A NUMERICAL METHOD FOR THE INVERSE STOCHASTIC SPECTRUM PROBLEM

MOODY T. CHU* AND QUANLIN GUO†

Abstract. Inverse stochastic spectrum problem involves the construction of a stochastic matrix with a prescribed spectrum. A differential equation aimed to bring forth the steepest descent flow in reducing the distance between isospectral matrices and nonnegative matrices, represented in terms of some general coordinates, is described. The flow is further characterized by an analytic singular value decomposition to maintain the numerical stability and to monitor the proximity to singularity. This flow approach can be used to design Markov chains with specified structure. Applications are demonstrated by numerical examples.

Key words. Stochastic Matrix, Least Squares, Steepest Descent, Isospectral Flow, Structured Markov chain, Analytic Singular Value Flow

AMS(MOS) subject classifications. 65F15, 65H15.

1. Introduction. Inverse eigenvalue problems concern the reconstruction of matrices from prescribed spectral data. The spectral data may involve complete or partial information of eigenvalues or eigenvectors. Generally, a problem without any restrictions on the matrix is of little interest. In order that the inverse eigenvalue problem be meaningful, it is often necessary to restrict the construction to special classes of matrices, such as symmetric Toeplitz matrices or matrices with other special structures. In this paper we limit our attention to the so called stochastic matrices, i.e., matrices with nonnegative elements where all its row sums are equal to one. We propose a numerical procedure for the construction of a stochastic matrix so that its spectrum agrees with a prescribed set of complex values. If the set of prescribed values turns out to be infeasible, the method produces a best approximation in the sense of least squares. To our knowledge, this inverse eigenvalue problem for stochastic matrices has not been studied extensively probably due to its difficulty as we shall discuss below. Nevertheless, for a variety of physical problems that can be described in the context of Markov chains, an understanding of the inverse eigenvalue problem for stochastic matrices and a capacity to solve the problem would make it possible to construct a system from its natural frequencies [7, 11]. The method proposed in the paper appears to be the first attemp at tackling this problem numerically with some success. Our technique can also be applied as a numerical way to solve the long standing inverse eigenvalue problems for nonnegative matrices.

Associated with every inverse eigenvalue problem are two fundamental questions — the theoretic issue on solvability and the practical issue on computability. The major effort in solvability has been to determine a necessary or a sufficient condition under which an inverse eigenvalue problem has a solution whereas the main concern

* Department of Mathematics, North Carolina State University, Raleigh, NC 27695-8205. This research was supported in part by the National Science Foundation under grant DMS-9422280.

† Department of Mathematics, North Carolina State University, Raleigh, NC 27695-8205.

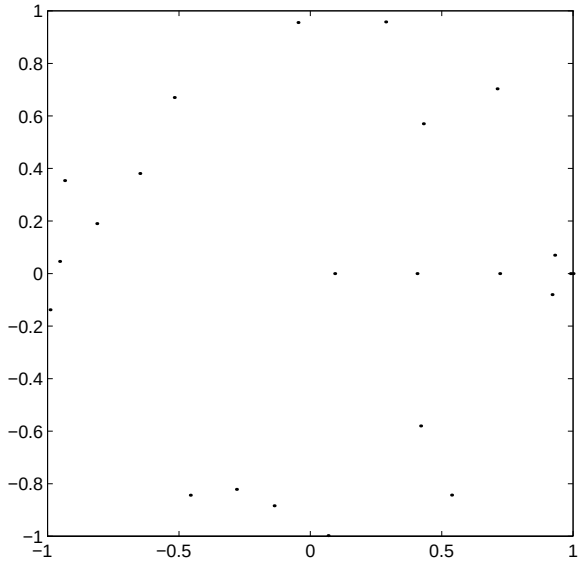

FIG. 1. Θ_4 *by the Karpelevič Theorem.*

in computability has been to develop an algorithm by which, knowing a priori that the given spectral data are feasible, a matrix can be constructed numerically. Both questions are difficult and challenging. Searching through the literature, we have found only a handful of inverse eigenvalue problems that have been completely understood or solved. The focus of this paper is on the computability for stochastic matrices.

For stochastic matrices, the inverse eigenvalue problem is particularly difficult as can be seen from the involvement in the best known result on existence by Karpelevič [14, 16]. Karpelevič completely characterized the set Θ_n of points in the complex plane that are eigenvalues of stochastic $n \times n$ matrices. In particular, the region Θ_n is symmetric about the real axis. It is contained within the unit circle and its intersections with the unit circle are points $z = e^{2\pi a/b}$ where a and b run over all integers satisfying $0 \leq a < b \leq n$. The boundary of Θ_n consists of these intersection points and of curvilinear arcs connecting them in circular order. These arcs are characterized by specific parametric equations whose formulas can be found in [14, 16]. For example, a complex number λ is an eigenvalue for a 4×4 stochastic matrix if and only if it belongs to a region Θ_4 such as the one shown in Figure 1. Complicated though it may seem, the Karpelevič theorem characterizes only one complex value a time and does not provide further insights into when two or more points in Θ_n are eigenvalues of the *same* stochastic matrix. Minc [16] distinctively called the problem we are considering, where the entire spectrum is given, *the inverse spectrum problem.*

It is known that the inverse eigenvalue problem for nonnegative matrices is virtually equivalent to that for stochastic matrices. For example, a complex nonzero number α is an eigenvalue of a nonnegative matrix with a positive maximal eigenvalue r if and only if α/r is an eigenvalue of a stochastic matrix. Our problem is much more complicated because it involves the entire spectrum. Fortunately, based on the following theorem we can proceed our computation once a nonnegative matrix is found.

THEOREM 1.1. *If A is a nonnegative matrix with positive maximal eigenvalue r and a positive maximal eigenvector x, then $D^{-1}r^{-1}AD$ is a stochastic matrix where $D := diag\{x_1, \ldots, x_n\}$.*

We thus should turn our attention to the inverse eigenvalue (or spectrum) problems for nonnegative matrices, a subject that has received considerable interest in the literature. Some necessary and a few sufficient conditions on whether a given set of complex numbers could be the spectrum of a nonnegative matrix can be found, for example, in [1, 3, 8, 9, 10, 12, 13, 17, 20] and the references contained therein. Yet numerical methods for constructing such a matrix, even if the spectrum is feasible, still need to be developed. Some discussion can be found in [6, 20]. Regardless of all the efforts, the inverse eigenvalue problem for nonnegative matrices has not been completely resolved to this date.

In an earlier paper [6] the first author has developed an algorithm that can construct symmetric nonnegative matrices with prescribed spectra by means of differential equations. Symmetry was needed there because the techniques by then were for flows in the group of orthogonal matrices only. Upon realizing the existence of an analytic singular value decomposition (ASVD) for a real analytic path of matrices [5, 15, 21], we are able to advance the techniques in [6] to general matrices in this paper.

This paper is organized as follows: We reformulate the inverse stochastic spectrum problem as that of finding the shortest distance between isospectral matrices and nonnegative matrices. In §2 we introduce a general coordinate system to describe these two types of matrices and discuss how this setting naturally leads to a steepest descent flow. This approach generalizes what has been done before, but requires the inversion of matrices that is potentially dangerous. In §3 we argue that the steepest descent flow is in fact analytic and hence an analytic singular value decomposition exists. We therefore are able to describe the flow by a more stable vector field. We illustrate the application of this differential equation to the inverse spectrum problem by numerical examples in §4.

2. Basic Formulation. The given spectrum $\{\lambda_1, \ldots, \lambda_n\}$ may be complex-valued. It is not difficult to create a simple, say tridiagonal, real-valued matrix Λ carrying the same spectrum. For multiple eigenvalues, one should also consider the possible real-valued Jordan canonical form, depending on the geometric multiplicity. Matrices in the set

$$\mathcal{M}(\Lambda) := \{P\Lambda P^{-1} | P \in R^{n\times n} \text{ is nonsingular}\} \tag{1}$$

obviously are isospectral to Λ. Let

$$\pi(R_+^n) := \{B \circ B | B \in R^{n\times n}\} \tag{2}$$

denote the cone of all nonnegative matrices where $\circ$ means the Hadamard product of matrices. Our basic idea is to find the intersection of $\mathcal{M}(\Lambda)$ and $\pi(R_+^n)$. Such an intersection, if exists, results in a nonnegative matrix isospectral to Λ. Furthermore, if the condition in Theorem 1.1 holds, i.e., if the eigenvector corresponding to the positive

maximal eigenvalue is positive, then we will have solved the inverse spectrum problem for stochastic matrices by a diagonal similarity transformation. The difficulty, as we have pointed earlier, is the lack of means to determine if the given spectrum is feasible. An arbitrarily given set of values $\lambda_1, \ldots, \lambda_n$, even if $\lambda_i \in \Theta_n$ for all i, may not be the spectrum of any nonnegative matrix. In this case, it is reasonable to ask for only the best possible approximation. To handle both problems at the same time, we reformulate the inverse spectrum problem as that of finding the shortest distance between $\mathcal{M}(\Lambda)$ and $\pi(R_+^n)$:

$$\text{minimize } F(P,R) := \frac{1}{2}\|P\Lambda P^{-1} - R \circ R\|^2 \tag{3}$$

where $\|\cdot\|$ represents the Frobenius matrix norm. Obviously, if Λ is feasible, then $F(P,R) = 0$ for some suitable P and R. Note that the variable P in (3) resides in the open set of nonsingular matrices whereas R is simple a general matrix in $R^{n\times n}$. The optimization in (3) subjects to no other significant constraint. Since the optimization is over unbounded open domain, it is possible that the minimum does not exist. We shall comment more on this point later.

The Fréchet derivative of F at (P,R) acting on (H,K) is calculated as follows:

$$\begin{aligned} F'(P,R)(H,K) &= \langle P\Lambda P^{-1} - R\circ R, H\Lambda P^{-1} - P\Lambda(P^{-1}HP^{-1}) - K\circ R - R\circ K\rangle \\ &= \langle (P\Lambda P^{-1} - R\circ R)P^{-T}\Lambda^T - P^{-T}\Lambda^T P^T(P\Lambda P^{-1} - R\circ R)P^{-T}, H\rangle \\ &\quad -\langle 2(P\Lambda P^{-1} - R\circ R)\circ R, K\rangle \end{aligned} \tag{4}$$

where $\langle\cdot,\cdot\rangle$ denotes the Frobenius inner product of two matrices. Define, for abbreviation,

$$M(P) := P\Lambda P^{-1} \tag{5}$$

$$\Delta(P,R) := M(P) - R\circ R. \tag{6}$$

The norm of $\Delta(P,R)$ represents how close we are able to solve the inverse spectrum problem. With respect to the product topology on $R^{n\times n}\times R^{n\times n}$, we can easily read off the gradient ∇F of the objective function F from (4):

$$\nabla F(P,R) = \left((\Delta(P,R)M(P)^T - M(P)^T\Delta(P,R))P^{-T}, -2\Delta(P,R)\circ R\right). \tag{7}$$

Therefore, the flow $(P(t),R(t))$ defined by the differential equations

$$\frac{dP}{dt} := [M(P)^T, \Delta(P,R)]P^{-T} \tag{8}$$

$$\frac{dR}{dt} := 2\Delta(P,R)\circ R, \tag{9}$$

where $[\cdot,\cdot]$ denotes the Lie bracket of two matrices, signifies in fact the steepest descent flow for the objective function F.

An important advance we have made here is that the gradient $\nabla F(P,R)$ no longer needs to be projected as was required in [6] since P needs not to be orthogonal. On the

other hand, a possible frailty of this advance is that the solution flow $P(t)$ is susceptible to becoming unbounded.

The differential system (8) and (9) has another interesting property that is useful for constructing Markov chains with designated structure. The Hadamard product in (9) implies that if $r_{ij} = 0$, then $\frac{dr_{ij}}{dt} = 0$. Thus the zero structure in the original matrix $R(0)$ is preserved throughout the integration. We may use this property to explore the possibility of constructing a Markov chain with prescribed linkages and spectrum.

3. ASVD flow. A somewhat worrisome feature in the differential system (8) and (9) is the involvement of P^{-1}. In this section we propose using the ASVD as a stable way to carry out the computation. Also, we have pointed out earlier that the minimization (3) over two open sets may not have a minimum. It is possible during the integration that the flow $P(t)$ from one particular starting value graduately moves toward the boundary, i.e., the closed subset of singular matrices in $R^{n\times n}$, and becomes more and more nearly singular. The ASVD technique allows us to monitor the situation. If the singular values indicate that $P(t)$ is nearly rank deficient, we can abort the integration and restart from a new initial value.

An analytic singular value decomposition of the path of matrices $P(t)$ is an analytic path of factorizations

$$P(t) = X(t)S(t)Y(t)^T \tag{10}$$

where $X(t)$ and $Y(t)$ are orthogonal and $S(t)$ is diagonal. In [5] Bunse-Gerstner et al prove that an ASVD exists if $P(t)$ is analytic. The fact that $P(t)$ defined by (8) and (9) is indeed analytic follows from the Cauchy-Kovalevskaya Theorem [18] since the coefficients of the vector field in (8) and (9) are analytic. With this understanding, we may proceed to describe the differential equations for the ASVD of $P(t)$.

It is worthy to point out that the two matrices P and R are used, respectively, as *coordinates* to describe the isospectral matrices and nonnegative matrices. We may have used more dimensions of variables than necessary to describe the underlying matrices, but that does no harm. When flows $P(t)$ and $R(t)$ are introduced, in a sense a flow in $\mathcal{M}(\Lambda)$ and a flow in $\pi(R^n_+)$ are also introduced. To stablize the computation, we further describe the motion of the coordinate P by three other variables X, S, and Y according to (10). The flows of $X(t)$, $S(t)$ and $Y(t)$ can be found in the following way due to Wright [15, 21].

Differentiating both sides of (10), we obtain the following equation after some suitable multiplications :

$$X^T\frac{dP}{dt}Y = X^T\frac{dX}{dt}S + \frac{dS}{dt} + S\frac{dY^T}{dt}Y. \tag{11}$$

Define

$$Q(t) := X^T\frac{dP}{dt}Y, \tag{12}$$

$$Z(t) := X^T\frac{dX}{dt}, \tag{13}$$

$$W(t) := \frac{dY^T}{dt} Y. \tag{14}$$

Note that $Q(t)$ is known from (8) where the inverse of $P(t)$ is calculated from

$$P^{-1} = Y S^{-1} X^T. \tag{15}$$

The diagonal entries of $S = \mathrm{diag}\{s_1, \ldots, s_n\}$ provide us with information about the proximity of $P(t)$ to singularity. On the other hand, comparing the diagonal entries on both sides of (11), we obtain the differential equation for $S(t)$:

$$\frac{dS}{dt} = \mathrm{diag}(Q), \tag{16}$$

since both $Z(t)$ and $W(t)$ are skew symmetric. Comparing the off-diagonal entries on both sides of (11), we obtain the linear system:

$$q_{jk} = z_{jk} s_k + s_j w_{jk}, \tag{17}$$

$$-q_{kj} = z_{jk} s_j + s_k w_{jk}. \tag{18}$$

If $s_k^2 \neq s_j^2$, we can solve this system and obtain

$$z_{jk} = \frac{s_k q_{jk} + s_j q_{kj}}{s_k^2 - s_j^2}, \tag{19}$$

$$w_{jk} = \frac{s_j q_{jk} + s_k q_{kj}}{s_j^2 - s_k^2} \tag{20}$$

for all $j > k$. Once $Z(t)$ and $W(t)$ are known, the differential equations for $X(t)$ and $Y(t)$ are given, respectively, by

$$\frac{dX}{dt} = XZ, \tag{21}$$

$$\frac{dY}{dt} = YW^T. \tag{22}$$

By now we have developed a complete coordinate system $(X(t), S(t), Y(t), R(t))$ for matrices in $\mathcal{M}(\Lambda) \times \pi(R_+^n)$. The differential equations (21), (16), (22) and (9) with the relationship (10) describe how these coordinates should be varied in t to produce the steepest descent flow for the objective function F. This flow is ready to be integrated numerically by any initial value problem solvers. We have thus proposed a numerical method for solving the inverse stochastic spectrum problem.

4. Convergence. When assessing the convergence properties of the foregoing approach, we must distinguish carefully the means used to measure the convergence.

First of all, the approach fails only at two occasions — either $P(t)$ becomes singular in finite time or that $F(P(t), R(t))$ converges to a nonzero constant. The former case, detected by examining the singular values of $P(t)$, requires a restart from a new initial value with the hope to avoid the singularity. The latter case indicates that a

least squares *local* solution has been found, but that solution has not yet solved the inverse spectrum problem. A restart may help to locate an exact solution, if the prescribed spectrum is feasible, or to move to another least squares approximation that may produce a different objective value.

In all cases, the function

$$G(t) := F(P(t), R(t)) \tag{23}$$

enjoys the property that

$$\frac{dG}{dt} = -\|\nabla F(P(t), R(t))\|^2 \leq 0 \tag{24}$$

along any solution curve $(P(t), R(t))$. It follows that $G(t)$ is monotone decreasing and that $\frac{dG}{dt} = 0$ only when a local stationary point of $F(P, R)$ is reached. Suppose that $P(t)$ remains nonsingular throughout the integration, an assumption that seems generic according to our experiences. Then $G(t)$ has to converge. It is in this sense that our method is globally convergent.

In [6] the coordinate matrix $P(t)$ is limited to be orthogonal, hence is bounded and exists for all t. This constraint is not imposed on the approach discussed in the current paper. Generally there is no guarantee that $P(t)$ is bounded. However, in the case that the solution flow $(P(t), R(t))$ corresponding to a certain initial value indeed is bounded and exists for all $t \geq 0$, then we can conclude from Lyapunov's second method [4] that ω-limit points of $P(t)$ exist and that each limit point satisfy $\nabla F(P, R) = 0$. In other words, limit points of the flow are necessarily stationary points. Because that the vector field always points to the steepest descent direction and that other types of stationary points are unstable, any limit point reached through numerical computation will most likely be a local minimizer for F. The structure of ω-limit set of the differential system (8) and (9) can be further analyzed in a way similar to that in [6]. For example, if the ω-limit set of a flow contains a point at which $F(P, R) = 0$, then that point is the only element in the ω-limit set. The flow hence converges to that limit point. We shall not repeat the detailed argument here. Our experiences seem to indicate that our method works reasonably well for solving the inverse spectrum problem.

5. Numerical Experiment. In this section, we report some experiences of our experiment with the differential equation applied to the inverse problem. The computation is carried out by MATLAB 4.2a on an ALPHA 3000/300LX workstation. The solvers used for the initial value problem are **ode113** and **ode15s** from the MATLAB ODE SUITE [19]. The code **ode113** is a PECE implementation of Adams-Bashforth-Moulton methods for non-stiff systems. The code **ode15s** is a quasi-constant step size implementation of the Klopfenstein-Shampine family of the numerical differential formulas for stiff systems. The statistics about the cost of integration can be obtained directly from the **odeset** option built in the integrator. More details of these codes can be found in the document [19]. The reason for using these two codes is simply for convenience and illustration. Any other ODE solvers can certainly be used instead.

In our experiments, the tolerance for both absolute error and relative error is set at 10^{-12}. This criterion is used to control the accuracy in following the solution path. The high accuracy we required here has little to do with the dynamics of the underlying vector field. We examine the output values at time interval of 10. The integration terminates automatically when the norm of $\Delta(P, R)$ or the relative improvement of $\Delta(P, R)$ between two consecutive output points is less than 10^{-9} indicating either a stochastic matrix with the prescribed spectrum or, in the case of an infeasible spectrum, a least squares solution has been found. So as to fit the data comfortably in the running text, we report only the case $n = 5$ and display all numbers with five digits.

Example 1 To ensure the feasiblity of test data, we start with a randomly generated stochastic matrix and use it eigenvalues as the objective spectrum. To demonstrate the robustness of our approach, the initial values of the differential equations are also generated randomly. Reported below is one typical run in our experiments.

The random matrix

$$A = \begin{bmatrix} 0.0596 & 0.2586 & 0.0838 & 0.3022 & 0.2958 \\ 0.0972 & 0.2833 & 0.3559 & 0.2545 & 0.0092 \\ 0.2015 & 0.1143 & 0.3645 & 0.2669 & 0.0528 \\ 0.2637 & 0.2116 & 0.1920 & 0.0333 & 0.2994 \\ 0.1785 & 0.3138 & 0.1386 & 0.2146 & 0.1545 \end{bmatrix}$$

is stochastic. Its spectrum $\{1.0000, -0.2403, 0.1186 \pm 0.1805i, -0.1018\}$, also random but feasible, is used as the target. We note that the presence of complex-conjugate pair(s) of eigenvalues in the spectrum is quite common. Orthogonal matrices X_0, Y_0 and the diagonal matrix S_0 from the singular value decomposition $P_0 = X_0 S_0 Y_0$ of the random matrix

$$P_0 = \begin{bmatrix} 0.2002 & 0.4213 & 0.9229 & 0.7243 & 0.4548 \\ 0.6964 & 0.0752 & 0.9361 & 0.2235 & 0.0981 \\ 0.7538 & 0.3620 & 0.2157 & 0.5272 & 0.2637 \\ 0.4366 & 0.3220 & 0.8688 & 0.1729 & 0.8697 \\ 0.8897 & 0.1436 & 0.7097 & 0.5343 & 0.7837 \end{bmatrix}$$

together with the matrix $R_0 = 0.8329\mathbf{1}$, where $\mathbf{1}$ is the matrix with all entries 1, are used as the initial values for $X(t)$, $Y(t)$, $S(t)$ and $R(t)$, respectively. Figure 1 depicts the history of $F(P(t), R(t))$ throughout the integration. As is expected, $F(P(t), R(t))$ is monotone decreasing in t. The flow $P(t)$ converges to a nonnegative matrix with the prescribed spectrum, that by Theorem 1.1 is converted into a stochastic matrix B:

$$B = \begin{bmatrix} 0.1679 & 0.0522 & 0.4721 & 0.0000 & 0.3078 \\ 0.1436 & 0.1779 & 0.4186 & 0.1901 & 0.0698 \\ 0.0000 & 0.1377 & 0.5291 & 0.3034 & 0.0299 \\ 0.0560 & 0.4690 & 0.2404 & 0.0038 & 0.2309 \\ 0.1931 & 0.1011 & 0.5339 & 0.1553 & 0.0165 \end{bmatrix}.$$

Note that B is not expected to be correlated to A other than the spectrum since no other information of A is used in the calculation. While the history of $F(P(t), R(t))$

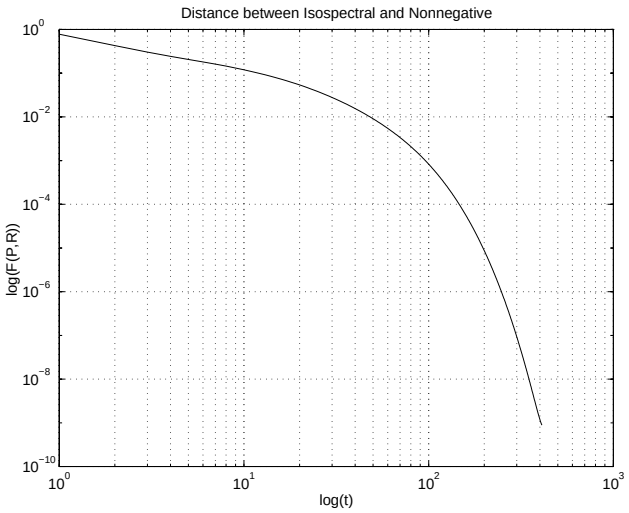

FIG. 2. *A log-log plot of $F(P(t), R(t))$ versus t for Example 1.*

is independent of the integrator used, Figure 2 indicates the number of steps taken in each interval of length 10 by the non-stiff solver **ode113** and by the stiff solver **ode15s**. Both solvers seem to work reasonably well, although the stiff solver clearly is advancing with much larger step sizes at the cost of solving implicit algebraic equations. Figure 3 summarizes the statistics of the cost when using **ode15s**. It should be pointed out that the numerical computation of the partial derivative (and the related function evaluations) could have been saved if the interval of output points had been larger [19].

Suppose we merely change the initial value R_0 in the above to another random matrix:

$$R_0 = \begin{bmatrix} 0.8329 & -0.9698 & 0.2274 & 0.9466 & -0.1409 \\ -0.6222 & 0.3131 & -0.7072 & 0.6990 & -0.6490 \\ 0.5684 & 0.4914 & 0.2558 & -0.2685 & -0.0901 \\ -0.9794 & 0.6124 & -0.4724 & -0.9758 & -0.8408 \\ -0.5250 & -0.9640 & 0.0399 & -0.0852 & 0.4312 \end{bmatrix}.$$

Then the resulting stochastic matrix C becomes

$$C = \begin{bmatrix} 0.1422 & 0.0310 & 0.8267 & 0.0000 & 0.0001 \\ 0.0016 & 0.5337 & 0.2791 & 0.0756 & 0.1099 \\ 0.0000 & 0.6413 & 0.1603 & 0.0000 & 0.1984 \\ 0.2549 & 0.7019 & 0.0139 & 0.0037 & 0.0255 \\ 0.0360 & 0.6595 & 0.2178 & 0.0315 & 0.0553 \end{bmatrix},$$

illustrating the non-uniqueness of the solution for the inverse spectrum problem and also the robustness of our differential equation approach.

Example 2. In this example, we illustrate the application of our approach to structured stochastic matrix. Suppose we want to find a stochastic matrix with eigenvalues $\{1.0000, -0.2608, 0.5046, 0.6438, -0.4483\}$. Furthermore, suppose we want the Markov

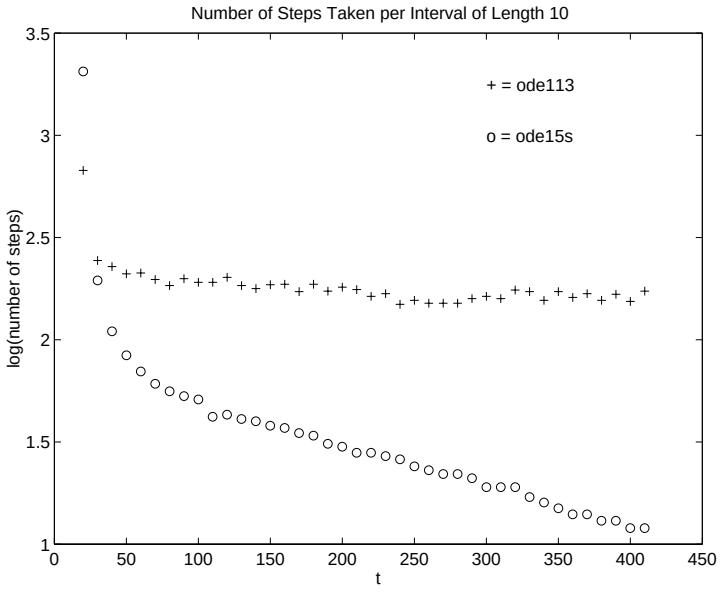

FIG. 3. *A comparison of steps taken by* **ode113** *and* **ode15s** *for Example 1.*

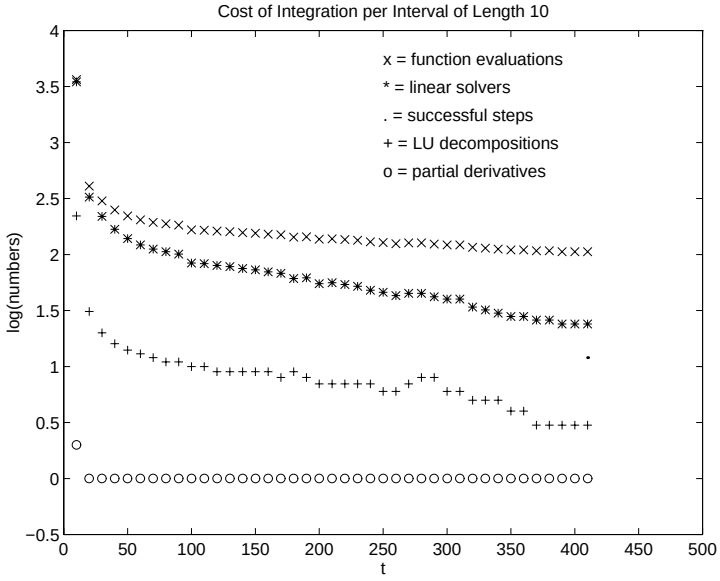

FIG. 4. *Cost of* **ode15s** *for Example 1.*

chain to be such that the states form a *ring* and that each state is linked at most to its two immediate neighbors. We begin with the initial matrices

$$P_0 = \begin{bmatrix} 0.1825 & 0.7922 & 0.2567 & 0.9260 & 0.9063 \\ 0.1967 & 0.5737 & 0.7206 & 0.5153 & 0.0186 \\ 0.5281 & 0.2994 & 0.9550 & 0.6994 & 0.1383 \\ 0.7948 & 0.6379 & 0.5787 & 0.1005 & 0.9024 \\ 0.5094 & 0.8956 & 0.3954 & 0.6125 & 0.4410 \end{bmatrix}$$

and $R_0 = 0.9210\hat{\mathbf{1}}$ where

$$\hat{\mathbf{1}} = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

As we pointed out earlier, the zeros in R_0 are invariant under the integration of (8) and (9). Thus we are maintaining the ring structure while searching for the one with matched spectrum. It turns out that the stochastic matrix

$$D = \begin{bmatrix} 0.0000 & 0.3094 & 0 & 0 & 0.6906 \\ 0.0040 & 0.5063 & 0.4896 & 0 & 0 \\ 0 & 0.0000 & 0.5134 & 0.4866 & 0 \\ 0 & 0 & 0.7733 & 0.2246 & 0.0021 \\ 0.4149 & 0 & 0 & 0.3900 & 0.1951 \end{bmatrix}$$

is the limit point of the solution flow and possesses the desirable spectrum.

Example 3. By a result of Dmitriev and Dynkin [16], a complex number α with $|\arg\alpha| \leq \frac{2\pi}{n}$ is an eigenvalue of an $n \times n$ stochastic matrix if and only α lies either in the triangle $\triangle(0, 1, e^{2\pi i/n})$ or in $\triangle(0, 1, e^{-2\pi i/n})$. The result by replacing the complex-conjugate pair in the spectrum of Example 1 with another pair of complex-conjugate values in these two triangles will not tamper the fact that every individual value is an eigenvalue of a certain stochastic matrix. However, whether these values are eigenvalues of the same stochastic matrix is difficult to confirm.

We experiment with, for instance, the eigenvalues $.3090 \pm 0.5000i$. Using the same initial values ($R_0 = 0.8329\mathbf{1}$) as in Example 1, we have experienced extremely slow convergence for this case. The history of $F(P, R)$ in Figure 4 clearly indicates this observation. The limit point, given by

$$E = \begin{bmatrix} 0.3818 & 0.0000 & 0.4568 & 0.0000 & 0.1614 \\ 0.5082 & 0.3314 & 0.0871 & 0.0049 & 0.0684 \\ 0.0000 & 0.0000 & 0.5288 & 0.4712 & 0.0000 \\ 0.0266 & 0.7634 & 0.0292 & 0.0310 & 0.1498 \\ 0.5416 & 0.0524 & 0.3835 & 0.0196 & 0.0029 \end{bmatrix},$$

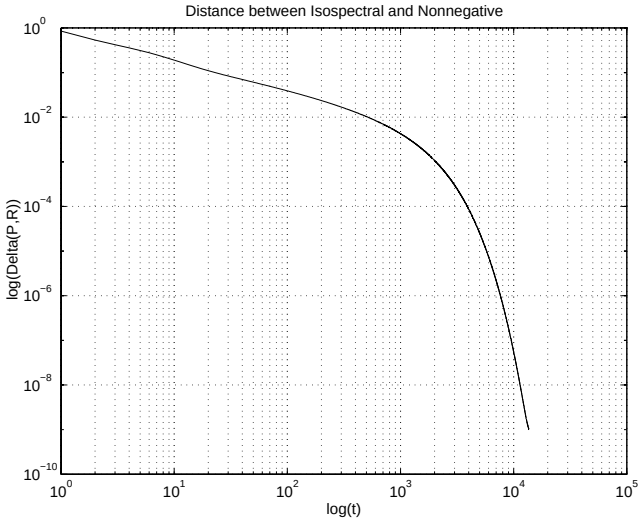

FIG. 5. *A log-log plot of $F(P(t), R(t))$ versus t for Example 3.*

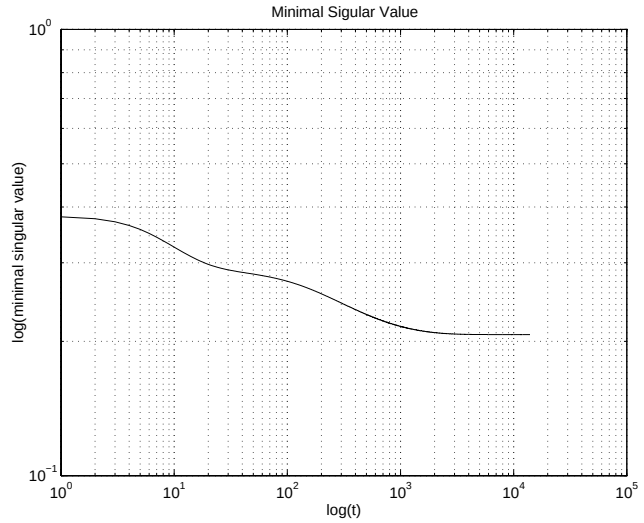

FIG. 6. *History of the smallest singular value for Example 3.*

exhibits an unexpected zero structure that we think is the cause of the slow convergence. The variation of the smallest singular value in the ASVD is plotted in Figure 5, indicating that matrices $P(t)$ stay away from singularity at a good distance. Suppose we modify the initial value to reflect the structure by simply setting the corresponding entries in the original R_0 zero. Then the flow converges to another limit point

$$F = \begin{bmatrix} 0.3237 & 0 & 0.4684 & 0 & 0.2079 \\ 0.4742 & 0.3184 & 0.1303 & 0.0007 & 0.0764 \\ 0 & 0.0000 & 0.5231 & 0.4769 & 0 \\ 0.0066 & 0.7536 & 0.0372 & 0.0958 & 0.1068 \\ 0.5441 & 0.0429 & 0.3959 & 0.0022 & 0.0149 \end{bmatrix}$$

at an almost equally slow pace. The spectra of both E and F agree with the desired data up to the integration error.

6. Conclusion. The theory of solvability on the inverse spectrum problem for stochastic or nonnegative matrices is yet to be developed, nevertheless we have proposed an ODE approach that is capable of constructing numerically stochastic or nonnegative matrices with the desired spectrum, if the spectrum is feasible. The method is easy to implement by existing ODE solvers. The method can also be used to approximate least squares solutions or linearly structured matrices.

7. Acknowledgement. The authors wish to thank Bart De Moor for bring this problem to their attention, and Carl Meyer for kindly pointing them to the reference [16].

REFERENCES

[1] W. W. Barrett and C. R. Johnson, Possible spectra of totally positive matrices, Linear Alg. Appl., 62(1984), 231-233.

[2] A. Berman and R. J. Plemmons, Nonnegative Matrices in the Mathematical Sciences, Classics in Applied Mathematics, 9, SIAM, Philadelphia, 1994.

[3] M. Boyle and D. Handelman, The spectra of nonnegative matrices via symbolic dynamics, Ann. Math., 133(1991), 249-316.

[4] F. Brauer and J. A. Hohel, Qualitative Theory of Ordinary Differential Equations, Benjamin, New York, 1969.

[5] A. Bunse-Gerstner, R. Byers, V. Mehrmann and N. K. Nichols, Numerical computation of an analytic singular value decomposition of a matrix valued function, Numer. Math. 60(1991), 1-40.

[6] M. T. Chu and K. R. Driessel, Constructing symmetric nonnegative matrices with prescribed eigenvalues by differential equations, SIAM J. Math. Anal., 22(1991), 1372-1387.

[7] B. De Moor, Private communication, 1994.

[8] M. Fiedler, Eigenvalues of nonnegative symmetric matrices, Linear Algebra and Appl., 9(1974), 119-142.

[9] S. Friedland, On an inverse problem for nonnegative and eventually nonnegative matrices, Israel J. Math., 29(1978), 43-60.

[10] S. Friedland, and A. A. Melkman, On the eigenvalues of nonnegative Jacobi matrices, Linear Algebra and Appl., 25(1979), 239-254.

[11] G. M. L. Gladwell, Inverse Problems in Vibration, Martinus Nijhoff Publishers, Dordrecht, Netherlands, 1986.
[12] D. Hershkowits, Existence of matrices with prescribed eigenvalues and entries, Linear and Multilinear Algebra, 14(1983), 315-342.
[13] R. Loewy and D. London, , A note on an inverse problems for nonnegative matrices, Linear and Multilinear Algebra, 6(1978), 83-90.
[14] F. I. Karpelevič, On the characteristic roots of matrices with nonnegative elements, Izv. Akad. Nauk SSSR Ser. Mat. 15(1951), 361-383 (in Russian).
[15] V. Mehrmann and W. Rath, Numerical methods for the computation of analytic singular value decompositions, Elec. J. Numer. Anal., 1(1993), 72-88.
[16] H. Minc, Nonnegative matrices, Wiley, New York, 1988.
[17] G. N. Oliveira, Nonnegative matrices with prescribed spectrum, Linear Alg. Appl., 54(1983), 117-121.
[18] M. Renardy and R. C. Rogers, An Introduction to Partial Differential Equations, Springer-Verlag, New York, 1993.
[19] L. F. Shampine and M. W. Reichelt, The MATLAB ODE suite, preprint, 1995.
[20] G. W. Soules, Constructing symmetric nonnegative matrices, Linear Alg. Appl., 13(1983), 241-251.
[21] K. Wright, Differential equations for the analytic singular value decomposition of a matrix, Numer. Math., 63(1992), 283-295.