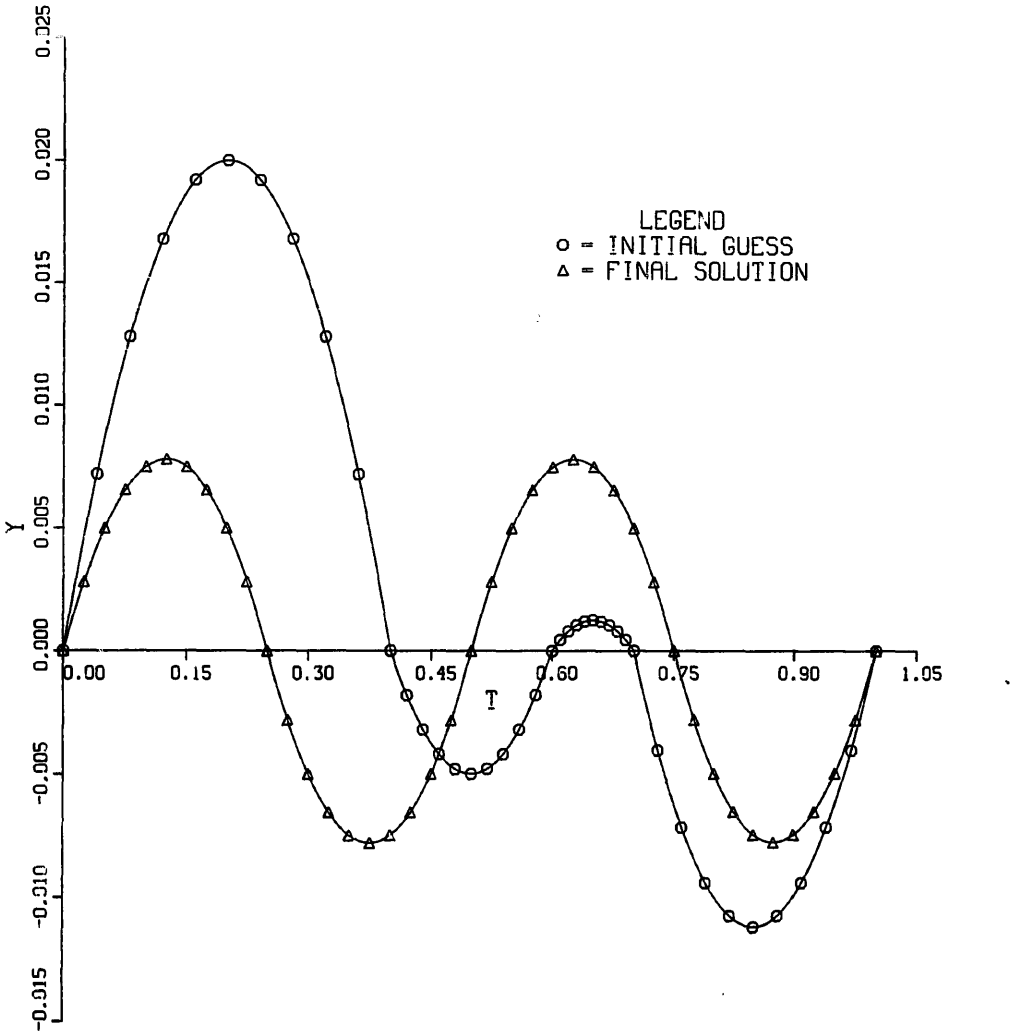

Fig. 2. Vibration problem with a constant restoring force.

where x_i are the interface points, and the value of F is obtained from a finite element method. In using this scheme, we have observed some interesting phenomena:

(a) Depending upon the number of elements used in computing the values of F, the above scheme may converge to different solutions of (23) even if all the other computing parameters are set up exactly the same. For example, with initial guesses of the interface points at {0.00, 0.21, 0.33, 0.76, 1.00}, the scheme converges to φ_2 if 80 elements are used, and converges to $-\varphi_3$ if 160 elements are used.

(b) Depending upon the location of initial guesses, some of the interface points may coalesce eventually even if all the other computing parameters are the same and the initial guesses are well separated. We demonstrate one example of two interface points in Table 3. The first column on the left contains the initial guesses for the first interface point and the first row on the top contains the initial guesses for the second interface point. The integer n in the table indicates that the scheme with the corresponding initial guesses returns a solution which has $n-1$ interface points. We do not see this coalescence happen in our algorithm.

Table 3
Effect of initial guesses on the convergence

	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.1	1	1	2	2	2	2	1	1
0.2		1	1	1	3	2	1	1
0.3			2	1	3	3	2	2
0.4				3	3	3	3	2
0.5					3	2	1	2
0.6						2	1	2
0.7							1	1
0.8								1

EXAMPLE 3. A damped free vibration problem with constant damping force.

We now consider another type of a nonlinear differential equation

$$mu'' + c\,\mathrm{sgn}(u') + ku = 0\,, \tag{29}$$

where m, c and k are all positive. The second term in (29) simply expresses that the friction opposes the motion but is independent of the speed. Multiplying both sides of (3.12) by u', we obtain $\mathrm{d}(m(u')^2/2 + ku^2)/\mathrm{d}t = -c|u'| \leqslant 0$. So the motion dies out with increasing time. The dynamics of (29) is similar to that of the well understood linear damped differential equation

$$mu'' + cu' + ku = 0\,. \tag{30}$$

After posing Dirichlet boundary conditions, however, the solutions to these two BVPs behave very differently.

As an example, suppose the boundary conditions are $u(0) = u(1) = 0$. For the BVP defined by (30), either there are infinitely many nonisolated solutions of the form $u(t) = Re^{-ct/(2m)}\sin(\mu t)$ with R arbitrary, if the value $\mu = (4km - c^2)^{1/2}/(2m)$ is an integer multiple of R, or there is no solution at all, otherwise. The locations of the interface points, if there are any, depend upon the values of m, c and k only and, in particular, are independent of the initial momentum.

In contrast, we claim that the BVP defined by (29) either has isolated solutions or has no solution, and that the value of the initial momentum affects the location of the interface points. To see this analytically, we break the solution of (29) into two phases denoted by u_+ and u_-, respectively, depending upon whether u' is positive or negative. Let $\nu = (k/m)^{1/2}$. Then the general solutions u_+ and u_- are of the forms

$$u_+(t) = R_+\cos(\nu t - \delta_+) - \frac{c}{k}\,, \tag{31}$$

and

$$u_-(t) = R_-\cos(\nu t - \delta_-) + \frac{c}{k}\,, \tag{32}$$

respectively, where R_+, δ_+, R_- and δ_- are determined from the boundary conditions and/or the interface conditions. For instance, suppose we start with positive momentum, then the initial value $u_+(0) = 0$ implies that

$$R_+ \cos \delta_+ = \frac{c}{k}\,, \tag{33}$$

whereas δ_+ is related to the initial momentum by the relation

$$u'_+(0) = \frac{\nu c}{k} \tan \delta_+\,. \tag{34}$$

The initial momentum is to be determined so that the other boundary condition is satisfied. The solution $u_+(t)$ can be continued until

$$t = t_1 = \frac{\delta_+}{\nu}\,, \tag{35}$$

which is the first time that $u'_+(t) = 0$. If $t_1 \leq 1$, it could be the location of the first interface point for our problem. Then the solution $u_-(t)$ with initial values $(u_-(0), u'_-(0)) = (0, c(\sec k - 1)/k)$ takes place and is continued for a time interval of length π/ν. If

$$t = t_2 = \frac{\delta_+ + \pi}{\nu} \tag{36}$$

is still less than 1, then it could be the location of the second interface point. This procedure continues to alternate between u_+ and u_-. It is now clear that the initial momentum continues to affect the location of the successive interface points. For instance, in the one interface point case, we see that the exact δ_+ (which determines the initial momentum) should satisfy the nonlinear equation

$$(\sec \delta_+ - 2)\cos(\nu - \delta_+) + 1 = 0\,. \tag{37}$$

In spite of the above theoretical analysis, let us now resort to the idea discussed in Section 2 to solve the BVP (29) with boundary conditions $u(0) = \alpha$, $\alpha > 0$ and $u(1) = 0$ numerically. Obviously one choice for the target equation comes from the (implicit) requirement of continuity of the solution at the interface points. Therefore, we are looking for the point t_1 so that

$$G(t_1) = u_1(t_1) - u_2(t_1) = 0\,, \tag{38}$$

where u_1 and u_2 solve the BVPs

$$mu''_1 + c + ku_1 = 0\,, \qquad u_1(0) = \alpha\ (\alpha > 0)\,, \quad u'_1(t_1) = 0\,, \tag{39}$$

and

$$mu''_2 - c + ku_2 = 0\,, \qquad u_2(1) = u'_2(t_1) = 0\,, \tag{40}$$

respectively.

Listed in Table 4 are the iterates of t_1 for the test case $\alpha = 0.5$, $m = 1$, $c = 0.25$ and $k = 9$ by our method. The solution segments for the initial guess as well as those for the final solution are sketched in Fig. 3. We remind the readers that the two BVPs (39) and (40) are solved

Table 4
Iterates of one interface point for the constant damping force vibration problem

Number of iterations	t_1
0	0.50000000
1	0.47433970
2	0.47420496
3	0.47419442
4	0.47419432
5	0.47419432

Table 5
Iterates of the interface between two liquids

Number of iterations	t_1
0	0.90000
1	0.79771
2	0.63659
3	0.50612
4	0.46919
5	0.46703

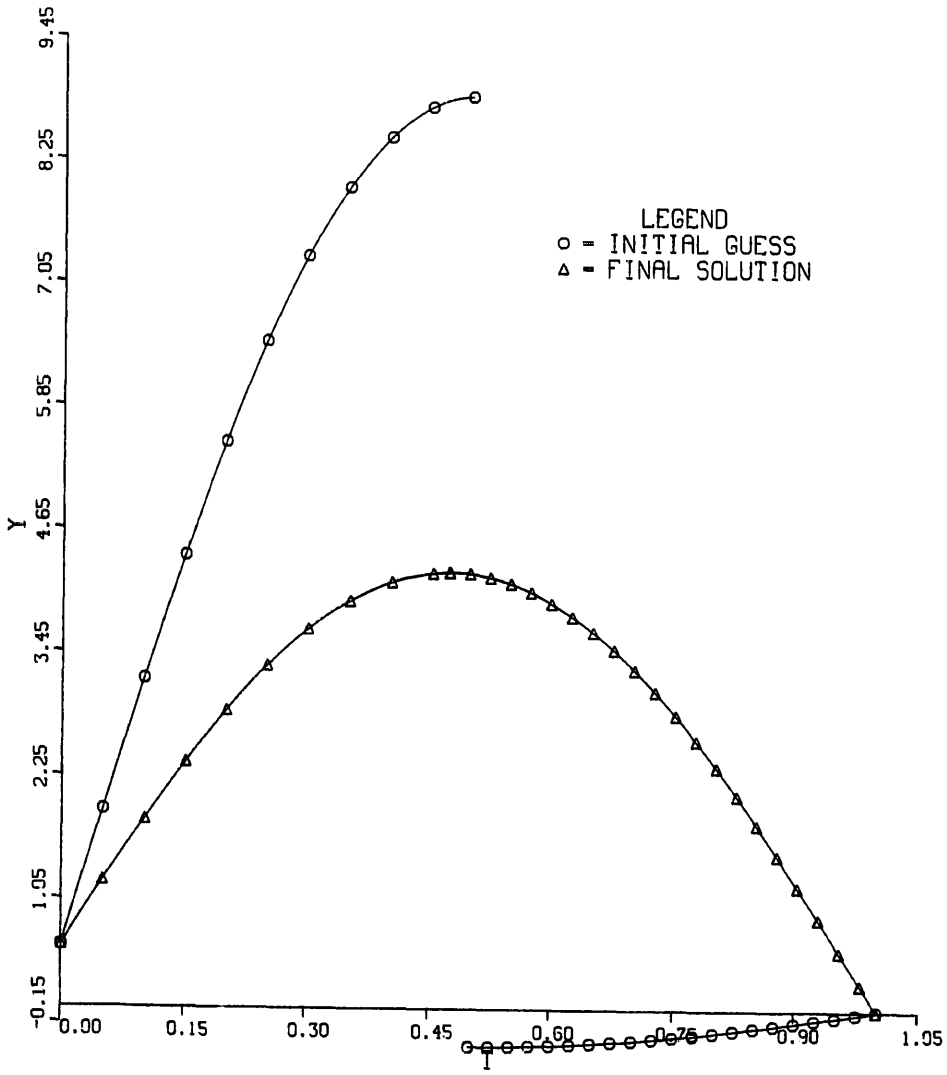

Fig. 3. Vibration problem with a constant damping force.

independently by a finite element method. Although the initial segments are highly apart from each other, finally the solution segments are patched together.

EXAMPLE 4. An overspecified boundary condition problem.

Consider the motion of a particle through a two-layered liquid column. We assume the particle begins to fall from position $u = 1$ at time $t = 0$ under the influence of gravity while its motion is retarded through viscous drag which is proportional to its velocity. At time $t = 1$, it reaches the bottom of the column $u = 0$. The problem is to locate the interface between the two layers of fluid. The equation of motion is described by

$$u_k'' = -g - \alpha_k u_k', \qquad u_1(0) = u_0, \quad u_1'(0) = 0, \quad u_2(1) = 0. \tag{41}$$

This problem is different from all the above examples in that u_1 is completely determined as an initial value problem. Physically we expect the transition of the motion at the interface to

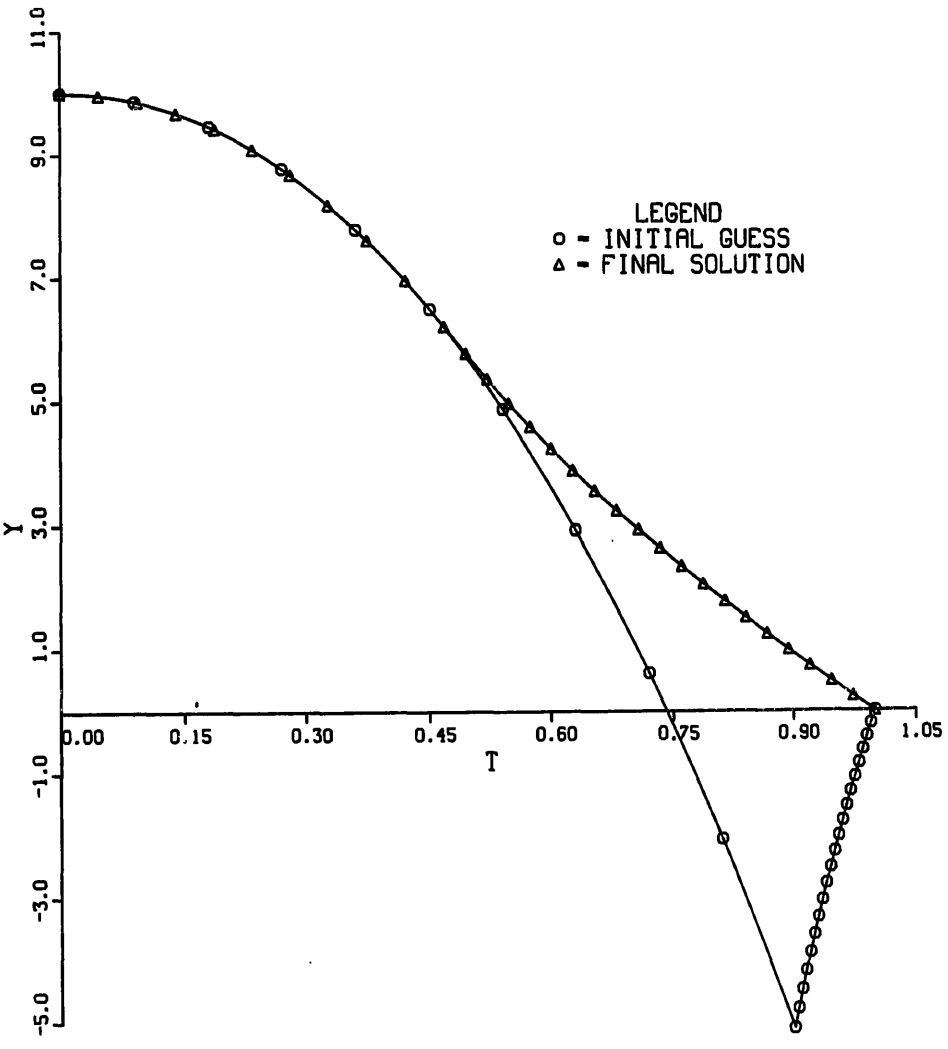

Fig. 4. Problem of interface between two liquids.

be smooth. So the target equation can be chosen to be either

$$G(t_1) = u_1(t_1) - u_2(t_1) = 0\,, \tag{42}$$

where u_2 satisfies the boundary conditions $u_2'(t_1) = u_1'(t_1)$ and $u_2(1) = 0$, or

$$G(t_1) = u_1'(t_1) - u_2'(t_1) = 0\,, \tag{43}$$

where u_2 satisfies the boundary conditions $u_2(t_1) = u_1(t_1)$ and $u_2(1) = 0$.

We have experimented with (43) by using linear elements for the BVP of u_2 and RKF45 for the IVP of u_1. With parameters $\alpha_1 = 0.5$, $\alpha_2 = 4.0$ and $u_0 = 10$, we obtain iterates of the interface t_1 as in Table 5. The initial and the final solutions are sketched in Fig. 4.

4. Conclusion

A numerical approach utilizing the secant method to iterate on the position of the interface points is suggested for solving interface problems for two-point boundary value problems. The advantages are that the interface conditions usually can be treated as the boundary conditions for the local problems, and that each of the local problems can be solved independently by standard numerical BVP techniques. We have applied the proposed method successfully to four different types of interface problems. In most cases, the target equation that characterizes the location of the interface points can be formulated easily from the natural structure in the interface problem. The mathematical theory of interface problems still needs to be strengthened. Meanwhile, we hope our approach provides a straightforward but feasible way to obtain the numerical solution.

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