INVERSE EIGENVALUE PROBLEMS

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Abstract. A collection of inverse eigenvalue problems are identified and classified according to their characteristics. Current developments in both the theoretic and the algorithmic aspects are summarized and reviewed in this paper. This exposition also reveals many open questions that deserves further study. An extensive bibliography of pertinent literature is attached.

Key words. Eigenvalue Problem, Inverse Problem, Parameter Estimation, System Reconstruction, Spectral Constraint, Structural Constraint, Iterative Methods, Direct Method, Continuous Methods, Least Squares.

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1. Introduction.

1.1. Objective. An inverse eigenvalue problem concerns the reconstruction of a matrix from prescribed spectral data. The spectral data involved may consist of the complete or only partial information of eigenvalues or eigenvectors. The objective of an inverse eigenvalue problem is to construct a matrix that maintains a certain specific structure as well as that given spectral property..

Associated with any inverse eigenvalue problem are two fundamental questions — the theoretic issue on *solvability* and the practical issue on *computability*. A major effort in solvability has been to determine a necessary or a sufficient condition under which an inverse eigenvalue problem has a solution. The main concern in computability, on the other hand, has been to develop a procedure by which, knowing a priori that the given spectral data are feasible, a matrix can be constructed numerically. Both questions are difficult and challenging.

Studies on inverse eigenvalue problems have been intensive, ranging from engineering application to algebraic theorization. Yet the results are scattered even within the same field of discipline. Despite the many efforts found in the literature, only a handful of the problems discussed in this paper have been completely understood or solved. Our goal in this work is to gather together a collection of inverse eigenvalue problems, to identify and classify their characteristics, and to summarize current developments in both the theoretic and the algorithmic aspects. We hope this presentation will help to better define the regimen of inverse eigenvalue problems as a whole and hence to stimulate further research.

1.2. Application. Inverse eigenvalue problems arise in a remarkable variety of applications. The list includes but is not limited to control design, system identification,

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seismic tomography, principal component analysis, exploration and remote sensing, antenna array processing, geophysics, molecular spectroscopy, particle physics, structure analysis, circuit theory, mechanical system simulation and so on.

To mention a few examples, we note that the state feedback as well as the output feedback pole assignment problems have been of major interest in system identification and control theory. There is a vast literature of research on this subject alone. An excellent recount of recent activities in this area can be found in the survey paper by Byrnes [35]. We shall see that pole assignment problems are a special case of what we call parameterized inverse eigenvalue problems in this article.

Also, one of the basic problems in classical vibration theory is to determine the natural frequencies and normal modes of the vibrating body. But inverse problems are concerned with the construction of a model of a given type, e.g., a mass-spring system, a string, and so on, with prescribed spectral data. Thus inverse problems have practical value to applied mechanics and structure design [9, 61, 90, 91, 92, 120, 156, 158, 161]. Discussion for higher dimensional problems can be found in [10, 137, 138, 198, 199, 200].

Applications to other types of engineering problems can be found in the books by Gladwell [93], Helmke and Moore [111], and articles such as [125, 188, 191, 196]. Examples of geophysics applications can be found in [153]. Examples of physics applications can be found in [8, 12, 59, 63, 72, 188]. Even within the field of numerical analysis where a specific algorithm is to be designed an inverse eigenvalue problem may arise. See, for example, [145].

Much of the discussion for inverse problems in the literature has been due to an interest in the inverse Sturm-Liouville problem [5, 107, 109, 152, 154, 202]. See also [57, 58, 107, 151] for a comprehensive study of the connection between the continuous problem and the matrix problem.

A significant common phenomenon in all these applications is that the physical parameters of a certain system are to be reconstructed from knowledge of its dynamical behavior, in particular its natural frequencies and/or normal modes. If the physical parameters can be (and often they are) described mathematically in the form of a matrix, then we have an inverse eigenvalue problem. In order to make the resulting model physically realizable, it should be noted that sometimes additional stipulations must be imposed upon the matrix.

1.3. Diversity. Depending on the application, inverse eigenvalue problems may be described in several different forms. Translated into mathematics, it is often necessary in order that the inverse eigenvalue problem be meaningful to restrict the construction to special classes of matrices, especially to those with specified structures. A problem without any restriction on the matrix generally is of little interest. The solution to an inverse eigenvalue problem therefore should satisfy two constraints — the *spectral constraint* referring to the prescribed spectral data and the *structural constraint* referring to the desirable structure. These constraints define a variety of inverse eigenvalue problems that will be surveyed in this paper.

In practice, it may occur that one of the two constraints of the problem should be enforced more critically than the other due to, for example, the physical realizability.

Without the realizability, the physical system simply cannot be built. There are also situations when one constraint could be more relaxed than the other due to, for example, the physical uncertainty. The uncertainty arises when there is simply no accurate ways to measure the spectrum or there is no reasonable means to obtain the entire information. When the two constraints cannot be satisfied simultaneously, sometimes we are interested in a least squares solution.

Gladwell suggests from the standpoint of engineering application that there should also be a distinction between *determination* and *estimation* in the nature of an inverse problem. He calls it an *essentially mathematical problem* when the given data is exact and complete so that the system can be precisely determined, and an *essentially engineering problem* when the the data is only approximate and often incomplete, and when only an estimation of the parameters of the system is sought so that the resulting behavior agrees approximately with the prescribed data [94, 97]. It is important to formulate the right question since research based on inappropriate or ill-chosen questions leads to unsatisfying and unnecessarily complicated answers.

1.4. Literature Overview. Classical approaches to determining the solvability of an inverse eigenvalue problems involve techniques developed from algebraic curves, degree theory or algebraic geometry. See, for example, [2, 16, 66, 80, 81, 86, 127]. Although in most cases the algebraic theory is still incomplete or missing, there are also numerical algorithms developed for computation purpose. A partial list includes, for example, [18, 28, 43, 54, 99, 104, 115, 128, 129, 144, 146, 184, 187].

A review of recent literature on inverse eigenvalue problems related exclusively to small vibrations of mechanical system can be found in [94] that is then updated in [97]. An early survey of direct methods for solving certain symmetric inverse eigenvalue problems was given by Boley and Golub [27]. Algorithms of iterative nature for more general problems were considered by Friedland et al [85]. This paper covers an even larger scope of inverse eigenvalue problems.

An earlier attempt similar to the objective of this paper was made by Zhou and Dai in their book [203] that greatly motivates this author to continue the current extension. We build our presentation upon that in [203] by bringing in the latest results known to this date. In particular, an extensive bibliography of pertinent literature is compiled. Regretfully, many Chinese references in [203] are not included because of difficulties in translation and availability. Other excellent resources for references, particularly those related to mechanical systems, can be found in [87, 93, 94, 97], and those to inverse Sturm-Liouville problems in [5].

We mention that entries of the matrix to be constructed usually represent physical parameters to be determined. So an inverse eigenvalue problem can generally be regarded as a parameter estimation problem. Each inverse eigenvalue problem, however, also carries it own characteristic. In the literature, the study usually is focused on one characteristic a time. Following the practice in the literature, we categorize inverse eigenvalue problems according to characteristics such as additive, multiplicative, parameterized, structured, partially described or least squares. This classification along with review articles by Gladwell [94, 97] who differentiates problems according to the

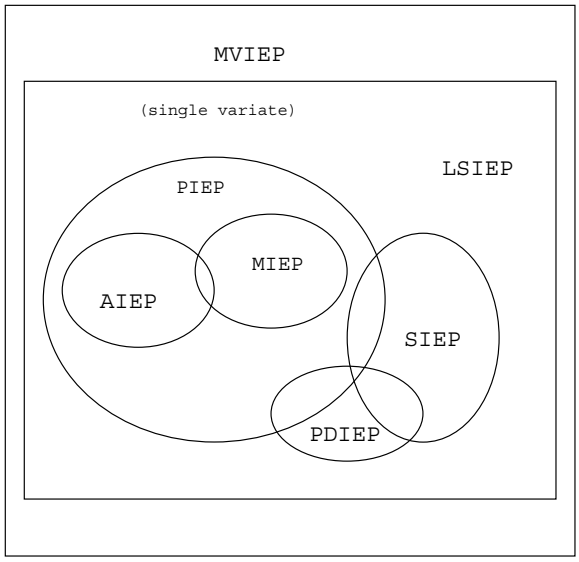

FIG. 1. *Classification of inverse eigenvalue problems.*

type of the mechanical system, i.e., continuous or discrete, damped or undamped, and the type of the prescribed data, i.e., spectral, modal, or nodal, complete or incomplete, should complement each other to offer a fairly broad view of research activities in this area.

1.5. Outline and Notation. This paper discusses explicitly thirty-seven inverse eigenvalue problems, not counting the many other implied variations. The forms and algorithms differ noticeably from problem to problem. Thus it is almost impossible to bring any unity into this collection. Also, an inverse eigenvalue problem often carries overlapping characteristics. It is sometimes difficult to determine which characteristic is the most prominent.

In an attempt to provide a better grasp of the scenarios, we shall adopt the name scheme ***IEP#** to identify a problem throughout the paper. Letter or letters " $*$ " in front of IEP indicate the type of the problem. The number "#" following IEP indicates the sequence of variation within type "*IEP". We first introduce the following acronyms:

MVIEP	=	Multi-Variate Inverse Eigenvalue Problem
LSIEP	=	Least Square Inverse Eigenvalue Problem
PIEP	=	Parameterized Inverse Eigenvalue Problem
SIEP	=	Structured Inverse Eigenvalue Problem
PDIEP	=	Partially Described Inverse Eigenvalue Problem
AIEP	=	Additive Inverse Eigenvalue Problem
MIEP	=	Multiplicative Inverse Eigenvalue Problem

The precise definition for each type of problems will be described in the sequel. We suggest using Figure 1 to lay down a possible inclusion relationship between the different problems. We hope readers will agree after perusing through our argument that this diagram, though not precise, provides a reasonable connection between the problems.

We intend to imply several points from Figure 1 that affect our presentation:

- The MVIEP basically is an unexplored territory because most of the studies in the literature have been for the single variate only. We shall touch upon its general setting in §6, but concentrate on the single variate problem for the rest of this paper. There should be plenty of new research topics in this area alone.
- All problems have a natural generalization to the least squares formulation.
- The AIEP and the MIEP are two extensively studied special cases of the PIEP.
- The relationship depicted in Figure 1 is not necessarily definite because many characteristics may overlap. We should not be surprised if there are other types of characterization overlooked in this classification.

In this survey, we choose to call attention to three major types of problems. In §2 we describe the PIEP where the emphasis is on the way that these parameters modulate the problem. In §3 we discuss the SIEP where the emphasis is on the structure that a solution matrix is supposed to maintain. In §4 we discuss the LSIEP where the best solution exists only in the sense of least squares approximation. We shall consider these three problems slightly more in breadth and depth with regard to the motivation, main results, and algorithmic issues.

In addition, we shall briefly discuss the PDIEP in §5 because it is difficult to place properly in Figure 1. The PDIEP arises when there are simply no reasonable tools available to evaluate the entire spectral information due to, for instance, the complexity or the size of the physical system. Often only partial data are readily obtainable and the engineers have to build the system based on that partial information.

To emphasize the modular representative in each category, we begin each section with a paradigmatic description of the problem. We then discuss variations by being more specific on conditions of the underlying matrices. It quickly becomes clear that we will not be able to give a full account on each of the problems in this presentation. We can only try to provide the readers with a few references whenever some kind of theory or algorithms have been derived. Despite our efforts, it is obvious that we will have left out some interesting problems from our collection. Yet we shall see that there are already more questions than answers in this exposition — the reason why this treatise is originally motivated.

Being tossed between the vast diversity of problems, theories, algorithms, and open questions, we find it very difficult to achieve any uniformity in this presentation. By presenting the discussion as a synthesis of subsections entitled Generic Form, Variations, Solvability Issues, and Numerical Methods, we hope we have provided a grasp of the different aspects of inverse eigenvalue problems.

Because of the scope of problems covered in this paper, we inevitably have to call upon lot of jargon in this presentation. We shall explain some of the unusual terms, but for most of the technical linear algebra terms we suggest that readers refer to the classical book by Horn and Johnson [118]. To facilitate the discussion, we shall adopt the following notation hereinafter:

$\mathbf{F}$ represents the scalar field of either real $\mathbb{R}$ or complex $\mathbb{C}$.
$A, B, \ldots$ denote matrices.
$b^{(\nu)}, q_i^{(\nu)}, v_i, \ldots$ denote vectors.
$\sigma(A)$ denotes the spectrum of A.
$\|\cdot\|$ denotes either the 2-norm of a vector or the Frobenius norm of a matrix.
$\mathcal{M}, \mathcal{N}, \ldots$ denote certain subsets of square matrices of which the size is clear from the context. In particular, we have:

$$\begin{aligned}
\mathcal{R}(n) &:= \mathbb{R}^{n\times n}, \\
\mathcal{S}(n) &:= \{\text{All symmetric matrices in } \mathcal{R}(n)\}, \\
\mathcal{O}(n) &:= \{\text{All orthogonal matrices in } \mathcal{R}(n)\}, \\
\mathcal{D}_{\mathcal{R}}(n) &:= \{\text{All diagonal matrices in } \mathcal{R}(n)\}, \\
\mathcal{C}(n) &:= \mathbb{C}^{n\times n}, \\
\mathcal{H}(n) &:= \{\text{All Hermitian matrices in } \mathcal{C}(n)\}, \\
\mathcal{D}_{\mathcal{C}}(n) &:= \{\text{All diagonal matrices in } \mathcal{C}(n)\}.
\end{aligned}$$

2. Parameterized Inverse Eigenvalue Problem.

2.1. Generic Form. Although almost every inverse eigenvalue problems can be regarded as an parameter estimation problem, the emphasis in this section is on the meticulous way that these parameters regulate the problem. A generic PIEP can be described as follows:

(PIEP) *Given a family of matrices $A(c) \in \mathcal{M}$ with $c = [c_1, \ldots, c_m] \in \mathbf{F}^m$ and scalars $\{\lambda_1, \ldots, \lambda_n\} \subset \mathbf{F}$, find a parameter c such that $\sigma(A(c)) = \{\lambda_1, \ldots, \lambda_n\}$.*

Note that the number m of parameters in c may be different from n. Depending upon how the family of matrices $A(c)$ is specifically defined in terms of c, the PIEP can appear and be solved very differently. Inverse eigenvalue problems in the above PIEP format arise frequently in discrete modeling [90, 107, 151] and factor analysis [110]. We shall illustrate several different aspects by examples in the following, but a common feature in all variations of the PIEP is that the parameter c is used as a "control" that modulates to the underlying problem in a certain specific, predestined way.

2.2. Variations. The inclusion of PIEP is quite broad. We mention a few interesting variations below:

The case when $A(c)$ is affine in c has attracted considerable attention recently:
(PIEP1) $A(c) = A_0 + \sum_{i=1}^n c_i A_i$ where $A_i \in \mathcal{R}(n)$, $\mathbf{F} = \mathbb{R}$.
(PIEP2) $A(c) = A_0 + \sum_{i=1}^n c_i A_i$ where $A_i \in \mathcal{S}(n)$, $\mathbf{F} = \mathbb{R}$ [85].

Also, the following two problems have been under extensive investigation in the literature:

(AIEP) *Given a matrix $A \in \mathcal{M}$, scalars $\{\lambda_1, \ldots, \lambda_n\} \subset \mathbf{F}$, and a class of matrices $\mathcal{N}$, find a matrix $X \in \mathcal{N}$ such that $\sigma(A + X) = \{\lambda_1, \ldots, \lambda_n\}$.*

(MIEP) *Given a matrix $A \in \mathcal{M}$, scalars $\{\lambda_1, \ldots, \lambda_n\} \subset \mathbf{F}$, and a class of matrices $\mathcal{N}$, find a matrix $X \in \mathcal{N}$ such that $\sigma(XA) = \{\lambda_1, \ldots, \lambda_n\}$.*

It is clear that the AIEP is a special case of the PIEP with $A(X) = A + X$ and X playing the role of c, and that the MIEP corresponds to the case where $A(X) = XA$. By being more specific on the class $\mathcal{N}$ of matrices, the problems themselves can be divided into further subclasses. Since both AIEP and MIEP have been of long and independent interest in various applications, we name them as separate types and shall examine them more carefully later.

The following example is yet another more complicated PIEP arising in descriptor systems:

(PIEP3) Given matrices $A \in \mathcal{C}(n)$, $B_i \in \mathbb{C}^{n\times m_i}$, $C_i \in \mathbb{C}^{l_i\times n}$, $i = 1, \ldots, q$, and scalars $\{\lambda_1, \ldots, \lambda_n\} \subset \mathbb{C}$, find matrices $K_i \in \mathbb{C}^{m_i\times l_i}$ such that $\sigma(A + \sum_{i=1}^{q} B_i K_i C_i) = \{\lambda_1, \ldots, \lambda_n\}$ [189].

When $q = 1$, the PIEP3 includes as special cases the state feedback as well the output feedback pole assignment problems. This problem stands alone as an important issue for decades. It has been studied extensively by different approaches ranging from linear system theory, combinatorics, complex function theory to algebraic geometry. See, for example, [34, 35, 111] and the references contained therein. Yet the results are still incomplete.

2.2.1. Additive Inverse Eigenvalue Problem. As indicated above, the emblem of an AIEP is that a given matrix A is perturbed by the *addition* of a specially structured matrix X in order to match the eigenvalues. The eigenvalue information can provide at most n equations, so sometimes it may be desirable to limit the number of free parameters in X. Other than this, the set $\mathcal{N}$ can be taken quite liberally. We may therefore use the set $\mathcal{N}$ to impose a certain structural constraint on the solution matrix X. For example, it may be that matrices in $\mathcal{N}$ are required to introduce no more non-zero entries (fill-in) than what are already in A, or that only certain positions of the matrix A are allowed to be added to. Structure on $\mathcal{N}$ sometimes arises naturally because of engineers' design constraints. When the number of unknown parameters in X and the number of equations provided through the eigenvalue information are not consistent, an AIEP may be considered in the context of least squares. See §4 for more details.

Thus far, most of the attention has been paid to the case where $\mathcal{N}$ contains only diagonal matrices. Even so, differences among the following special cases should be carefully distinguished:

(AIEP1) $\mathcal{M} = \mathcal{R}(n)$, $\mathbf{F} = \mathbb{R}$, $\mathcal{N} = \mathcal{D}_{\mathcal{R}}(n)$.
(AIEP2) $\mathcal{M} = \mathcal{S}(n)$, $\mathbf{F} = \mathbb{R}$, $\mathcal{N} = \mathcal{D}_{\mathcal{R}}(n)$.
(AIEP3) $\mathcal{M} = \mathcal{C}(n)$, $\mathbf{F} = \mathbb{C}$, $\mathcal{N} = \mathcal{D}_{\mathcal{C}}(n)$ [81].
(AIEP4) $\mathcal{M} = \mathcal{H}(n)$, $\mathbf{F} = \mathbb{R}$, $\mathcal{N} = \mathcal{D}_{\mathcal{R}}(n)$ [69].

The AIEP4 was first posed by Downing and Householder [69]. Its special case AIEP2 with A being a Jacobi matrix is of particular interest because the discretization of the boundary value problem, for example,

$$(1) \qquad \begin{aligned} -u''(x) + p(x)u(x) &= \lambda u(x), \\ u(0) = u(\pi) &= 0, \end{aligned}$$

by the central difference formula with uniform mesh $h = \frac{\pi}{n+1}$ naturally leads to the eigenvalue problem in tridiagonal structure,

$$\left(\frac{1}{h^2} \begin{bmatrix} 2 & -1 & 0 & & & \\ -1 & 2 & -1 & & & \\ 0 & -1 & 2 & \cdots & & 0 \\ \vdots & & & \ddots & & \\ 0 & & & & 2 & -1 \\ 0 & & & & -1 & 2 \end{bmatrix} + X \right) u = \lambda u, \tag{2}$$

where X is a diagonal matrix representing the discretization of $p(x)$. Thus an AIEP2 may be interpreted as a discrete analog of the inverse Strum-Liouville problem, a classical subject where the potential $p(x)$ is to be found so that the system possesses a prescribed spectrum.

Another interesting variant of the AIEP arises in, e.g., control or algorithm design, where the stability is at issue. In such a problem it is more practically critical to have eigenvalues located in a certain region than at a certain points. One such problem can be stated as follows:

(AIEP5) Given $A \in \mathcal{R}(n)$, find $X \in \mathcal{N}$ with $\sigma(A+X)$ lies in a certain fixed region, say the right-half, of the complex plane.

Related to the AIEP5, for example, is the nearest unstable matrix problem [33]. The problem concerns the distance from a given matrix, stable in the sense that all its eigenvalues have negative real part, to the nearest matrix with one eigenvalue on the imaginary axis. Also related is the communality problem in factor analysis [110] and the educational testing problem [53, 77]. The former concerns finding a diagonal matrix D so that the sum $A + D$ in which A is a given real symmetric matrix with zero diagonal entries has as many zero eigenvalues as possible. The latter concerns finding a positive diagonal matrix D so that the difference $A - D$ in which A is a given real symmetric positive definite matrix remains positive semi-definite while the trace trace(D) is maximized.

2.2.2. Multiplicative Inverse Eigenvalue Problem. In contrast to the AIEP, a MIEP stands out when the task is to *pre-multiply* a given matrix A by a specially structured matrix X to reposition or to precondition the distribution of its eigenvalues. This is very similar to but more general than the idea of preconditioning the matrix A where it is desired to find an efficient preconditioner M for A so that the product $M^{-1}A$ *approximates* the identity. It is known that preconditioning plays a very important role in many computational issues. Although the sense in which $M^{-1}A$ should approximate the identity differs according to the underlying method to be used, the general setting in the MIEP can be applied to the optimal preconditioning of a given matrix A. The set $\mathcal{N}$ can be used particularly to exploit a certain sparsity pattern of the preconditioner [101, 102].

Similar to the AIEP, perhaps the simplest possible preconditioners are the diagonal scalings:

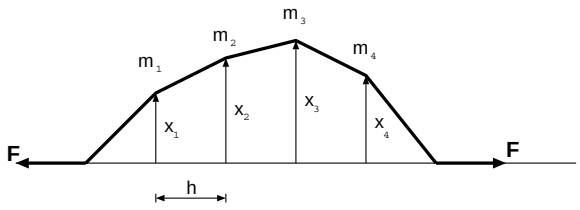

FIG. 2. *Vibration of particles on a string.*

(MIEP1) $\mathcal{M} = \mathcal{R}(n)$, $\mathbf{F} = \mathbb{R}$, $\mathcal{N} = \mathcal{D}_{\mathcal{R}}(n)$.
(MIEP2) $\mathcal{M} = \mathcal{S}(n)$, $\mathbf{F} = \mathbb{R}$, $\mathcal{N} = \mathcal{D}_{\mathcal{R}}(n)$
(MIEP3) $\mathcal{M} = \mathcal{C}(n)$, $\mathbf{F} = \mathbb{C}$, $\mathcal{N} = \mathcal{D}_{\mathcal{C}}(n)$ [80].

We illustrate one example of MIEP2 arising from engineering application. Consider the vibration of particles on a string sketched in Figure 2. Suppose four particles, each with mass m_i, are uniformly spaced with distance h and are vibrating vertically subject to the horizontal tension F. Then the equation of motion is given by [203]:

$$\begin{aligned} m_1\frac{d^2x_1}{dt^2} &= -F\frac{x_1}{h} + F\frac{x_2 - x_1}{h} \\ m_2\frac{d^2x_2}{dt^2} &= -F\frac{x_2 - x_1}{h} + F\frac{x_3 - x_2}{h} \\ m_3\frac{d^2x_3}{dt^2} &= -F\frac{x_3 - x_2}{h} + F\frac{x_4 - x_3}{h} \\ m_4\frac{d^2x_4}{dt^2} &= -F\frac{x_4 - x_3}{h} - F\frac{x_4}{h} \end{aligned}$$

which can be summarized as the system

$$\frac{d^2x}{dt^2} = -DAx \tag{3}$$

where $x = [x_1, x_2, x_3, x_4]^T$, $A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$, and $D = \mathrm{diag}(d_1, d_2, d_3, d_4)$ with $d_i = \frac{F}{m_i h}$. To solve (3), we typically consider the eigenvalue problem

$$DAx = \lambda x$$

where λ is the square of the so called natural frequency of the system. The inverse problem then amounts to calculating the mass m_i, $i = 1, \ldots, 4$, so that the resulting system vibrates at a prescribed natural frequency.

Similarly, a discretization of the boundary value problem

$$-u''(x) = \lambda\rho(x)u(x) \tag{4}$$

yields the eigenvalue problem

$$Au = \lambda Xu \tag{5}$$

where X is a positive diagonal matrix representing $\rho(x)$. Thus an MIEP is to determine the density function $\rho(x) > 0$ from the prescribed spectrum.

A conservative, n degrees of freedom mass-spring system with mass matrix X and stiffness matrix A also ends with the formulation (5). Since the physical realizability of the stiffness matrix A usually is more complex than the mass matrix X, a practical way of ensuring the overall physical realizability in engineering design is to determine A from static constraints and then to find a positive diagonal matrix X so that some desired natural frequencies are achieved.

There are other types of multiplicative inverse eigenvalue problems:

(MIEP4) Given a matrix $A \in \mathcal{H}_n$ and scalars $\{\lambda_1, \ldots, \lambda_n\} \subset \mathbb{R}$, find a matrix $X \in \mathcal{D}_{\mathcal{R}}(n)$ such that $\sigma(X^{-1}AX^{-1}) = \{\lambda_1, \ldots, \lambda_n\}$ [69].

(MIEP5) Given $A \in \mathcal{R}(n)$, find $X \in \mathcal{D}_{\mathcal{R}}(n)$ with positive entries such that $\sigma(XA)$ lies in the right-half complex plane.

2.3. Solvability Issues. It would be nice to be able to address the solvability issue of the PIEP by one major theorem. But such a result simply does not, and probably will never, exist because the description of PIEP is too general. Searching through the literatures, on the other hand, reveals that scattered around are pieces of understanding of its individual variations. The information in fact is so diverse and massive that we find it extremely difficult to condense the results here. We can only summarize some of the major developments problem by problem.

It is easy to construct examples, even in $\mathcal{R}(2)$, that the PIEP1 and the PIEP2 may have no solution at all. In this case, a least squares formulation becomes more desirable. We shall discuss this issue in §4.

Considerable advances toward the understanding of the AIEP have been made over the years. There is a rich literature on both the theoretic and the numerical aspects for this type of problems. To see a few necessary and some sufficient conditions on the solvability, we refer to results in articles [20, 36, 79, 103, 106, 126, 131, 132, 142, 143, 147, 148, 164, 173, 180, 192, 193]. Notably we have the following main result addressing the existence question for the AIEP3 by Friedland [2, 81]:

THEOREM 2.1. *For any specified $\{\lambda_1, \ldots, \lambda_n\}$, the AIEP3 is solvable. The number of solutions is finite and does not exceed $n!$. Moreover, for almost all $\{\lambda_1, \ldots, \lambda_n\}$, there are exactly $n!$ solutions.*

We hasten to point out that the existence question when $\mathbf{F} = \mathbb{R}$, including the AIEP1 or the AIEP2, has yet to be settled. It would be an interesting research topic to study the theoretical and the algorithmic aspects of the more general AIEP when the constraint set $\mathcal{N}$ imposes a structure other than the diagonal. We are not aware of any discussion in this regard.

By using degree theory, Friedland has also proved the following theorem for the MIEP3:

THEOREM 2.2. *If all principal minors of A are distinct from zero, then the MIEP3 is solvable for arbitrary $\{\lambda_1, \ldots, \lambda_n\}$ and there exist at most $n!$ distinct solutions.*

Although in practice one does not need a preconditioner that exactly repositions the eigenvalues, an understanding of the MIEP might shed some insights into the design

of a good preconditioner. Friedland's result suggests that in the complex context the matrix A can be perfectly conditioned by a diagonal matrix. Despite its elegance in mathematical theory, one should not overlook the practicality of Theorem 2.2. What is missing is an efficient algorithm for implementing Theorem 2.2. Indeed, we are not even aware of any numerical procedure to do so. In fact, we are more interested in only the real arithmetic and thus the MIEP1 or MIEP5. Unfortunately, despite the many preconditioners that have been proposed or used in practice, there are not as many theoretic results known for these types of problems. General discussions on the MIEP can be found in [67, 105, 149, 170, 181]. In the context of minimizing the condition number $\kappa(M^{-1}A)$, one classical result due to Forsythe and Strauss [78, 101] is worth mentioning.

THEOREM 2.3. *Assume A is symmetric, positive definite, and has property-A, i.e., A can be symmetrically permuted into the form $\begin{bmatrix} D_1 & B \\ B^T & D_2 \end{bmatrix}$ where D_1 and D_2 are diagonal matrices. Let D denote the diagonal of A. Then*

$$\kappa(D^{-1/2}AD^{-1/2}) \le \kappa(\hat{D}A\hat{D}) \tag{6}$$

for any other positive definite diagonal matrix $\hat{D}$. In other words, D is the optimal diagonal preconditioner for the matrix A.

The following theorem is a standard result addressing the solvability issue of PIEP3 when $q = 1$ [122, 179].

THEOREM 2.4. *Given $A \in \mathcal{R}(n)$, $B \in \mathbb{R}^{n\times m}$, and a set of n complex numbers $\{\lambda_1, \ldots, \lambda_n\}$, closed under complex conjugation, then there exist a matrix $K \in \mathbb{R}^{m\times n}$ such that $\sigma(A + BK) = \{\lambda_1, \ldots, \lambda_n\}$ if and only if the pair (A, B) is controllable, that is, if and only if the following condition holds:*

$$\{y^T A = \mu y^T \text{ and } y^T B = 0\} \Longleftrightarrow y^T = 0.$$

Moreover, in the single-input case, i.e., $m = 1$, if a solution exists, then it is unique.

We shall refer readers to a recent survey paper by Byrnes [35] for pole assignment problems and not give any more reviews here.

2.4. Sensitivity Analysis. Associated with any PIEP, and indeed any inverse eigenvalue problem, is the important issue of sensitivity analysis. That is, we need to determine how a solution matrix is subject to change with respect to the perturbation of the prescribed eigenvalues. This is the inverse problem of the classical matrix perturbation theory [17, 176].

The difficulty of this inverse sensitivity analysis even for symmetric matrices can be illustrated from the generalized Wielandt-Hoffman theorem [17, Theorem 8.5] that, for any two Hermitian matrices A and B, we have

$$\|\mathrm{Eig}^{\downarrow}(A) - \mathrm{Eig}^{\downarrow}(B)\|_2 \le \|A - B\|_2 \le \|\mathrm{Eig}^{\downarrow}(A) - \mathrm{Eig}^{\uparrow}(B)\|_2 \tag{7}$$

where $\mathrm{Eig}^{\downarrow}(A)$ (and similarly $\mathrm{Eig}^{\uparrow}(A)$) means the diagonal matrix with eigenvalues of A arranged in descending (ascending) order. The classical theory concerns the sensitivity

of the eigenpairs to random perturbation. The first inequality in (7) provides an upper bound on the variation of eigenvalues. The inverse problem, however, concerns the structural modification, say, of a physical system due to spectral adjustment. Even if the adjustment is relatively minor, the second inequality in (7) does not necessarily provide a good bound on the variation of solution matrices. In fact, it is fundamental that eigenvalues are continuous functions in the entries of a matrix [177]. But the converse sometimes even does not make any sense because the inverse problem may have no solution at all if the data are changed. This important yet difficult question so far as we know has not been addressed thoroughly in the literature.

Some recent work on the validation of a numerical solution to an AIEP can be found in [1, 178, 195] and the thesis [7]. Some immediate application and related discussion can be found in, for example, [34, 122, 179] for robust pole assignment problems, and [15, 133, 155] for incomplete modal analysis. For other situations, the analysis perhaps needs to be carried out individually. Any advance in this direction certainly is welcomed.

2.5. Numerical Methods. Even though the existence theory or a sensitivity analysis for a PIEP may still be incomplete or missing, it does not necessarily imply that the problem is untouchable by some numerical means. In this section we review some of the methods for the PIEP.

Numerical algorithms for solving the AIEP2 and the AIEP4 can be found, for example, in [18, 24, 27, 69, 85, 107, 130, 146, 187]. Most methods for symmetric or Hermitian problems depend heavily on the fact that the eigenvalues are real valued and, hence, can be totally ordered. In this case, the ith eigenvalue $\lambda_i(X)$ of $A + X$ for each fixed i is continuous and piecewise differentiable in X. Standard techniques for solving nonlinear algebraic systems may be used. We shall illustrate one such iterative method for the AIEP under the context of PIEP in a later part of this section.

When eigenvalues are complex valued, including the case $\mathbf{F} = \mathbb{C}$ or even the AIEP1 in general, it becomes more difficult to track the evolution of eigenvalues because complex numbers do not form an ordered field and one cannot explicitly identify which value in the spectrum is the ith eigenvalue. An existence proof for the AIEP3 by the homotopy method which, in return, gives rise to a numerical method for finding all solutions of the AIEP3 can be found in [43]. See also [194]. Except for the homotopy method that tracks each individual eigenvalue by a homotopy curve determined by its initial value, it seems that other methods for solving a complex-valued AIEP will inevitably involve some kind of matching mechanism [30, 31, 41].

The MIEP1 may be written in the form of the PIEP1 by, for example, selecting $A_0 = 0$ and $A_k = e_k a_k^T$ for $k = 1, \ldots, n$ where a_k^T is the kth row of A and e_k denotes the kth standard basis in $\mathbb{R}^n$. The matrices A_k in this setting, of course, are not symmetric. If A is symmetric and positive definite, then the matrix XA is similar to $L^T X L$ where L is the Cholesky factor of $A = LL^T$. We may then convert an MIEP2 to a PIEP2 by using symmetric matrices $A_0 = 0$ and $A_k = L^T E_k L$ with $E_k := \mathrm{diag}(e_k)$.

Assuming the existence of a solution, several numerical methods for the PIEP2 have been studied in [85]. The geometric interpretation of one of these method, Method III in [85], as a variant of the Newton method is particularly useful and interesting [47],

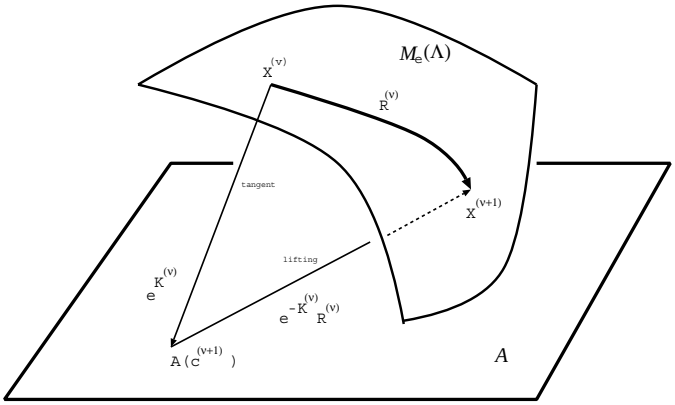

FIG. 3. *Geometry of PIEP.*

since many inverse eigenvalue problems can be written in the PIEP2 form. We discuss the basic idea in a little bit more detail below.

For illustration, we shall consider the case that all eigenvalues $\lambda_1, \ldots, \lambda_n$ are distinct. Let

$$\Lambda := diag\{\lambda_1, \ldots, \lambda_n\} \tag{8}$$

and let $\mathcal{A}$ denote the affine subspace

$$\mathcal{A} := \{A(c) | c \in \mathbb{R}^n\} \tag{9}$$

where $A(c)$ is defined in PIEP2. It can be proved that the set

$$\mathcal{M}_e(\Lambda) := \{Q\Lambda Q^T | Q \in \mathcal{O}(n)\}, \tag{10}$$

where $\mathcal{O}(n)$ denotes the group of all $n \times n$ orthogonal matrices, is a smooth manifold of dimension $\frac{n(n-1)}{2}$. Any tangent vector $T(X)$ to $\mathcal{M}_e(\Lambda)$ at a point $X \in \mathcal{M}_e(\Lambda)$ must be of the form

$$T(X) = XK - KX \tag{11}$$

for some skew-symmetric matrix $K \in \mathbb{R}^{n \times n}$ [45, 48] . We recall the elementary fact that the new iterate $x^{(\nu+1)}$ of a classical Newton step

$$x^{(\nu+1)} = x^{(\nu)} - (f'(x^{(\nu)}))^{-1} f(x^{(\nu)}) \tag{12}$$

for a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is precisely the x-intercept of the line which is tangent to the graph of f at $(x^{(\nu)}, f(x^{(\nu)}))$. If we think of the surface $\mathcal{M}_e(\Lambda)$ as playing the role of the graph of f and the affine subspace $\mathcal{A}$ as playing the role of the x-axis, then an iterative process analogous to the Newton method can be developed for the PIEP2. The geometry is illustrated in Figure 3.

Given $X^{(\nu)} \in \mathcal{M}_e(\Lambda)$, there exist a $Q^{(\nu)} \in \mathcal{O}(n)$ such that

$$Q^{(\nu)^T} X^{(\nu)} Q^{(\nu)} = \Lambda. \tag{13}$$

From (11), we know $X^{(\nu)}+X^{(\nu)}K-KX^{(\nu)}$ with any skew-symmetric matrix K represents a tangent vector to $\mathcal{M}_e(\Lambda)$ emanating from $X^{(\nu)}$. We thus seek an $\mathcal{A}$-intercept $A(c^{(\nu+1)})$ of such a vector with the affine subspace $\mathcal{A}$. That is, we want to find a skew-symmetric matrix $K^{(\nu)}$ and a vector $c^{(\nu+1)}$ such that

$$X^{(\nu)} + X^{(\nu)}K^{(\nu)} - K^{(\nu)}X^{(\nu)} = A(c^{(\nu+1)}). \tag{14}$$

The unknowns $K^{(\nu)}$ and $c^{(\nu+1)}$ in equation (14) can be solved separately as follows. Using (13), we transform (14) into the system

$$\Lambda + \Lambda\tilde{K}^{(\nu)} - \tilde{K}^{(\nu)}\Lambda = {Q^{(\nu)}}^T A(c^{(\nu+1)})Q^{(\nu)} \tag{15}$$

where

$$\tilde{K}^{(\nu)} := {Q^{(\nu)}}^T K^{(\nu)} Q^{(\nu)} \tag{16}$$

is still skew-symmetric. Because $\tilde{K}^{(\nu)}$ has zero diagonal, the scaling by Λ does not effect a change of the diagonal entries on the left-hand side of (15). This observation effectively separates $c^{(\nu+1)}$ from $K^{(\nu)}$. More precisely, a comparison of the diagonals on both sides of (15) gives rise to the linear system of n equations:

$$J^{(\nu)}c^{(\nu+1)} = \lambda^* - b^{(\nu)} \tag{17}$$

where

$$J_{ij}^{(\nu)} := {q_i^{(\nu)}}^T A_j q_i^{(\nu)}, \text{ for } i,j = 1,\ldots,n \tag{18}$$

$$\lambda^* := (\lambda_1,\ldots,\lambda_n)^T \tag{19}$$

$$b_i^{(\nu)} := {q_i^{(\nu)}}^T A_0 q_i^{(\nu)}, \text{ for } i = 1,\ldots,n \tag{20}$$

and $q_i^{(\nu)}$ is the i-th column of the matrix $Q^{(\nu)}$. The vector $c^{(\nu+1)}$, therefore, can be solved from (17). Note that $K^{(\nu)}$ is not involved in (17) at all. Note also that in the setup of (17) we have used the fact that $A(c)$ is linear in c, i.e., $A(c^{(\nu+1)}) = A_0 + \sum_{j=1}^n c_j^{(\nu+1)} A_j$. Once $c^{(\nu+1)}$ is obtained, the skew-symmetric matrix $\tilde{K}^{(\nu)}$ (and, hence, the matrix $K^{(\nu)}$) can be determined from the off-diagonal equations of (15). In fact,

$$\tilde{K}_{ij}^{(\nu)} = \frac{{q_i^{(\nu)}}^T A(c^{(\nu+1)}) q_j^{(\nu)}}{\lambda_i - \lambda_j}, \tag{21}$$

for $1 \le i < j \le n$. In this way, the equation (14) is completely solved.

In the classical Newton method the new iterate $x^{(\nu+1)}$ is "lifted up" naturally along the y-axis to the the point $(x^{(\nu+1)}, f(x^{(\nu+1)}))$ from which the next tangent line will begin. We note that $(x^{(\nu+1)}, f(x^{(\nu+1)}))$ is a point on the graph of f. Analogously, we now need a way to "lift up" the point $A(c^{(\nu+1)}) \in \mathcal{A}$ to a point $X^{(\nu+1)} \in \mathcal{M}_e(\Lambda)$. The difficulty here is that there is no obvious coordinate axis to follow. One possible way of this lifting can be motivated as follows: It is clear that solving the PIEP2 is equivalent

to finding an intersection of the two sets $\mathcal{M}_e(\Lambda)$ and $\mathcal{A}$. Suppose all the iterations are taking place near a point of intersection. Then we should have

$$X^{(\nu+1)} \approx A(c^{(\nu+1)}). \tag{22}$$

But from (14), we also should have

$$A(c^{(\nu+1)}) \approx e^{-K^{(\nu)}} X^{(\nu)} e^{K^{(\nu)}}. \tag{23}$$

High accuracy calculation of the exponential matrix $e^{K^{(\nu)}}$ in (23) is expensive and is not needed. So, instead, we define the Cayley transform

$$R^{(\nu)} := (I + \frac{K^{(\nu)}}{2})(I - \frac{K^{(\nu)}}{2})^{-1} \tag{24}$$

which happens to be the $(1,1)$ Padé approximation of the matrix $e^{K^{(\nu)}}$. It is well known that $R^{(\nu)} \in \mathcal{O}(n)$, and that

$$R^{(\nu)} \approx e^{K^{(\nu)}} \tag{25}$$

if $\|K^{(\nu)}\|$ is small. Motivated by (22) and (23), we now define

$$X^{(\nu+1)} := {R^{(\nu)}}^T X^{(\nu)} R^{(\nu)} \in \mathcal{M}_e(\Lambda) \tag{26}$$

and the next iteration is ready to begin. It is interesting to note that

$$X^{(\nu+1)} \approx {R^{(\nu)}}^T e^{K^{(\nu)}} A(c^{(\nu+1)}) e^{-K^{(\nu)}} R^{(\nu)} \approx A(c^{(\nu+1)}) \tag{27}$$

represents what we mean by a lifting of the matrix $A(c^{(\nu+1)})$ from the affine subspace $\mathcal{A}$ to the surface $\mathcal{M}_e(\Lambda)$.

In summary, we realize that (14) is the equation for finding the $\mathcal{A}$-intercept of a tangent line passing $X^{(\nu)}$ and that (26) is the equation for lifting the $\mathcal{A}$-intercept to a point on $\mathcal{M}_e(\Lambda)$. The above process is identical to Method III proposed in [85], but the geometric meaning should be clearer now. We may thus say that Method III is precisely equivalent to the Newton method applied to $f(x) = 0$, for some specified $f(x)$. In [85] Method III is proved to converge quadratically.

The lift can also be done by using the Wielandt-Hoffman theorem (See Theorem 4.2 and [44]). More specifically, we may take the lift to be the nearest point on $\mathcal{M}_e(\Lambda)$ to $A(c^{(\nu+1)})$. It can be shown that $X^{(\nu+1)}$ must be given by

$$X^{(\nu+1)} := \hat{Q}^{(\nu+1)} \tilde{\Lambda}^{(\nu+1)} {\hat{Q}^{(\nu+1)}}^T, \tag{28}$$

provided that

$$A(c^{(\nu+1)}) = \hat{Q}^{(\nu+1)} \Sigma^{(\nu+1)} {\hat{Q}^{(\nu+1)}}^T$$

is the spectral decomposition of $A(c^{(\nu+1)})$ and that $\tilde{\Lambda}^{(\nu+1)}$ is the diagonal matrix whose elements are a rearrangement of those of Λ in the same ordering as those in $\Sigma^{(\nu+1)}$. It

can be shown that the rate of convergence for this case is still quadratic [183, Theorem 2.4].

The above tangent-and-lift idea can further be explored to refine other types of parameterized inverse eigenvalue problems. Once such success is in the refinement for the inverse Toeplitz eigenvalue problem [52]. The idea leads to the introduction of three coordinate-free lifting schemes that can handle multiple eigenvalue cases in a way that methods in [85] cannot. See the SIEP1 in §3. The same idea can also be applied to the inverse singular value problem [47]. See also the PISVP in §3. It is worth pursuing to generalize this idea to other types of inverse eigenvalue problem, especially to the much more complicated case when $A(c)$ is not linear in c.

3. Structured Inverse Eigenvalue Problem.

3.1. Generic Form. Perhaps the most focused inverse eigenvalue problems are the structured problem where a matrix with a specified *structure* as well as a designated spectrum is sought after. A generic structured inverse eigenvalue problem may be stated as follows:

(SIEP) *Given scalars* $\{\lambda_1, \ldots, \lambda_n\} \in \mathbf{F}$, *find* $X \in \mathcal{N}$ *which consists of specially structured matrices such that* $\sigma(X) = \{\lambda_1, \ldots, \lambda_n\}$.

By demanding X to belong to $\mathcal{N}$, where a structure is defined, the SIEP is required to meet both the spectral constraint and the structural constraint. The structural constraint usually is imposed due to the realizability of the underlying physical system.

3.2. Variations. Many types of structures have been considered for the SIEP, among which the following problems are most interesting:

(SIEP1) $\mathbf{F} = \mathbb{R}$ and $\mathcal{N} = \{$All Toeplitz matrices in $\mathcal{S}(n)\}$ [66, 86, 127, 184].

(SIEP2) $\mathbf{F} = \mathbb{R}$ and $\mathcal{N} = \{$All per-symmetric Jacobi matrices in $\mathcal{S}(n)\}$ [26, 28, 116].

(SIEP3) $\mathbf{F} = \mathbb{R}$ and $\mathcal{N} = \{$All nonnegative matrices in $\mathcal{S}(n)\}$ [45].

(SIEP4) $\mathbf{F} = \mathbb{R}$ and $\mathcal{N} = \{$All nonnegative matrices in $\mathcal{R}(n)\}$.

(SIEP5) $\mathbf{F} = \mathbb{C}$ and $\mathcal{N} = \{$All row-stochastic matrices in $\mathcal{R}(n)\}$ [54].

The spectra of structured matrices may also be structured. So sometimes additional spectral information is given. For example, the following problems have been discussed extensively in the literature.

(SIEP6a) Given scalars $\{\lambda_1, \ldots, \lambda_n\}$ and $\{\mu_1, \ldots, \mu_{n-1}\} \subset \mathbb{R}$ that satisfy the interlacing property $\lambda_i \leq \mu_i \leq \lambda_{i+1}$ for $i = 1, \ldots, n-1$, find a Jacobi matrix J so that $\sigma(J) = \{\lambda_1, \ldots, \lambda_n\}$ and $\sigma(\tilde{J}) = \{\mu_1, \ldots, \mu_{n-1}\}$ where $\tilde{J}$ is the leading $(n-1) \times (n-1)$ principal submatrix of J [27, 28, 92, 100, 108, 116, 117].

The SIEP6a enjoys an interesting physical interpretation in vibrations. It may be regarded as identifying the spring configurations of an undamped system from its spectrum and the spectrum of the constrained system where the last mass is restricted to have no motion [92]. When the damper comes into the system, the question becomes an inverse eigenvalue problem for symmetric quadratic pencil:

(SIEP6b) Given scalars $\{\lambda_1, \ldots, \lambda_{2n}\}$ and $\{\mu_1, \ldots, \mu_{2n-2}\} \in \mathbb{C}$, find tridiagonal symmetric matrices C and K such that the determinant $\det(Q(\lambda))$ of the λ-matrix

$Q(\lambda) = \lambda^2 I + \lambda C + K$ has zeros precisely $\{\lambda_1, \ldots, \lambda_{2n}\}$ and $\det(\tilde{Q}(\lambda))$ has zeros precisely $\{\mu_1, \ldots, \mu_{2n-2}\}$ where $\tilde{Q}(\lambda)$ is obtained by deleting the last row and the last column of $Q(\lambda)$ [161].

Generalizations and variations of SIEP6a include the following problems. Algorithms developed for the SIEP6a can easily be adopted to solve these problems.

(SIEP7) Given scalars $\{\lambda_1, \ldots, \lambda_n\}$ and $\{\mu_1, \ldots, \mu_{n-1}\} \subset \mathbb{R}$ satisfying $\lambda_i \leq \mu_i \leq \lambda_{i+1}$ for $i = 1, \ldots, n-1$, and a positive number β, find a periodic Jacobi matrix J of the form

$$J = \begin{bmatrix} a_1 & b_1 & & & & b_n \\ b_1 & a_2 & b_2 & & & 0 \\ 0 & b_2 & a_3 & & & 0 \\ \vdots & & & \ddots & & \\ & & & & a_{n-1} & b_{n-1} \\ b_n & & & & b_{n-1} & a_n \end{bmatrix}$$

so that $\sigma(J) = \{\lambda_1, \ldots, \lambda_n\}$ and $\sigma(\tilde{J}) = \{\mu_1, \ldots, \mu_{n-1}\}$ where $\tilde{J}$ is the leading $(n-1) \times (n-1)$ principal submatrix of J, and $\prod_1^n b_i = \beta$ [26, 27, 76].

(SIEP8) Given scalars $\{\lambda_1, \ldots, \lambda_n\}$ and $\{\mu_1, \ldots, \mu_n\} \subset \mathbb{R}$ satisfying $\lambda_i \leq \mu_i \leq \lambda_{i+1}$ for $i = 1, \ldots, n$ and $\lambda_{n+1} = \infty$, construct Jacobi matrices J and $\overline{J}$ so that $\sigma(J) = \{\lambda_1, \ldots, \lambda_n\}$ and $\sigma(\overline{J}) = \{\mu_1, \ldots, \mu_n\}$ where J and $\overline{J}$ differ only in the (n, n) position [28].

(SIEP9) Given a Jacobi matrix $J_n \in \mathcal{R}(n)$ and distinct scalars $\{\lambda_1, \ldots, \lambda_{2n}\} \subset \mathbb{R}$, construct a Jacobi matrix $J_{2n} \in \mathcal{R}(2n)$ so that $\sigma(J_{2n}) = \{\lambda_1, \ldots, \lambda_{2n}\}$ and that the leading $n \times n$ principal submatrix of J_{2n} is exactly J_n [117].

Obviously, banded matrices are another important structure that frequently arises from applications. A symmetric banded matrix with bandwidth $2r+1$ contains $\sum_{k=n-r}^{n} k$ entries. Thus one type of inverse eigenvalue problem for banded matrices is as follows:

(SIEP10) Given scalars $\{\lambda_1^{(k)}, \ldots, \lambda_k^{(k)}\}$, $k = n-r, \ldots, n$ satisfying the interlacing property $\lambda_i^{(k)} \leq \lambda_i^{(k-1)} \leq \lambda_{i+1}^{(k)}$ for $i = 1, \ldots, k-1$ and $k = n-r+1, \ldots, n$, construct a symmetric banded matrix A with bandwidth $2r+1$ such that each leading $k \times k$ principal submatrix of A has spectrum precisely $\{\lambda_1^{(k)}, \ldots, \lambda_k^{(k)}\}$ [19, 25, 27, 136, 160].

Many different types of structural constraints can be imposed upon an inverse eigenvalue problem. For instance, the structure could be an unitary Hessenberg matrix [3], a block Jacobi matrix [204], or others [55, 83, 96, 160, 182]. Readers are referred to the literature for the cause of why these special structures are of interest.

Similar to the PIEP, there is also the parameterized inverse singular value problem:

(PISVP) Given a family of matrices $A(c) \in \mathbb{R}^{m \times n}$, with $c = [c_1, \ldots, c_n] \in \mathbb{R}^n$, $m \geq n$ and a set of nonnegative real values $\{\sigma_1, \ldots, \sigma_n\}$, find a parameter c such that the singular values of $A(c)$ are precisely $\{\sigma_1, \ldots, \sigma_n\}$.

Note that the significant difference between the PIEP and the PISVP is that the matrices involved in the PISVP can be rectangular. Since eigenvalues of the symmetric matrix $\begin{bmatrix} 0 & A(c) \\ A(c)^T & 0 \end{bmatrix}$ are plus and minus of singular values of the matrix $A(c)$, the

PISVP can be solved by conversion to a special parameterized SIEP. In fact, each of the inverse problems discussed in this paper for eigenvalues have a counterpart problem for singular values. Just like many of the inverse eigenvalue problems, the existence question for the inverse singular value problem remains open. Indeed, to our knowledge, this interesting research topic has never been thoroughly explored before. The first work seems to be in the paper [47] where two numerical methods for the case when $A(c)$ is linear in c are proposed.

Some of the entries in a SIEP could also be specified beforehand. Sometimes a certain submatrix is specified [65, 171]. Sometimes the characteristic polynomial is prescribed [172] Here is an example that plays an important role under the notion of majorization [6, 135].

(SHIEP) Given two sets of real values $\{a_1, \ldots, a_n\}$ and $\{\lambda_1, \ldots, \lambda_n\}$, construct a Hermitian matrix H with diagonal $\{a_1, \ldots, a_n\}$ such that $\sigma(H) = \{\lambda_1, \ldots, \lambda_n\}$.

3.3. Solvability Issues. The SIEP1 is the well-known inverse Toeplitz eigenvalue problem. This seemingly easy problem has intrigued researchers for years [66, 86]. We first note that eigenstructure of Toeplitz (and, in fact, centro-symmetric) matrices is quite special [39, 114]. In particular, there are exactly $\lfloor \frac{n}{2} \rfloor$ skew-symmetric eigenvectors and $\lceil \frac{n}{2} \rceil$ symmetric eigenvectors where we say a vector v is symmetric if $Jv = v$, and skew-symmetric if $Jv = -v$, and J is the "backward identity" [118]. The corresponding eigenvalues are said, respectively, to have odd and even *parity*. Only very recently Landau [127] reported that the solvability issue was finally completely settled. More specifically, let $T(c_1, \ldots, c_n)$ denote the symmetric Toeplitz matrix whose first row is $(c_1, \ldots, c_n)$. Consider the mapping $\phi : \mathbb{R}^{n-2} \longrightarrow \mathbb{R}^{n-2}$ defined by

$$\phi(t_3, \ldots, t_n) = (y_2, \ldots, y_{n-1}) \tag{29}$$

with

$$y_i = -\frac{\lambda_i}{\lambda_1}, \ i = 2, \ldots, n-1, \tag{30}$$

where $\lambda_1 \leq \ldots \leq \lambda_n$ are the eigenvalues of the Toeplitz matrix $T(0, 1, t_3, \ldots, t_n)$. Note that $\sum_{i=1}^{n} \lambda_i = 0$. Hence $\lambda_1 < 0$, else all eigenvalues vanish, and $\lambda_n = -\sum_{i=1}^{n-1} \lambda_i$. It follows that the range of ϕ resides in the simplex

$$\Delta := \{(y_2, \ldots, y_{n-1}) | -1 \leq y_2 \leq \ldots \leq y_{n-1}, \ y_2 + \ldots + y_{n-2} + 2y_{n-1} \leq 1\}. \tag{31}$$

A matrix $T(c_1, \ldots, c_n)$ is said to be *regular* provided that every principal submatrix $T(c_1, \ldots, c_k)$, $1 \leq k \leq n$, has the property that its eigenvalues are distinct and alternate parity with the largest one having even parity. Landau argues that the set

$$\mathcal{F} := \{M = T(0, 1, t_3, \ldots, t_n) | t_3, \ldots, t_n \in \mathbb{R}, \text{ and } M \text{ is regular}\}$$

is not empty and proves the following theorem.

THEOREM 3.1. *The restriction of ϕ (See definition (29)) to values $(t_3, \ldots, t_n) \in \mathbb{R}^{n-2}$ such that $T(0, 1, t_3, \ldots, t_n) \in \mathcal{F}$, a special subclass of Toeplitz matrices, is a surjective map onto Δ.*

Any given arbitrary $y_1 \leq y_2 \leq \ldots \leq y_n$ corresponds after shifting and scaling to a unique point on Δ. It follows from Theorem 3.1 that the SIEP1 is always solvable. The argument, using the topological degree theory, unfortunately was not constructive. The search for an efficient way of constructing a Toeplitz matrix is still not completely satisfactory.

There is a wealth of applications involving nonnegative or positive matrices. Many references concerning properties of nonnegative or positive matrices are available. See, for example, [16, 118, 140]. In the understanding of nonnegative matrices, perhaps one of the most significant results is the Perron-Frobenius theory. For reference, we only state the first part of the Perron-Frobenius theorem for irreducible matrices [16, 118]:

THEOREM 3.2. *Suppose the matrix $A \in \mathbb{R}^{n\times n}$ is nonnegative and irreducible. Then*

1. *The spectral radius $\rho(A)$ of A is a positive eigenvalue, called the Perron value, of A;*
2. *There is positive vector, call the Perron vector, such that $Ax = \rho(A)x$;*
3. *$\rho(A)$ has algebraic multiplicity 1.*

Since the Perron-Frobenius theorem concerns the spectrum of nonnegative matrices, there has been great interest in studying the inverse problems, i.e., the SIEP3, the SIEP4 and the SIEP5 [13, 16, 29, 75, 82, 84, 134, 140, 150, 175]. Thus far, most of the discussion in the literature for the SIEP3 or the SIEP4 have been centered around the establishment of a sufficient or a necessary condition to qualify whether a given set of values is the spectrum of a non-negative matrix [16, 75, 140]. For example, let s_k denote the k-th moment

$$s_k = \sum_{i=1}^{n} \lambda_i^k.$$

The following necessary condition is due to Loewy and London [134].

THEOREM 3.3. *If $\lambda_1, \ldots, \lambda_n$ are eigenvalues of an $n \times n$ nonnegative matrix, then*

$$s_k^m \leq n^{m-1} s_{km}$$

for all $k, m = 1, 2, \ldots$.

A nonnegative matrix M such that M^m is positive for some nonnegative integer m is called a primitive matrix. A fundamental result due to Boyle and Handelman concerns the inverse eigenvalue problem for primitive matrices [29].

THEOREM 3.4. *Let S be a subring of $\mathbb{R}$ containing the unity 1. If $\{\lambda_1, \ldots, \lambda_n\}$ forms the nonzero spectrum of a primitive matrix (whose size could be larger than n) over S, then the following conditions necessarily hold:*

1. *One of $\lambda_1, \ldots, \lambda_n$ is the Perron value, i.e., there exists i such that $\tilde{\lambda} := \lambda_i > |\lambda_j|$ for $j \neq i$,*
2. *Coefficients of $\prod_{i=1}^{n}(t - \lambda_i)$ are in S.*
3. *If $S = \mathbf{Z}$ the ring of all integers, then*

$$\sum_{k|m} \mu(\frac{m}{k}) s_k \geq 0$$

for all $m = 1, 2, \ldots$, *where* μ *is the Möbius function for natural numbers defined by*

$$\mu(d) = \begin{cases} 1 & \text{if } d = 1, \\ 0 & \text{if } d \text{ is not square free,} \\ (-1)^t & \text{if } d \text{ is a product of } t \text{ distinct primes.} \end{cases}$$

4. *If* $S \neq \mathbf{Z}$, *then for all* $k, m = 1, 2, \ldots$,

$$\begin{aligned} s_k &\geq 0 \\ s_k > 0 &\Rightarrow s_{km} > 0. \end{aligned}$$

Conversely, if $\{\lambda_1, \ldots, \lambda_n\}$ *satisfies the above conditions and if one of its subsets containing* $\tilde{\lambda}$ *is the nonzero spectrum of a primitive matrix over* S, *then* $\{\lambda_1, \ldots, \lambda_n\}$ *itself is the nonzero spectrum of a primitive matrix over* S.

Very few of these theoretical results are ready for implementation to actually compute this matrix. The most constructive result we have seen is the sufficient condition studied by Solues [175]. But the condition there is still limited because the construction depends on the specification of the Perron vector — in particular, the components of the Perron eigenvector need to satisfy certain inequalities in order for the construction to work.

The SIEP5 is closely related to the SIEP4 by the following theorem [140]:

THEOREM 3.5. *If* A *is a nonnegative matrix with positive maximal eigenvalue* r *and a positive maximal eigenvector* x, *then* $D^{-1}r^{-1}AD$ *is a stochastic matrix where* $D := diag\{x_1, \ldots, x_n\}$.

Thus once a SIEP4 is solved and if the eigenvector corresponding to the positive maximal eigenvalue is positive, then we will have solved the SIEP5 by a diagonal similarity transformation.

On the other hand, it is worthy to mention an existence theorem by Karpelevič [121, 140]. Karpelevič completely characterized the set Θ_n of points in the complex plane that are eigenvalues of stochastic $n \times n$ matrices. In particular, the region Θ_n is symmetric about the real axis. It is contained within the unit circle and its intersections with the unit circle are points $z = e^{2\pi ia/b}$ where a and b run over all integers satisfying $0 \leq a < b \leq n$. The boundary of Θ_n consists of these intersection points and of curvilinear arcs connecting them in circular order. These arcs are characterized by specific parametric equations whose formulas are too complicated to described here but can be found in [121, 140]. For example, a complex number λ is an eigenvalue for a 4×4 stochastic matrix if and only if it belongs to a region Θ_4 shown in Figure 4. Complicated though it may seem, it should be noted that the Karpelevič theorem characterizes only one complex value at a time and does not provide further insights into when two or more points in Θ_n are eigenvalues of the *same* stochastic matrix. Minc [140] distinctively called the problem SIEP5, where the entire spectrum is given, *the inverse spectrum problem* for row-stochastic matrices.

A survey of the basic theory and numerical methods for SIEP6a can be found in the article by Boley and Golub [27]. In particular, we have the following result.

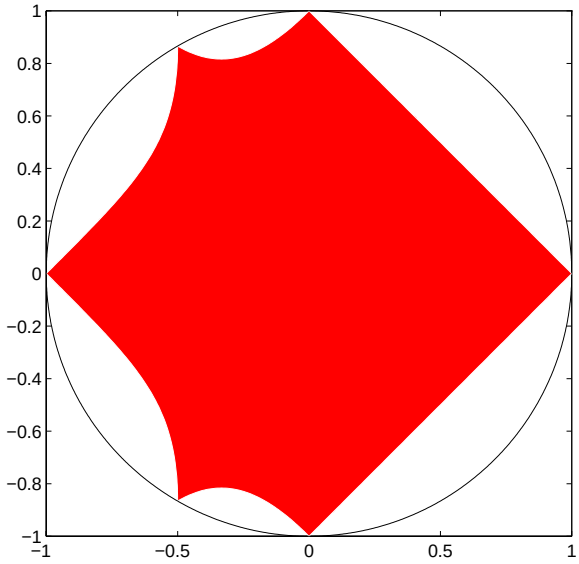

FIG. 4. Θ_4 *by the Karpelevič Theorem.*

THEOREM 3.6. *Suppose all the* μ_i, $i = 1, \ldots, n-1$, *are distinct. Then*

1. *The SIEP6a is guaranteed to have a solution.*
2. *The SIEP6a can be solved in finitely many steps.*

The following theorem, due to Duarte [70], generalizes the SIEP6a to a much larger class of matrices.

THEOREM 3.7. *Suppose the given real numbers* $\{\lambda_1, \ldots, \lambda_n\}$ *and* $\{\mu_1, \ldots, \mu_{n-1}\}$ *satisfy the interlacing property* $\lambda_i \leq \mu_i \leq \lambda_{i+1}$ *for* $i = 1, \ldots, n-1$. *Let* k *be a fix integer between* 1 *and* n *and let* Γ *be a specified collection of unordered pairs of nodes* (i_s, i_t) *for* $1 \leq i_s, i_t \leq n$. *Then there exists a Hermitian matrix* A *such that* $\sigma(A) = \{\lambda_1, \ldots, \lambda_n\}$, $\sigma(A_k) = \{\mu_1, \ldots, \mu_{n-1}\}$ *where* A_k *is the* $(n-1) \times (n-1)$ *submatrix of* A *by deleting the* k*-th row and column, and* $a_{ij} = 0$ *whenever* $i \neq j$ *and* (i, j) *is not an edge of* Γ.

Ram and Elhay [161] give an account of solutions to the SIEP6b.

THEOREM 3.8. *If the given eigenvalues are all distinct, then the SIEP6b is always solvable over the complex field and there are at most* $2^n(2n-3)!/(n-2)!$ *different solutions.*

In contrast to Theorem 3.6, Ram and Elhay also show that apart from finding the roots of certain polynomials, the problem can be solved in a finite number of steps. For physical realizability, however, the matrices C and K in the SIEP6b should further be required to be real-valued, to have positive diagonal elements and negative off-diagonal elements, and to be weakly diagonally dominant. So far as we know, there is no general result in this regard. For the special case of a simply connected mass-spring system, the damping matrix C is of rank one, i.e., $C = cc^T$ for some column vector c. Veselić [185, 186] shows in this case that one set of eigenvalues, closed under complex conjugation, suffice to determine a unique solution.

The existence question for the SHIEP can be completely settled by the Schur-Horn theorem [118]:

THEOREM 3.9. *(Schur-Horn Theorem)*

1. Let H be a Hermitian matrix. Let $\lambda = [\lambda_i] \in R^n$ and $a = [a_i] \in R^n$ denote the vectors of eigenvalues and diagonal entries of H, respectively. If the entries are arranged in increasing order $a_{j_1} \leq \ldots \leq a_{j_n}$, $\lambda_{m_1} \leq \ldots \leq \lambda_{m_n}$, then

$$\sum_{i=1}^{k} a_{j_i} \geq \sum_{i=1}^{k} \lambda_{m_i}, \tag{32}$$

for all $k = 1, 2, \ldots, n$ with equality for $k = n$.

2. Given any $a, \lambda \in R^n$ satisfying (32), there exists a Hermitian matrix H with eigenvalues λ and diagonal entries a.

The notion of (32) is also known as *a majorizing* λ, which has arisen as the precise relationship between two sets of numbers in many areas of disciplines, including matrix theory and statistics. The theorem asserts that $\{a_1, \ldots, a_n\}$ majorizes $\{\lambda_1, \ldots, \lambda_n\}$ if and only if there exists a Hermitian matrix H with eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$ and diagonal entries $\{a_1, \ldots, a_n\}$.

3.4. Numerical Methods. The SIEP1 can be written as a PIEP2, i.e., we can write $T(t_1, \ldots, t_n) = \sum_{i=1}^{n} t_i T_i$ where $T_i = T(0, \ldots, 0, 1, 0, \ldots, 0)$ is the fixed Toeplitz matrix of which the first row is identically zero except 1 at the i-th entry. Thus any of the locally convergent methods such as the Newton method described in [85] can be applied. Other types of iterative methods can be found in [52, 128, 129], and most recently [184]. The idea in the latter approach involves simultaneous approximation of the matrix as well as its eigenvectors.

As an alternative approach the author has proposed a continuous realization procedure to solve the SIEP1 based on the projected gradient idea. In the projected gradient approach the goal is to minimize the distance between $Q^T \Lambda Q$ and its projection onto the subspace $\mathcal{N}$ of all symmetric Toeplitz matrices while allowing the variable matrix Q to change among $\mathcal{O}(n)$. It can be shown that the projected gradient of the objective function can be formulated explicitly. This gives rise to the construction of a descent flow that can be followed numerically. The explicit form also facilitates the computation of the second-order optimality conditions. A full account of discussion on this differential equation approach and its generalization can be found in [42, 44, 48]. Among the several alternatives, we suggest in particular this autonomous initial value problem

$$\begin{cases} \frac{dX}{dt} &= [X, k(X)] \\ X(0) &= \Lambda \end{cases} \tag{33}$$

where $[A, B] := AB - BA$ denotes the Lie bracket and $k(X) = [k_{ij}(X)]$ is the Toeplitz annihilator matrix defined by

$$k_{ij}(X) := \begin{cases} x_{i+1,j} - x_{i,j-1} & \text{if } 1 \leq i < j \leq n, \\ 0 & \text{if } 1 \leq i = j \leq n, \\ x_{i,j-1} - x_{i+1,j} & \text{if } 1 \leq j < i \leq n. \end{cases} \tag{34}$$

It has been observed that the solution orbit of (33) always converges to a geometrically isolated equilibrium point. Thus it seems to suggest a global method for the SIEP1.

A surprising discovery, after fully analyzing the stability of equilibrium points for the case $n = 3$ recently, reveals that the dynamical system does possess periodic solutions that have never been detected numerically before [50]. What has happened is that these periodic orbits are orbitally unstable [56] and thus, due to round-off errors, no integral curves can ever be attracted to them. It is further observed that the isospectral properties have never been lost despite this instability. So the ODE approach does offer a reasonable global method. A more rigorous mathematical analysis is missing to suggest a constructive proof of existence for the SIEP1. On the other hand, the study in [52] sheds light on how the parity of eigenvalues affects the solvability. The periodic solution mentioned earlier turns out to have the wrong parity assignment. It appears plausible, according to Theorem 3.1 and our experience, that a suitable parity assignment and the ODE may give rise to a globally convergent method. See also the discussion in [184].

Using a similar idea of gradient flow to systematically reduce the distance between the isospectral surface and the cone of non-negative matrices, we can formulate a nice numerical method for solving the SIEP4 [45, 54]. We are not aware of any other techniques for the SIEP4, so we describe the basic idea of the continuation approach below.

Since the emphasis in a SIEP4 is on the positivity of the entries, not on the symmetry of the matrix, it is likely that the given eigenvalues $\lambda_1, \ldots, \lambda_n$ are complex valued, though closed under conjugation. Let Λ now denote a real-valued matrix, possibly tridiagonal, whose spectrum is $\{\lambda_1, \ldots, \lambda_n\}$. Note that matrices in the set

$$\mathcal{M}(\Lambda) := \{P\Lambda P^{-1} | P \in \mathcal{R}(n) \text{ is nonsingular}\} \tag{35}$$

obviously are isospectral to Λ. Let

$$\pi(\mathbb{R}^n_+) := \{B \circ B | B \in \mathcal{R}(n)\} \tag{36}$$

denote the cone of all nonnegative matrices where $\circ$ means the Hadamard product of matrices. The goal is to find the intersection of $\mathcal{M}(\Lambda)$ and $\pi(\mathbb{R}^n_+)$. Such an intersection, if it exists, results in a nonnegative matrix isospectral to Λ. We formulate the inverse spectrum problem as finding the shortest distance between $\mathcal{M}(\Lambda)$ and $\pi(\mathbb{R}^n_+)$:

$$\text{minimize } F(P, R) := \frac{1}{2}\|P\Lambda P^{-1} - R \circ R\|^2. \tag{37}$$

Note that the variable P in (37) resides in the open set of nonsingular matrices whereas R is simply a general matrix in $\mathbb{R}^{n \times n}$. Since the optimization is over an unbounded open domain, it is possible that the minimum does not exist. The gradient ∇F of the objective function F is given by:

$$\nabla F(P, R) = \Big((\Delta(P, R)M(P)^T - M(P)^T\Delta(P, R))P^{-T}, -2\Delta(P, R) \circ R\Big), \tag{38}$$

where

$$\begin{aligned} M(P) &:= PJP^{-1} \\ \Delta(P, R) &:= M(P) - R \circ R. \end{aligned}$$

Therefore, the flow $(P(t), R(t))$ defined by the differential equations

$$\frac{dP}{dt} := [M(P)^T, \Delta(P,R)]P^{-T} \tag{39}$$

$$\frac{dR}{dt} := 2\Delta(P,R) \circ R, \tag{40}$$

where $[\cdot,\cdot]$ again denotes the Lie bracket of two matrices, signifies in fact the steepest descent flow for the objective function F. It is worth noting that the two matrices P and R are used, respectively, as *coordinates* to describe the isospectral matrices and nonnegative matrices. We may have used more dimensions of variables than necessary to describe the underlying matrices, but that does no harm. The involvement of P^{-1} in the differential system (39) and (40), however, is worrisome. To remedy this problem, we observe that the coefficients of the vector field in (39) and (40) are analytic in t. By the well known Cauchy-Kovalevskaya Theorem [162] it follows that $P(t)$ is analytic in t as well. We may thus further describe the motion of $P(t)$ in terms of its analytic singular value decomposition [32]:

$$P(t) = X(t)S(t)Y(t)^T \tag{41}$$

where $X(t)$ and $Y(t)$ are orthogonal and $S(t)$ is diagonal. The governing equations for $X(t), S(t)$ and $Y(t)$ can be obtained from the following [139, 190]: Differentiating both sides of (41), we obtain the following equation after some suitable multiplications :

$$X^T\frac{dP}{dt}Y = X^T\frac{dX}{dt}S + \frac{dS}{dt} + S\frac{dY^T}{dt}Y. \tag{42}$$

Define

$$Q(t) := X^T\frac{dP}{dt}Y, \tag{43}$$

$$Z(t) := X^T\frac{dX}{dt}, \tag{44}$$

$$W(t) := \frac{dY^T}{dt}Y. \tag{45}$$

Note that $Q(t)$ is known from (39) where the inverse of $P(t)$ is calculated from

$$P^{-1} = YS^{-1}X^T. \tag{46}$$

The diagonal entries of $S = \mathrm{diag}\{s_1, \ldots, s_n\}$ provide us with information about the proximity of $P(t)$ to singularity. On the other hand, comparing the diagonal entries on both sides of (42), we obtain the differential equation for $S(t)$:

$$\frac{dS}{dt} = \mathrm{diag}(Q), \tag{47}$$

since both $Z(t)$ and $W(t)$ are skew symmetric. Comparing the off-diagonal entries on both sides of (42), we obtain the linear system:

$$q_{jk} = z_{jk}s_k + s_j w_{jk}, \tag{48}$$

$$-q_{kj} = z_{jk}s_j + s_k w_{jk}. \tag{49}$$

If $s_k^2 \neq s_j^2$, we can solve this system and obtain

$$z_{jk} = \frac{s_k q_{jk} + s_j q_{kj}}{s_k^2 - s_j^2}, \tag{50}$$

$$w_{jk} = \frac{s_j q_{jk} + s_k q_{kj}}{s_j^2 - s_k^2} \tag{51}$$

for all $j > k$. Even in the situation where $s_k^2 = s_j^2$, the existence of an ASVD guarantees that the equations must be consistent and so z_{jk} and w_{jk} still can be obtained. Detailed discussion of this case can be found in [190]. Once $Z(t)$ and $W(t)$ are known, the differential equations for $X(t)$ and $Y(t)$ are given, respectively, by

$$\frac{dX}{dt} = XZ, \tag{52}$$

$$\frac{dY}{dt} = YW^T. \tag{53}$$

By now we have developed a complete coordinate system $(X(t), S(t), Y(t), R(t))$ for matrices in $\mathcal{M}(\Lambda) \times \pi(\mathbb{R}_+^n)$. The differential equations (47), (52), (53) and (40) with the relationship (41) describe how these coordinates should be varied in t to produce the steepest descent flow for the objective function F. This flow is ready to be integrated numerically by many initial value problem solvers. We think those special-purpose integrators developed in [40, 68, 123, 166] are of particular value to this continuous approach. See also [88, 89] for general discussion in this regard. We shall review some special tools in §3.4.1. By using these available solvers, we have thus developed a numerical method for solving the inverse eigenvalue problems for nonnegative matrices.

While the Schur-Horn theorem is regarded as classical by now, most of the known proofs have been non-constructive or difficult to implement. It has been an interesting and challenging inverse eigenvalue problem to develop a numerical way of constructing such a Hermitian matrix. See, for example, [112, 141]. Recently an algorithm using the continuation idea has been proposed with some success [51]. The solution to the differential equation

$$\frac{dX}{dt} = [X, [\mathrm{diag}(X) - \mathrm{diag}\{a_1, \ldots, a_n\}, X]] \tag{54}$$

from any initial point $X_0 \in \mathcal{M}_e(\Lambda)$ defines an isospectral flow on $\mathcal{M}_e(\Lambda)$ whose limit point is a solution of the SHIEP. The argument for convergence of this method also provides a constructive proof of the theorem. A more recent iterative method can be found in [201].

3.4.1. Special Tools for Continuation. We have observed in the above some advantages of using continuous realization methods to tackle difficult inverse eigenvalue problems. A key issue in the continuation method is that the solution flow stays on a certain invariant manifold. When integrating these differential systems by numerical methods, the loss of that invariance becomes significant because the properties that

we intend to acquire from that manifold might have been lost. It is therefore important to study numerical methods for the integration of these dynamical system that maintain the corresponding invariance. As far as the inverse eigenvalue problems are concerned, the invariance needed to be preserved is either the isospectral property or the orthogonality.

In addition to the very vigorous ongoing research in the area of differential algebraic equations, there has been considerable interest in recent years in structure-preserving methods for systems like the ones we have discussed. To mention a few examples, there are numerical Hamiltonian methods by Sanz-Serna [166, 167], automatic and projected unitary schemes by Dieci, Russel and Van Vleck [68], gradient algorithms by Helmke and Moore [111], modified Gauss-Legendre Runge-Kutta methods for isospectral flows by Calvo, Iserles and Zanna [37, 38], systolic algorithms and adaptive neural networks by Dehaene [64], and methods of iterated commutators of ordinary differential equations on Lie groups by Zanna [197]. Most of these results are fairly new. It is not surprising that the current status of these methods is still quite primitive since most discussion is still limited to fixed-step analysis. Any advances of these methods will certainly benefit the computation of the dynamical system that we have proposed and, in return, benefit the applications that we have mentioned.

In our opinion, a more widely applicable and immediately available approach for the parameter dynamics is to apply a standard integrator and regularly replace the approximate solution by an appropriate "projection". For example, suppose Q is an approximate solution satisfying

$$Q^T Q = I + O(h^r)$$

where r represents the order of the numerical method. Let $Q = \tilde{Q}R$ be the unique QR decomposition of Q with $\mathrm{diag}(R) > 0$. Then

$$\tilde{Q} = Q + O(h^r) \tag{55}$$

and $\tilde{Q} \in \mathcal{O}(p,q)$ [88, 68]. The condition $\mathrm{diag}(R) > 0$ is important to ensure the transition of $Q(t)$ is smooth in t [163]. Higham even points out that the optimal replacement is given by the orthonormal polar factor [113]. Furthermore, this factor can be computed by quadratically convergent iteration schemes without significantly degrading the finite time global error bound for the original integrator. We have used this *nonlinear* projection idea alone with traditional variable-order variable-step methods in many of our latest studies. (The ODE Suite [165] is particularly suitable and convenient because of the matrix manipulations involved in the dynamical systems, but any ODE integrator will do.) Numerical experiments indicate that the implementation is almost free of trouble and that the error bound is consistent with that estimated in theory [113].

There is plenty of room for improving the implementation. One common feature in the continuous realization methods is that the desired solution usually appears as the asymptotically stable equilibrium point of the system. Thus it is desirable to develop a fast method that can trace the qualitative behavior efficiently without losing the asymptotically stable equilibrium. This idea of entropy is particularly feasible for the

gradient flows because the objective function naturally serves as a Lyapunov function. In other words, the conventional concept of stability for a numerical ODE method might be relaxed somewhat because we are only interested in the limit point, not the evolution process itself.

Finally, we want to point out that deriving higher order iterative schemes, not in the context of discretization of a differential equation, but in the context of a Newton method, is possible. We have already outlined an idea in the context of the PIEP2. We have also experimented this idea successfully with other types of inverse eigenvalue problems. See, for example, [41, 47, 52]. We certainly can apply similar ideas to other problems.

4. Least Squares Approximation with Spectral Constraint.

4.1. Generic Form. It is known that an inverse eigenvalue problem, especially for the real-valued case, may not necessarily have an exact solution. It is also known that the spectral information, in practice, often is obtained by estimation and hence needs not to be rigorously complied with. That is, there are situations where an approximate solution best in the sense of *least squares* would be satisfactory. In this section we review how the least squares solution can be obtained.

All the problems discussed hitherto have a natural generalization to the least squares formulation. However, recall that any inverse eigenvalue problem has two constraints. Thus depending upon which constraint is to be enforced explicitly, we should clarify two ways of defining a least squares approximation.

One natural way is to measure and to minimize the discrepancy among the eigenvalues, i.e.:

(LSIEPa) *Given a set of scalars* $\{\lambda_1^*, \ldots, \lambda_m^*\} \subset \mathbf{F}$ *($m \leq n$), find a matrix* $X \in \mathcal{N}$ *and a set* $\sigma = \{\sigma_1, \ldots, \sigma_m\}$ *of indices with* $1 \leq \sigma_1 < \ldots < \sigma_m \leq n$ *such that the function*

$$F(X, \sigma) := \frac{1}{2} \sum_{i=1}^{m} (\lambda_{\sigma_i}(X) - \lambda_i^*)^2, \tag{56}$$

where $\lambda_i(X)$, $i = 1, ..., n$, *are eigenvalues of the matrix* X, *is minimized.*

Note that the set of prescribed eigenvalues has cardinality m which might be less than n. Consequently, associated with the LSIEPa for each fixed X is always a combinatorics problem

$$\min_{1 \leq \sigma_1 < \ldots < \sigma_m \leq n} \sum_{i=1}^{m} (\lambda_{\sigma_i}(X) - \lambda_i^*)^2, \tag{57}$$

that looks for the closest match between a subset of spectrum of X and the prescribed eigenvalues.

Another way to formulate the least squares approximation is to measure and to minimize the discrepancy between the matrices, i.e.:

(LSIEPb) *Given a set* $\mathcal{M}$ *whose elements satisfy a certain spectral constraint and a set* $\mathcal{N}$ *that defines a structural constraint, find* $X \in \mathcal{M}$ *that minimizes the function*

$$F(X) := \frac{1}{2} \|X - P(X)\|^2 \tag{58}$$

where $P(X)$ is the projection of X onto $\mathcal{N}$.

The spectral constraint could be, for example, the isospectral surface

$$\mathcal{W}(\Lambda) := \{X \in \mathcal{R}(n) | X = Q^T \Lambda Q, Q \in \mathcal{O}(n)\} \subset \mathcal{S}(n)$$

where the complete spectrum $\Lambda := \text{diag}\{\lambda_1, \ldots, \lambda_n\}$ is given, or the set

$$\mathcal{W}(\Gamma, V) := \{X \in \mathcal{R}(n) \text{ or } \mathcal{S}(n) | XV = V\Gamma\}$$

where only a portion of eigenvalues $\Gamma := \text{diag}\{\lambda_1, \ldots, \lambda_k\}$ and eigenvectors $V := [v_1, \ldots, v_k]$ are given. We shall discuss the latter case in §5, but we mention it here to remind readers of its least squares formulation. Note that if $F(X) = 0$ at a least squares solution, then we have also solved the inverse eigenvalue problem of finding $X \in \mathcal{N}$ that satisfies $\mathcal{M}$. So a general SIEP can be solved through the setup of an LSIEPb. We have already seen a similar setting in the discussion of SIEP4.

For engineering applications, it is mostly the case that the realizability of the physical system is more critical an issue than the accuracy of the eigenvalues. That is, the structural constraint $\mathcal{N}$ has to be enforced in order that the construction of a physical system be realizable whereas a discrepancy in the eigenvalues is sometimes tolerable because often these eigenvalues are an estimate anyway.

4.2. Variations. In the LSIEPa, it should be noted that the number of available parameters for adjusting the matrix X, i.e., the degree of freedom in $\mathcal{N}$, could be different from the dimension n. We mention one special case of LSIEPa where the number ℓ of free parameters might also differ from the number m of the partially prescribed eigenvalues:

(LSIEPa1) $\mathcal{N} = \{A(d) = A_0 + \sum_{i=1}^{\ell} d_i A_i | A_0, A_1, \ldots, A_\ell \in S(n) \text{ given.}\}$, $\mathbf{F} = \mathbb{R}$.

Under the context of LSIEPb, the problem LSIEPa1 has a new face. For a given $\Lambda_m^* := \text{diag}\{\lambda_1^*, \ldots, \lambda_m^*\}$, consider the subset

$$\Gamma := \left\{Q\text{diag}(\Lambda_m^*, \Lambda_c)Q^T \mid Q \in \mathcal{O}(n), \Lambda_c \in \mathcal{D}_{\mathcal{R}}(n-m)\right\} \tag{59}$$

and the affine subspace

$$\mathcal{A} := \{A(d) | d \in \mathbb{R}^\ell\} \tag{60}$$

with $A(d)$ defined in LSIEPa1. Since Γ contains all symmetric matrices in $\mathbb{R}^{n \times n}$ with $\lambda_1^*, ..., \lambda_m^*$ as part of the spectrum, finding the shortest distance between $\mathcal{A}$ and Γ would be another meaningful least squares approximation. We formulate the problem as follows:

(LSIEPb1). Find $d \in \mathbb{R}^\ell$, $Q \in \mathcal{O}(n)$, and $\Lambda_c \in \mathcal{D}_{\mathcal{R}}(n-m)$ such that the function

$$G(d, Q, \Lambda) := \frac{1}{2}\|A(d) - Q\text{diag}\left(\Lambda_m^*, \Lambda_c\right) Q^T\|_F^2, \tag{61}$$

is minimized.

The setting of LSIEPb can have other applications, including :

(LSIEPb2) $\mathcal{M} = \mathcal{W}(\Lambda)$. $\mathcal{N} = \{A\}$ [44].
(LSIEPb3) $\mathcal{M} = \mathcal{W}(\Lambda)$ and $\mathcal{N} = \{\text{All Toeplitz matrices in } \mathcal{S}(n)\}$.
(LSIEPb4) $\mathcal{M} = \mathcal{W}(\Gamma, V)$, $\mathcal{N} = \{A\}$ and $\mathcal{N} = \mathcal{R}(n)$ or $\mathcal{S}(n)$.

In view of the fact that the spectral constraint and the structural constraint are often inconsistent with each other and the fact that the spectral information often is incomplete or inexact, we think that the least squares formulation of inverse eigenvalue problems is a very important area that deserves further study.

4.3. Solvability Issues. At the first glance, the LSIEPa1 and the LSIEPb1 appear to be very different. In particular, it appears that no permutation of eigenvalues is involved in the LSIEPb1 whereas the complementary spectrum Λ_c in the LSIEPb1 is not mentioned in the LSIEPa1. However, a process of implicit sorting is indeed happening inside the LSIEPb1 and Λ_c is somehow settled in the LSIEPa1. In fact, it can be shown that the LSIEPa1 and LSIEPb1 are equivalent in the following sense [41].

THEOREM 4.1. *Suppose (d^*, σ^*) and (d_+, Q_+, Λ_+) are the global minimizers of the LSIEPa1 and the LSIEPb1, respectively. Let $\overline{\sigma^*}$ denote the complement of σ^* over the set $\{1, \ldots, n\}$. Then*

1. *The permutation σ^* solves (57) with $d = d^*$.*
2. *$d^* = d_+$.*
3. *The columns of Q_+ are orthonormal eigenvectors of $A(d^*)$ arranged in such a way that $Q_+^T A(d^*) Q_+ = diag\left(\Lambda_{\sigma^*}(d^*), \Lambda_{\overline{\sigma^*}}(d^*)\right)$.*
4. *$\Lambda_+ = \Lambda_{\overline{\sigma^*}}(d^*)$.*
5. *$F(d^*, \sigma^*) = G(d_+, Q_+, \Lambda_+)$.*

While the formulation of LSIEPa1 is a natural generalization of the conventional inverse eigenvalue problems, the formulation of LSIEPb1 enjoys a simple geometric intuition. We shall see below how this geometric interpretation motivates a numerical method.

The seemingly insignificant LSIEPb2 is closely related to the work by Brockett [31] who relates a number of finite automata to a smooth flow defined by the so called double bracket equation. The answer to its solvability issue can also be interpreted as the important Wielandt-Hoffman theorem [31, 48, 118]. We describe a generic case where all eigenvalues involved are distinct below [44].

THEOREM 4.2. *Assume the given eigenvalues are arranged in the order $\lambda_1 > \lambda_2 > \ldots > \lambda_n$. Let the eigenvalues of the given symmetric matrix A be $\mu_1 > \mu_2 > \ldots > \mu_n$. Then $Q \in \mathcal{O}(n)$ is a local minimizer of the function $\|Q^T \Lambda Q - A\|_F$ if and only if the columns $q_1, \ldots, q_n$ of the matrix Q^T are the normalized eigenvectors of A corresponding respectively to $\mu_1, \ldots, \mu_n$. The solution to the LSIEPb2 is unique and is given by*

$$X = \lambda_1 q_1 q_1^T + \ldots + \lambda_n q_n q_n^T. \tag{62}$$

The sorting properties of eigenvalues acquired in Theorem 4.2 (see also Theorem 4.1) has incited several other important research efforts, including the link between Toda lattice and gradient flow [22] and hence the sorting of eigenvalues observed in the

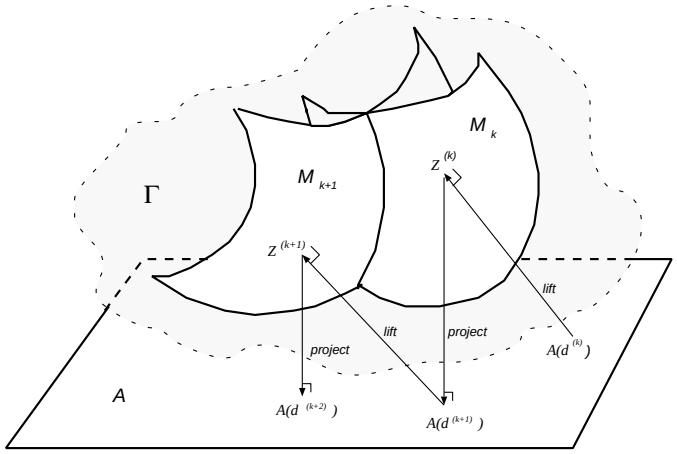

FIG. 5. *Geometric sketch of lift and projection.*

QR algorithm, the link with the total least squares problem [21], and applications in linear programming and, in particular, to the interior point methods [21, 14, 73, 74]. A glimpse of the progress in this fascinating area can be found in [23].

4.4. Numerical Methods. Numerical methods for LSIEPa1 and LSIEPb1 are discussed in [41]. The so called lift and project method is particularly worth mentioning. We sketch the idea below.

The idea is to alternate between Γ and $\mathcal{A}$, that is, for each given $d^{(k)} \in R^\ell$, we iterate the following two steps:

1. **(Lift)** Find the point $Z^{(k)} \in \Gamma$ such that $\mathrm{dist}(A(d^{(k)}), Z^{(k)}) = \mathrm{dist}(A(d^{(k)}), \Gamma)$. We call $Z^{(k)}$ a *lift* of $A(d^{(k)})$ onto Γ.
2. **(Projection)** Find the point $d^{(k+1)} \in R^\ell$ such that $\mathrm{dist}(A(d^{(k+1)}), Z^{(k)}) = \mathrm{dist}(Z^{(k)}, \mathcal{A})$. The point $A(d^{(k+1)}) \in \mathcal{A}$ is called a *projection* of $Z^{(k)}$ onto $\mathcal{A}$.

A schematic diagram of the iteration is illustrated in Figure 5. We use the shaded region in Figure 5 to symbolize that the topology of Γ could be much more complicated than one of its substructures $\mathcal{M}_k$ that will be defined later. The method only needs to work with the much simpler set $\mathcal{M}_k$.

The projection of $Z^{(k)} \in R^{n\times n}$ onto $\mathcal{A}$ is easy to do. The vector $d^{(k+1)}$ is the solution of the linear system

$$\sum_{i=1}^{\ell} \langle A_i, A_j \rangle d_i^{(k+1)} = \langle Z^{(k)} - A_0, A_j \rangle, \ , j = 1, \ldots, \ell. \tag{63}$$

where $\langle A, B \rangle := \mathrm{trace}(A^T B)$ is the Frobenius inner product for matrix A and B. Note that the coefficient matrix in (63) is independent of k. So the left-hand side of (63) needs to be factorized only once.

The lift step is more involved because elements in Γ involve $n - m$ undetermined eigenvalues Λ_c. Motivated by Theorem 4.1, however, the step can proceed as follows: Suppose $A(d^{(k)}) = Q(d^{(k)})\mathrm{diag}(\Lambda_{\sigma^{(k)}}(d^{(k)}), \Lambda_{\overline{\sigma^{(k)}}}(d^{(k)}))\ Q(d^{(k)})^T$ is the spectral decomposition of $A(d^{(k)})$ where $\sigma^{(k)} = \sigma^{(k)}(d^{(k)})$ is the permutation that solves the combinatorics problem (57) with $d = d^{(k)}$ and $Q(d^{(k)})$ is the corresponding orthogonal matrix

of eigenvectors. Then the shortest distance between $A(d^{(k)})$ and Γ is attained, by the Wielandt-Hoffman theorem [31, 48], at the point

$$Z^{(k)} := Q(d^{(k)})\mathrm{diag}\left(\Lambda_m^*, \Lambda_{\overline{\sigma^{(k)}}}(d^{(k)})\right) Q(d^{(k)})^T. \tag{64}$$

In other words, in order to find the shortest distance from $A(d^{(k)})$ to Γ, it suffices to find the shortest distance from $A(d^{(k)})$ to a *substructure* $\mathcal{M}_k$ of Γ, where the substructure

$$\mathcal{M}_k := \left\{Q\mathrm{diag}\left(\Lambda_M^*, \Lambda_{\overline{\sigma^{(k)}}}(d^{(k)})\right) Q^T | Q \in \mathcal{O}(n)\right\} \tag{65}$$

has a much simpler topology than Γ because the diagonal elements are fixed. (See Figure 5.) The cost for this lift is to solve (57) per step. Clearly, when the iterates are reaching convergence the permutations $\sigma^{(k)}$ should become stabilized.

THEOREM 4.3. *The lift and project method is a descent method in the sense that*

$$\|A(d^{(k+1)}) - Z^{(k+1)}\|_F^2 \leq \|A(d^{(k+1)}) - Z^{(k)}\|_F^2 \leq \|A(d^{(k)}) - Z^{(k)}\|_F^2. \tag{66}$$

Thus the method generates a sequence of matrix pairs $\{(Z^{(k)}, A(d^{(k)}))\}$ *that converges to a local stationary point for the problem of minimizing (61).*

5. Partially Described Inverse Eigenvalue Problem.

5.1. Generic Form. In the reconstruction of a system, instead of knowing the complete spectrum, there are also situations where only a *portion* of eigenvalues and eigenvectors are available. This is especially the case when due to the complexity or the size of the physical system, no reasonable analytical tools are available to evaluate the entire spectral information. Through the vibration test where the excitation and the response of the structure at many points are measured experimentally, there are identification techniques that can extract a part of the eigenpairs of the structure from the measurements [15, 71]. A generic partially described inverse eigenvalue problem is as follows:

(PDIEP) *Given vectors* $\{v^{(1)}, \ldots, v^{(k)}\} \subset \mathbf{F}^n$ *and scalars* $\{\lambda_1, \ldots, \lambda_k\} \subset \mathbf{F}$ *where* $1 \leq k < n$, *find a matrix* $X \in \mathcal{N}$ *such that* $Xv^{(i)} = \lambda_i v^{(i)}$ *for* $i = 1, \ldots, k$.

5.2. Variations. More specifically we could consider problems such as:
(PDIEP1) $\mathbf{F} = \mathbb{R}$. $\mathcal{N} = \{$All Toeplitz matrices in $\mathcal{S}(n)\}$ [46].
(PDIEP2) $\mathbf{F} = \mathbb{R}$. $\mathcal{N} = \{$All Jacobi matrices in $\mathcal{S}(n)\}$ [203].
(PDIEP3) $\mathbf{F} = \mathbb{R}$, $\mathcal{N} = \{$All per-symmetric Jacobi matrices in $\mathcal{S}(n)\}$ [203].
As an example of another type of PDIEP, consider the dynamical system

$$M\frac{d^2}{dt^2}v + C\frac{d}{dt}v + Kv = 0, \tag{67}$$

where M, C, K are symmetric and M is positive definite, that arises in a wide range of applications. Upon separation of variables, the system naturally leads to the quadratic λ-matrix problem:

$$P(\lambda)x = 0 \tag{68}$$

with

$$P(\lambda) = M\lambda^2 + C\lambda + K. \tag{69}$$

Suppose now a state feedback forcing function of the form

$$u(t) = b(f^T \frac{d}{dt} v(t) + g^T v(t)), \tag{70}$$

where $b, f, g \in \mathbb{R}^n$ are constant vectors, is applied to the system. The resulting closed loop system leads to the λ-matrix problem with pencil

$$Q(\lambda) = M\lambda^2 + (C - bf^T)\lambda + (K - bg^T). \tag{71}$$

The goal of this feedback control $u(t)$ is to relocate those bad eigenvalues in (69) that either are unstable or lead to large vibration phenomena in the system (67) while maintaining those good eigenvalues. This notion gives rise to the following partial pole assignment problem:

(PDIEP4) Given matrices M, C, K, its associated eigenvalues $\{\lambda_1, \ldots, \lambda_{2n}\}$ of the pencil (69), a fixed vector $b \in \mathbb{R}^n$, and m complex numbers $\{\mu_1, \ldots, \mu_m\}$, $m \leq n$, find $f, g \in \mathbb{C}^n$ such that the spectrum of the closed loop pencil (71) has spectrum $\{\mu_1, \ldots, \mu_m, \lambda_{m+1}, \ldots, \lambda_{2n}\}$.

Other variations of problems include:

(PDIEP5) Given two distinct scalars $\lambda, \mu \in R$ and two non-zero vectors $x, y \in \mathbb{R}^n$, find two Jacobi matrices J and $\overline{J}$ so that $Jx = \lambda x$ and $\overline{J}y = \mu y$, where J and $\overline{J}$ differ only in the (n, n) position [203].

(PDIEP6) Given distinct scalars $\{\lambda_1, \ldots, \lambda_n\} \subset \mathbb{R}$ and a non-zero vector $x \in \mathbb{R}^n$, find a Jacobi matrix J such that $\lambda(J) = \{\lambda_1, \ldots, \lambda_n\}$ and that either $Jx = \lambda_1 x$ or $Jx = \lambda_n x$ [203].

(PDIEP7) Construct an $n \times n$ symmetric band matrix of bandwidth p from the knowledge of all the eigenvalues and the first p components of all the normalized eigenvectors [95].

5.3. Solvability Issues. Regarding the PDIEP1, it is known that eigenvectors of a Toeplitz matrix have a special structure [4, 39, 66], i.e., eigenvectors of any symmetric and centro-symmetric matrix must be either symmetric or skew-symmetric. It is first proved by Cybenko that the dimension of Toeplitz matrices with a single prescribed eigenvector in $\mathbb{R}^n$ should be at least $[\frac{n+1}{2}]$ [60]. Thus the fact that the dimension of Toeplitz matrices with two prescribed eigenvectors is independent of n appears a little bit surprising. More precisely, let

$$S(v) := \{c \in \mathbb{R}^n | T(c)v = \lambda v \text{ for some } \lambda \in \mathbb{R}\}$$

denote the affine subspace of all symmetric Toeplitz matrices with v as one of its eigenvectors. Then the PDIEP1 with $k = 2$ can be answered as follows [46].

THEOREM 5.1. *Let the entry $\pi(u, v)/\eta$ in the table below represent the pair of numbers where $\pi(u, v) := dim(S(u) \bigcap S(v))$ is the dimension of the affine subspace of all*

symmetric Toeplitz matrices with u, v as two of its eigenvectors and η is the dimension of the affine subspace of solutions to the PDIEP1 with $k = 2$. Then depending upon whether n is odd (or even) and the symmetry of the eigenvectors, for almost all eigenvectors u and v and for any eigenvalues λ_1 and λ_2, the following table is true.

n = odd (even)	*symmetric*	*skew-symmetric*
symmetric	*2/0 (3/1)*	*2/0 (2/0)*
skew-symmetric	*2/0 (2/0)*	*4/0 (3/1)*

Thus, for example, if n is odd and if at least one of the given eigenvectors is symmetric, or if n is even and one eigenvector is symmetric and the other is skew-symmetric, then the Toeplitz matrix is uniquely determined. That is, two eigenvectors and two eigenvalues can uniquely determine a Toeplitz matrix in these cases.

The solution to the PDIEP4 is given in the following theorem [62, 161]:

THEOREM 5.2. *Let the eigenvector matrix and eigenvalue matrix of (69) be partitioned into $X = [X_1, X_2]$ and $\Lambda = diag(\Lambda_1, \Lambda_2)$, respectively, where $X_1 \in \mathbb{C}^{n\times m}, X_2 \in \mathbb{C}^{n\times(2n-m)}$, $\Lambda_1 \in D_{\mathcal{C}}(m)$, and $\Lambda_2 \in D_{\mathcal{C}}(2n-m)$. Define $\beta = [\beta_1, \ldots, \beta_m]^T \in \mathbb{C}^m$ by*

$$\beta_j := \frac{1}{b^T x_j} \frac{\mu_j - \lambda_j}{\lambda_j} \prod_{i=1, i\neq j}^{m} \frac{\mu_i - \lambda_j}{\lambda_i - \lambda_j}. \tag{72}$$

Then the pair of vectors

$$f := MX_1\Lambda_1\beta \tag{73}$$

$$g := -KX_1\beta \tag{74}$$

solve the PDIEP4.

6. Multivariate Inverse Eigenvalue Problem. A multivariate eigenvalue problem is to find real scalars $\lambda_1, \ldots, \lambda_m$ and a real vector $x \in R^n$ such that equations

$$Ax = \Lambda x \tag{75}$$

$$||x_i|| = 1,\ i = 1, \ldots, m. \tag{76}$$

are satisfied, in which $A \in \mathcal{S}(n)$ is a given positive definite matrix partitioned into blocks:

$$A = \begin{bmatrix} A_{11} & A_{12} & \ldots & A_{1m} \\ A_{21} & A_{22} & \ldots & A_{2m} \\ \vdots & \vdots & & \vdots \\ A_{m1} & A_{m2} & \ldots & A_{mm} \end{bmatrix},$$

Λ is the diagonal matrix

$$\Lambda = \text{diag}\{\lambda_1 I^{[n_1]}, \ldots, \lambda_m I^{[n_m]}\}$$

with $I^{[n_i]}$ the identity matrix of size n_i, and $x \in R^n$ is partitioned into blocks

$$x = [x_1^T, \ldots, x_m^T]^T$$

with $x_i \in R^{n_i}$. Trivially, the single variate case when $m = 1$ is simply a classical symmetric eigenvalue problem. The general problem arises from multivariate canonical analysis in statistics [49, 110, 119]. In the context of factor analysis, for example, the original n random variables are divided into m "factors" each of which consists of n_i variables, blocks in A represent covariance matrices between these factors, and x_i determines how these n_i variables should be combined into one simple factor. The equations (75) and (76) represent necessary conditions where coefficients are to be determined so that the resulting linear combinations of sets of random variables are maximally correlated.

If $m = 2$, the problem can still be handled by using the SVD decomposition. But for $m > 2$, only a heuristic iterative method has been proposed by Horst [119], but the convergence theory has been proved only recently [49]. It is further proved that the number of solutions is $\prod_{i=1}^{m}(2n_i)$. Quite surprisingly, it appears that this problem has never been studied in the numerical linear algebra community. Neither the algebraic theory nor numerical methods are in place by any standard.

The multivariate eigenvalue problem is interesting in its own right and is intricate with many possible new research directions. On the other hand, following the spirit of all the other inverse eigenvalue problems discussed above, we can formulate various kinds of multivariate inverse eigenvalue problems, i.e., given the partition pattern and spectral information we want to determine whether a sample matrix A can be constructed. This study would be useful for constructing statistical models. Conceivably these problems would be far more challenging to handle than the already difficult single-variate inverse eigenvalue problems. We are not aware of any work in this area.

7. Conclusion. A physical process is often described by a mathematical model of which the parameters represent important physical quantities. An important step in the construction of a mathematical model for engineering applications is to verify the model by comparing the predicted behavior of the model with experimental results and then to update the model to more accurately represent the physical process. An inverse eigenvalue problem amounts to one such modeling process in which quantities are represented in terms of matrices whereas the comparison is based upon the spectral information and the update is governed by the underlying structure constraint.

We have briefly discussed in this paper a variety of inverse eigenvalue problems. These problems are identified and classified according to their mathematical attributes. Some of the problems have immediate engineering application while others are perhaps more mathematical abstraction. Regardless, these inverse eigenvalue problems raise some fundamental questions including issues of solvability, numerical reconstruction, sensitivity of the reconstruction to noisy data, and so on. We have reviewed some of the known results, but more importantly we have pointed out many more unsettled open problems. The following table offers a quick glimpse at the current status of problems surveyed in this paper. The results that have been discussed specifically in this presentation are identified by theorem and/or section numbers; otherwise, we try to list some references for further study. Be cautioned that by no means the list is complete. We have attached an extensive bibliography of pertinent literature in this

area for further reference. The question mark "?" in the table indicates that we are not aware of any result in that regard. On other hand, a listing of references does not necessarily imply that the problem is completely solved. Quite on the contrary, often the listing represents only some partial results. The check mark "$\surd$" indicates that some results are available but either the author fails to locate the source or these results can be derived from other established facts.

It is hoped that this presentation will serve to stimulate further research in this direction.

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Variation	Solvability	Computability	Relation to Others
PIEP1	[170, 180]	?	
PIEP2	?	§2.5, [47, 85]	MIEP2, SIEP1
PIEP3	Thm. 2.4	[189]	
AIEP1	[142, 143, 147, 148]	[144]	
AIEP2	[103, 106, 142, 143, 147, 148]	[18, 24, 27, 69, 85, 107, 130, 146, 187]	
AIEP3	Thm. 2.1	[43, 194]	
AIEP4	?	[69]	
AIEP5	?	[33]	
MIEP1	[181, 149]	[104, 168]	PIEP1
MIEP2	[105, 106, 168]	$\surd$	PIEP2
MIEP3	Thm. 2.2	?	
MIEP4	?	[69]	
MIEP5	?	?	
SIEP1	Thm. 3.1	§3.4, [52, 128, 129, 184]	PIEP2
SIEP2	[28]	[26, 116]	
SIEP3	?	[45]	
SIEP4	Thm. 3.3 & 3.4	§3.4	SIEP5
SIEP5	?	§3.4 & Thm. 3.5, [54]	SIEP4
SIEP6a	Thm. 3.6	[27, 28, 92, 100, 108, 116, 117]	
SIEP6b	Thm. 3.8	[161]	
SIEP7	$\surd$	[26, 27, 76]	
SIEP8	$\surd$	[28]	
SIEP9	[27]	[117]	
SIEP10	$\surd$	[19, 25, 27, 136, 160]	
PISVP	?	[47]	PIEP2
SHIEP	Thm. 3.9	[51, 201]	
LSIEPa1	$\surd$	§4.4 & Thm. 4.1	LSIEPb1
LSIEPb1	$\surd$	§4.4	LSIEPa1
LSIEPb2	Thm. 4.2	$\surd$	
LSIEPb3	$\surd$	§ 3.4	SIEP1
LSIEPb4	?	?	
PDIEP1	Thm. 5.1	[62, 161]	
PDIEP2	[203]	[203]	
PDIEP3	[203]	[203]	
PDIEP4	Thm. 5.2	$\surd$	
PDIEP5	[203]	[203]	
PDIEP6	[203]	[203]	
PDIEP7	$\surd$	[95]	
MVIEP	?	?	?

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