Numerical Methods
for Inverse Singular Value Problems

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Abstract

Two numerical methods — one continuous and the other discrete — are proposed for solving inverse singular value problems. The first method consists of solving an ordinary differential equation obtained from an explicit calculation of the projected gradient of a certain objective function. The second method generalizes an iterative process proposed originally by Friedland et al. for solving inverse eigenvalue problems. With the geometry understood from the first method, it is shown that the second method (also, the method proposed by Friedland et al. for inverse eigenvalue problems) is a variation of the Newton method. While the continuous method is expected to converge globally at a slower rate (in finding a stationary point of the objective function), the discrete method is proved to converge locally at a quadratic rate (if there is a solution). Some numerical examples are presented.

1. Introduction.

For decades there has been considerable discussion about inverse eigenvalue problems. Some theoretical results and computational methods can be found, for example, in the articles [1, 2, 8, 9, 10] and the references contained therein. Recently Friedland et al. [10] have surveyed four quadratically convergent numerical methods for the following inverse eigenvalue problem:

> **(IEP)** Given real symmetric matrices $A_0, A_1, \ldots, A_n \in R^{n\times n}$ and real numbers $\lambda_1^* \geq \ldots \geq \lambda_n^*$, find values of $c := (c_1, \ldots, c_n)^T \in R^n$ such that the eigenvalues of the matrix
>
> $$A(c) := A_0 + c_1 A_1 + \ldots + c_n A_n \tag{1}$$
>
> are precisely $\lambda_1^*, \ldots, \lambda_n^*$.

In particular, the so called Method III proposed in [10] has been suggested to be a new method. Also included in [10] is a good collection of interesting applications where the (IEP) may arise.

In this paper we want to consider the inverse singular value problem, a question very analogous to the (IEP). The problem is stated as follows:

> **(ISVP)** Given real general matrices $B_0, B_1, \ldots, B_n \in R^{m\times n}$, $m \geq n$ and non-negative real numbers $\sigma_1^* \geq \ldots \geq \sigma_n^*$, find values of $c := (c_1, \ldots, c_n)^T \in R^n$ such that the singular values of the matrix
>
> $$B(c) := B_0 + c_1 B_1 + \ldots + c_n B_n \tag{2}$$
>
> are precisely $\sigma_1^*, \ldots, \sigma_n^*$.

At the present time, we do not know of any physical application of the (ISVP). But we think the problem is of interest in its own right.

Using the fact that the eigenvalues of the symmetric matrix $\begin{bmatrix} 0 & A \\ A^T & 0 \end{bmatrix}$ are plus and minus of the singular values of the matrix A, an (ISVP) can always be solved by conversion to an (IEP). On the other hand, it can easily be argued by counterexamples that the (IEP) will not always have a solution. Existence questions for the (IEP), therefore, should be considered under more restricted condition. The inverse Toeplitz eigenvalue problem (ITEP), for example, is a special case of the (IEP) where $A_0 = 0$ and $A_k := (A_{ij}^{(k)})$ with

$$A_{ij}^{(k)} := \begin{cases} 1, & \text{if } |i-j| = k-1; \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

Even though the (ITEP) is so specially structured, the question of whether symmetric Toeplitz matrices can have arbitrary real eigenvalues is still an open problem for $n \geq 5$ [3, 7, 15]. Likewise, the existence question for the (ISVP) might also be an interesting research topic. As yet we have not been aware of any result in the literature. The present paper is devoted to the numerical computation only.

The following notations will be used throughout the discussion: $\mathcal{O}(n)$ stands for the manifold of all orthogonal matrices in $R^{n\times n}$; $\Sigma := (\Sigma_{ij}) \in R^{m\times n}$ stands for the "diagonal" matrix in which

$$\Sigma_{ij} := \begin{cases} \sigma_i^*, & \text{if } 1 \leq i = j \leq n; \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

The set $\mathcal{M}_s(\Sigma)$ defined by

$$\mathcal{M}_s(\Sigma) := \{U\Sigma V^T | U \in \mathcal{O}(m), V \in \mathcal{O}(n)\} \tag{5}$$

obviously is equal to the set of all matrices in $R^{m\times n}$ whose singular values are precisely $\sigma_1^*, \ldots, \sigma_n^*$. We use $\mathcal{B}$ to denote the affine subspace

$$\mathcal{B} := \{B(c) | c \in R^n\}. \tag{6}$$

Clearly solving the (ISVP) is equivalent to finding an intersection of the two sets $\mathcal{M}_s(\Sigma)$ and $\mathcal{B}$. In this paper we propose two different ways to find such an intersection, if it exists.

Our first approach is motivated by a recent study of the projected gradient method [4]. The (ISVP) is cast as an equality-constrained optimization problem in which the distance (measured in the Frobenius norm) between $\mathcal{M}_s(\Sigma)$ and $\mathcal{B}$ is minimized. We show that the gradient of the distance function can be projected explicitly onto the feasible set without using Lagrange multipliers. Consequently, we are able to derive an ordinary differential equation which characterizes a steepest descent flow for the distance function. The steepest descent flow is easy to follow by using any available ODE software. Our first method for the (ISVP) is embedded in this continuous realization process. The formulation of the differential system is presented in Section 2. A similar approach for the (IEP) has already been discussed in [3].

Our second approach is simply a generalization of the so called Method III in [10]. Method III has been thought to be a new method. In the course of trying to understand why Method III works, we begin to realize, based on the knowledge we learn from [3, 4], that Method III can be interpreted geometrically as a "classical" Newton method. We emphasize the word "classical" because the geometry involved in Method III is closely related to that of the Newton method for one dimensional nonlinear equations. This interpretation (for the (IEP)) will be explained in Section 3. Once the geometry is understood, Method III can easily be generalized to an iterative process for the (ISVP). Furthermore, the method can be shown to converge quadratically. The discussion of our second approach is presented in Section 4.

One other important result of Friedland et al. is the modification of Method III so as to retain quadratic convergence when multiple eigenvalues are present. We certainly can do similar modification in our method when multiple singular values are present. This modification is described in Section 5. The behavior of our modified method is expected to be similar to that in [10]. Indeed, a proof of quadratic convergence can be established in the same spirit as in [10]. We shall not provide the proof in this paper. The numerical examples reported in Section 6, however, should illustrate our results.

Both of the continuous approach and the iterative approach generate sequences of matrices in the manifold $\mathcal{M}_s(\Sigma)$. But schematically, the continuous approach evolves explicitly in the manifold $\mathcal{M}_s(\Sigma)$ whereas the iterative approach is an implicit lifting of evolution in the affine subspace $\mathcal{B}$. It is also worth noting that the continuous method converges globally but slowly whereas the iterative method converges quadratically but locally. These features can, therefore, be taken advantage of in such a way that the continuous method is used first to a low order of accuracy to help get into the domain of convergence of the discrete method which, then, will be turned on to improve the accuracy.

2. A Continuous Approach for ISVP.

In this section we shall solve the (ISVP) by minimizing the distance between $\mathcal{M}_s(\Sigma)$ and $\mathcal{B}$. The distance is measured in term of the norm induced by the Frobenius inner product

$$< A, B > := trace(AB^T) = \sum_{i,j} A_{ij} B_{ij} \tag{7}$$

for any $A = (A_{ij})$ and $B = (B_{ij})$ in $R^{m \times n}$. We shall derive an ordinary differential equation that characterizes a steepest descent flow for the distance function.

For clarity, we shall assume that the given matrices $B_1, \ldots, B_n \in R^{m \times n}$ in the (ISVP) are linearly independent. A classical Gram-Schmidt process may be applied to produce a new sequence $\tilde{B}_1, \ldots, \tilde{B}_n$ that are mutually orthonormal with respect to the Frobenius inner product. Furthermore, these matrices are related by

$$\tilde{B}_k = \Gamma_{1k} B_1 + \ldots + \Gamma_{kk} B_k \tag{8}$$

for some real numbers Γ_{ij}. Obviously, if

$$\sum_{k=1}^{n} \tilde{c}_k \tilde{B}_k = \sum_{k=1}^{n} c_k B_k, \tag{9}$$

then $\tilde{c} := (\tilde{c}_1, \ldots, \tilde{c}_n)^T$ and $c := (c_1, \ldots, c_n)^T$ are related by

$$c = \Gamma \tilde{c} \tag{10}$$

where Γ is the upper triangular matrix

$$M := \begin{bmatrix} \Gamma_{11} & \Gamma_{12} & \ldots & & \Gamma_{1n} \\ 0 & \Gamma_{22} & \ldots & & \\ \vdots & & & & \vdots \\ 0 & \ldots & & 0 & \Gamma_{nn} \end{bmatrix}. \tag{11}$$

It is clear, therefore, that without loss of generality we may assume the matrices $B_1, \ldots, B_n$ are mutually orthonormal to begin with. We may further assume that B_0 is perpendicular to all B_k, for $k = 1, \ldots, n$. It will become clear in the sequel that these assumptions facilitate the computations.

Given an arbitrary $X \in R^{m \times n}$, it is not difficult to see that the distance between X and the affine subspace $\mathcal{B}$ is given by

$$dist(X, \mathcal{B}) = \|X - (B_0 + P(X))\| \tag{12}$$

where, due to the orthonormality,

$$P(X) = \sum_{k=1}^{n} < X, B_k > B_k \tag{13}$$

is simply the projection of X onto the linear subspace spanned by $B_1, \ldots, B_n$. For any $U \in R^{m \times m}$ and $V \in R^{n \times n}$, we define a residual matrix $R(U, V)$ by

$$R(U, V) := U \Sigma V^T - (B_0 + P(U \Sigma V^T)). \tag{14}$$

Our first approach of solving the (ISVP) is to consider the optimization problem:

$$\text{(15)} \qquad \begin{array}{ll} \text{Minimize} & F(U,V) := \frac{1}{2}\|R(U,V)\|^2 \\ \text{Subject to} & (U,V) \in \mathcal{O}(m) \times \mathcal{O}(n). \end{array}$$

We note in (15) that $U\Sigma V^T \in \mathcal{M}_s(\Sigma)$ so long as $(U,V) \in \mathcal{O}(m) \times \mathcal{O}(n)$. We note also that the feasible set $\mathcal{O}(m) \times \mathcal{O}(n)$ is a smooth manifold [6].

By introducing the product topology on $R^{m\times m} \times R^{n\times n}$ with the induced inner product

$$\text{(16)} \qquad < (A_1, B_1), (A_2, B_2) > := < A_1, A_2 > + < B_1, B_2 >,$$

we calculate the Fréchet derivative of F as follows:

$$\begin{aligned} F'(U,V)(H,K) &= < R(U,V), H\Sigma V^T + U\Sigma K^T - P(H\Sigma V^T + U\Sigma K^T) > \\ &= < R(U,V), H\Sigma V^T + U\Sigma K^T > \\ \text{(17)} \qquad &= < R(U,V)V\Sigma^T, H > + < \Sigma^T U^T R(U,V), K^T > . \end{aligned}$$

In the middle line of (17) we have used the fact that the range of R is always perpendicular to the range of P. The gradient ∇F of F, therefore, may be interpreted as the pair of matrices:

$$\text{(18)} \qquad \nabla F(U,V) = (R(U,V)V\Sigma^T, R(U,V)^T U\Sigma) \in R^{m\times m} \times R^{n\times n}.$$

Because of the product topology, we know

$$\text{(19)} \qquad \mathcal{T}_{(U,V)}(\mathcal{O}(m) \times \mathcal{O}(n)) = \mathcal{T}_U\mathcal{O}(m) \times \mathcal{T}_V\mathcal{O}(n)$$

where $\mathcal{T}_{(U,V)}(\mathcal{O}(m) \times \mathcal{O}(n))$ stands for the tangent space to the manifold $\mathcal{O}(m) \times \mathcal{O}(n)$ at $(U,V) \in \mathcal{O}(m) \times \mathcal{O}(n)$ and so on. The projection of $\nabla F(U,V)$ onto $\mathcal{T}_{(U,V)}(\mathcal{O}(m) \times \mathcal{O}(n))$, therefore, is the product of the projection of the first component of $\nabla F(U,V)$ onto $\mathcal{T}_U\mathcal{O}(m)$ and the projection of the second component of $\nabla F(U,V)$ onto $\mathcal{T}_V\mathcal{O}(n)$. Each of these projections can easily be calculated by using a technique developed in [4]. In particular, we claim that the projection $g(U,V)$ of the gradient $\nabla F(U,V)$ onto $\mathcal{T}_{(U,V)}(\mathcal{O}(m) \times \mathcal{O}(n))$ is given by the pair of matrices:

$$\begin{aligned} g(U,V) = \quad & \left(\frac{R(U,V)V\Sigma^T U^T - U\Sigma V^T R(U,V)^T}{2} U, \right. \\ \text{(20)} \qquad & \left. \frac{R(U,V)^T U\Sigma V^T - V\Sigma^T U^T R(U,V)}{2} V \right). \end{aligned}$$

Thus, the vector field

$$\text{(21)} \qquad \frac{d(U,V)}{dt} = -g(U,V)$$

defines a steepest descent flow on the manifold $\mathcal{O}(m) \times \mathcal{O}(n)$ for the objective function $F(U,V)$.

Corresponding to the flow (21) on the manifold $\mathcal{O}(m) \times \mathcal{O}(n)$, there is a flow on the set $\mathcal{M}_s(\Sigma)$ defined by

$$\text{(22)} \qquad X(t) := U(t)\Sigma V(t)^T.$$

Upon differentiating both sides of (22) with respect to t and substituting (21) for $\frac{dU}{dt}$ and $\frac{dV}{dt}$, we see that $X(t)$ is governed by an ordinary differential system:

$$\begin{aligned}\frac{dX}{dt} &= X\frac{X^T(B_0+P(X))-(B_0+P(X))^TX}{2} \\ &\quad -\frac{X(B_0+P(X))^T-(B_0+P(X))^TX}{2}X. \end{aligned} \tag{23}$$

Once an initial value of $X(0) \in \mathcal{M}_s(\Sigma)$ is specified, the equation (23) defines an initial value problem whose orbit moves in the steepest descent direction toward minimizing $dist(X(t), \mathcal{B})$. This is what we call the first method for the (ISVP). As $t \longrightarrow \infty$, the solution $X(t)$ of (23) moves toward a local stationary point for the distance function. A special case which we hope will occurœ is, of course, $dist(X(\infty), \mathcal{B}) = 0$. In this case, the coefficients needed in (ISVP) are determined from $c_i =< X(\infty), B_i >$. If this case does not happen, then one needs to change to another initial value and repeat the integration.

In the above derivation, no assumption on the multiplicity of singular values is needed. If $\sigma_1^*, \ldots, \sigma_n^*$ are all distinct, then it can be proved that $\mathcal{M}_s(\Sigma)$ is indeed a smooth manifold of dimension $\frac{m(m-1)+n(n-1)}{2}$. Otherwise, $\mathcal{M}_s(\Sigma)$ is a union of finitely many submanifolds with different dimensions. In either case, we note that any tangent vector $T(X)$ to $\mathcal{M}_s(\Sigma)$ at a point $X \in \mathcal{M}_s(\Sigma)$ about which a local chart can be defined must be of the form [3]

$$T(X) = XK - HX \tag{24}$$

for some skew symmetric matrices $H \in R^{m\times m}$ and $K \in R^{n\times n}$. The right-hand side of (23) obviously is a special case of (24). We will see in the next two sections that this observation helps us to generalize a result of Friedland et al. [10] from the (IEP) to the (ISVP).

3. A Newton Method for IEP.

In this section, we introduce a geometric interpretation of Method III proposed by Friedland et al. [10] for the (IEP) before we go on to discuss our second method for the (ISVP). We think such an interpretation is worth mentioning because the geometry behind Method III is very simple and not mentioned in [10], and also because this geometry sheds light on how the (ISVP) should be handled. For clarity, we shall consider the unmodified version of Method III only, that is, all eigenvalues $\lambda_1^*, \ldots, \lambda_n^*$ are assumed to be distinct. Let

$$\Lambda := diag\{\lambda_1^*, \ldots, \lambda_n^*\} \tag{25}$$

and let $\mathcal{A}$ denote the affine subspace

$$\mathcal{A} := \{A(c) | c \in R^n\} \tag{26}$$

where $A(c)$ is defined by (1). It can be proved that the set

$$\mathcal{M}_e(\Lambda) := \{Q\Lambda Q^T | Q \in \mathcal{O}(n)\} \tag{27}$$

is a smooth manifold of dimension $\frac{n(n-1)}{2}$. Similar to (24), any tangent vector $T(X)$ to $\mathcal{M}_e(\Lambda)$ at a point $X \in \mathcal{M}_e(\Lambda)$ must be of the form [3]

$$T(X) = XK - KX \tag{28}$$

for some skew-symmetric matrix $K \in R^{n\times n}$.

We recall the elementary fact that the new iterate $x^{(\nu+1)}$ of a classical Newton step

$$x^{(\nu+1)} = x^{(\nu)} - (f'(x^{(\nu)}))^{-1} f(x^{(\nu)}) \tag{29}$$

for a function $f : R \longrightarrow R$ is precisely the x-intercept of the line which is tangent to the graph of f at $(x^{(\nu)}, f(x^{(\nu)}))$. If we think of the surface $\mathcal{M}_e(\Lambda)$ as playing the role of the graph of f and the affine subspace $\mathcal{A}$ as playing the role of the x-axis, then an iterative process analogous to the Newton method can be developed for the (IEP).

Given $X^{(\nu)} \in \mathcal{M}_e(\Lambda)$, there exist a $Q^{(\nu)} \in \mathcal{O}(n)$ such that

$$Q^{(\nu)^T} X^{(\nu)} Q^{(\nu)} = \Lambda. \tag{30}$$

From (28), we know $X^{(\nu)} + X^{(\nu)}K - KX^{(\nu)}$ with any skew-symmetric matrix K represents a tangent vector to $\mathcal{M}_e(\Lambda)$ emanating from $X^{(\nu)}$. We thus seek an $\mathcal{A}$-intercept $A(c^{(\nu+1)})$ of such a vector with the affine subspace $\mathcal{A}$. That is, we want to find a skew-symmetric matrix $K^{(\nu)}$ and a vector $c^{(\nu+1)}$ such that

$$X^{(\nu)} + X^{(\nu)}K^{(\nu)} - K^{(\nu)}X^{(\nu)} = A(c^{(\nu+1)}). \tag{31}$$

The geometry is illustrated in Figure 1. [Insert Figure 1 in this space]

We now explain how the equation (31) can be solved. Using (30), it follows that

$$\Lambda + \Lambda\tilde{K}^{(\nu)} - \tilde{K}^{(\nu)}\Lambda = Q^{(\nu)^T} A(c^{(\nu+1)}) Q^{(\nu)} \tag{32}$$

where

$$\tilde{K}^{(\nu)} := Q^{(\nu)^T} K^{(\nu)} Q^{(\nu)} \tag{33}$$

is still a skew-symmetric matrix. The n diagonal equations of (32) give rise to the linear system

$$J^{(\nu)} c^{(\nu+1)} = \lambda^* - b^{(\nu)} \tag{34}$$

where

$$J_{ij}^{(\nu)} := q_i^{(\nu)^T} A_j q_i^{(\nu)}, \text{ for } i, j = 1, \ldots, n \tag{35}$$

$$\lambda^* := (\lambda_1^*, \ldots, \lambda_n^*)^T \tag{36}$$

$$b_i^{(\nu)} := q_i^{(\nu)^T} A_0 q_i^{(\nu)}, \text{ for } i = 1, \ldots, n \tag{37}$$

and $q_i^{(\nu)}$ is the i-th column of the matrix $Q^{(\nu)}$. The vector $c^{(\nu+1)}$, therefore, can be solved from (34). Once $c^{(\nu+1)}$ is obtained, the skew-symmetric matrix $\tilde{K}^{(\nu)}$ (and, hence, the matrix $K^{(\nu)}$) can be determined from the off-diagonal equations of (32). In fact,

$$\tilde{K}_{ij}^{(\nu)} = \frac{q_i^{(\nu)^T} A(c^{(\nu+1)}) q_j^{(\nu)}}{\lambda_i^* - \lambda_j^*}, \tag{38}$$

for $1 \leq i < j \leq n$. In this way, the equation (31) is completely solved.

In the classical Newton method the new iterate $x^{(\nu+1)}$ is "lifted up" naturally along the y-axis to the the point $(x^{(\nu+1)}, f(x^{(\nu+1)}))$ from which the next tangent line will begin. We note that $(x^{(\nu+1)}, f(x^{(\nu+1)}))$ is a point on the graph of f. Analogously, we now need a way to "lift up" the point $A(c^{(\nu+1)}) \in \mathcal{A}$ to a point $X^{(\nu+1)} \in \mathcal{M}_e(\Lambda)$. The difficulty here is that there is no obvious coordinate axis to follow. One possible way of this lifting can be motivated as follows: It is clear that solving the (IEP) is equivalent to finding an intersection of the two sets $\mathcal{M}_e(\Lambda)$ and $\mathcal{A}$. Suppose all the iterations are taking place near a point of intersection. Then we should have

$$X^{(\nu+1)} \approx A(c^{(\nu+1)}). \tag{39}$$

But from (31), we also should have

$$A(c^{(\nu+1)}) \approx e^{-K^{(\nu)}} X^{(\nu)} e^{K^{(\nu)}}. \tag{40}$$

High accuracy calculation of the exponential matrix $e^{K^{(\nu)}}$ in (40) is expensive and is not needed. So, instead, we define the Cayley transform

$$R := (I + \frac{K^{(\nu)}}{2})(I - \frac{K^{(\nu)}}{2})^{-1} \tag{41}$$

which happens to be the $(1, 1)$ Padé approximation of the matrix $e^{K^{(\nu)}}$. It is well known that $R \in \mathcal{O}(n)$, and that

$$R \approx e^{K^{(\nu)}} \tag{42}$$

if $\|K^{(\nu)}\|$ is small. Motivated by (39) and (40), we now define

$$X^{(\nu+1)} := R^T X^{(\nu)} R \in \mathcal{M}_e(\Lambda) \tag{43}$$

and the next iteration is ready to begin. It is interesting to note that

$$X^{(\nu+1)} \approx R^T e^{K^{(\nu)}} A(c^{(\nu+1)}) e^{-K^{(\nu)}} R \approx A(c^{(\nu+1)}) \tag{44}$$

represents what we mean by a lifting of the matrix $A(c^{(\nu+1)})$ from the affine subspace $\mathcal{A}$ to the surface $\mathcal{M}_e(\Lambda)$.

In summary, we realize that (31) is the equation for finding the $\mathcal{A}$-intercept of a tangent line passing $X^{(\nu)}$ and that (43) is the equation for lifting the $\mathcal{A}$-intercept to a point on $\mathcal{M}_e(\Lambda)$. The above process is identical to Method III proposed in [10], but the geometric meaning should be clearer now. We may thus say that Method III is precisely equivalent to the Newton method applied to $f(x) = 0$, for some specified $f(x)$. In [10] Method III is proved to converge quadratically. In the next section we shall use the same idea to develop a quadratically convergent method for the (ISVP).

4. An Iterative Approach for ISVP.

In this section we come back to the (ISVP) and develop a quadratically convergent iterative method. The given matrices $B_0, B_1, \ldots, B_n$ are no longer required to be orthonormal as assumed in Section 2. We shall assume, however, that all singular values $\sigma_1^*, \ldots, \sigma_n^*$ are positive and distinct. The multiple singular values case will be discussed in the next section.

Given $X^{(\nu)} \in \mathcal{M}_s(\Sigma)$, there exist $U^{(\nu)} \in \mathcal{O}(m)$ and $V^{(\nu)} \in \mathcal{O}(n)$ such that

$$U^{(\nu)^T} X^{(\nu)} V^{(\nu)} = \Sigma. \tag{45}$$

We now seek a $\mathcal{B}$-intercept of a line that is tangent to the manifold $\mathcal{M}_s(\Sigma)$ at $X^{(\nu)}$. According to (24) and the discussion in the proceeding section, this amounts to finding two skew-symmetric matrices $H^{(\nu)} \in R^{m\times m}$, $K^{(\nu)} \in R^{n\times n}$ and a vector $c^{(\nu+1)} \in R^n$ such that

$$X^{(\nu)} + X^{(\nu)}K^{(\nu)} - H^{(\nu)}X^{(\nu)} = B(c^{(\nu+1)}) \tag{46}$$

or equivalently

$$\Sigma + \Sigma\tilde{K}^{(\nu)} - \tilde{H}^{(\nu)}\Sigma = U^{(\nu)^T} B(c^{(\nu+1)})V^{(\nu)} \tag{47}$$

with

$$\tilde{H}^{(\nu)} := U^{(\nu)^T} H^{(\nu)} U^{(\nu)}, \tag{48}$$

$$\tilde{K}^{(\nu)} := V^{(\nu)^T} K^{(\nu)} V^{(\nu)}. \tag{49}$$

We note that the system (47) involves $\frac{m(m-1)}{2} + \frac{n(n-1)}{2} + n$ unknowns all together, namely the vector $c^{(\nu+1)}$ and the skew matrices $\tilde{H}^{(\nu)}$ and $\tilde{K}^{(\nu)}$. But there are only mn equations. Fortunately, a closer look at (47) shows that the $\frac{(m-n)(m-n-1)}{2}$ unknowns $\tilde{H}^{(\nu)}_{ij}$, $n+1 \le i \ne j \le m$, at the lower-right corner of $\tilde{H}^{(\nu)}$ are not bound to any equations at all. For simplicity, we shall set

$$\tilde{H}^{(\nu)}_{ij} = 0 \text{ for } n+1 \le i \ne j \le m. \tag{50}$$

For convenience, we denote

$$W^{(\nu)} := U^{(\nu)^T} B(c^{(\nu+1)})V^{(\nu)}. \tag{51}$$

Then (47) is equivalent to

$$W^{(\nu)}_{ij} = \Sigma_{ij} + \Sigma_{ii}\tilde{K}^{(\nu)}_{ij} - \tilde{H}^{(\nu)}_{ij}\Sigma_{jj}, \text{ for } 1 \le i \le n, 1 \le j \le n \tag{52}$$

where $\tilde{K}^{(\nu)}_{ij}$ is understood to be zero if $i \ge n+1$. We analyze these equations as follows:

(a) For $1 \le i = j \le n$, the equations from (52) can be rewritten as a linear system

$$J^{(\nu)}c^{(\nu+1)} = \sigma^* - b^{(\nu)} \tag{53}$$

where

$$J^{(\nu)}_{st} := u^{(\nu)^T}_s B_t v^{(\nu)}_s, \text{ for } 1 \le s,t \le n, \tag{54}$$

$$\sigma^* := (\sigma^*_1, \ldots, \sigma^*_n)^T, \tag{55}$$

$$b^{(\nu)}_s := u^{(\nu)^T}_s B_0 v^{(\nu)}_s, \text{ for } 1 \le s \le n. \tag{56}$$

and $u^{(\nu)}_s$ and $v^{(\nu)}_s$ are the column vectors of $U^{(\nu)}$ and $V^{(\nu)}$, respectively. If the matrix $J^{(\nu)}$ is nonsingular, then the vector $c^{(\nu+1)}$ is obtained from solving (53). Once $c^{(\nu+1)}$ is known, the matrix $W^{(\nu)}$ is determined.

(b) For $n+1 \le i \le m$ and $1 \le j \le n$, we have from (52)

$$\tilde{H}_{ij}^{(\nu)} = -\tilde{H}_{ji}^{(\nu)} = -\frac{W_{ij}^{(\nu)}}{\sigma_j^*}. \tag{57}$$

(c) For $1 \le i < j \le n$, we have from (52)

$$W_{ij}^{(\nu)} = \Sigma_{ii}\tilde{K}_{ij}^{(\nu)} - \tilde{H}_{ij}^{(\nu)}\Sigma_{jj}, \tag{58}$$

$$W_{ji}^{(\nu)} = \Sigma_{jj}\tilde{K}_{ji}^{(\nu)} - \tilde{H}_{ji}^{(\nu)}\Sigma_{ii} = -\Sigma_{jj}\tilde{K}_{ij}^{(\nu)} + \tilde{H}_{ij}^{(\nu)}\Sigma_{ii}. \tag{59}$$

Solving for $\tilde{H}_{ij}^{(\nu)}$ and $\tilde{K}_{ij}^{(\nu)}$ yields

$$\tilde{H}_{ij}^{(\nu)} = -\tilde{H}_{ji}^{(\nu)} = \frac{\sigma_i^* W_{ji}^{(\nu)} + \sigma_j^* W_{ij}^{(\nu)}}{(\sigma_i^*)^2 - (\sigma_j^*)^2}, \tag{60}$$

$$\tilde{K}_{ij}^{(\nu)} = -\tilde{K}_{ji}^{(\nu)} = \frac{\sigma_i^* W_{ij}^{(\nu)} + \sigma_j^* W_{ji}^{(\nu)}}{(\sigma_i^*)^2 - (\sigma_j^*)^2}. \tag{61}$$

By now, the equations in (47) are completely solved. The skew-symmetric matrices $H^{(\nu)}$ and $K^{(\nu)}$ are determined according to (48) and (49).

The next step is to lift the matrix $B(c^{(\nu+1)}) \in \mathcal{B}$ to a point $X^{(\nu+1)} \in \mathcal{M}_s(\Sigma)$ in a way similar to that for the (IEP). First we define two orthogonal matrices

$$R := (I + \frac{H^{(\nu)}}{2})(I - \frac{H^{(\nu)}}{2})^{-1}, \tag{62}$$

$$S := (I + \frac{K^{(\nu)}}{2})(I - \frac{K^{(\nu)}}{2})^{-1}. \tag{63}$$

Then the lifted matrix on $\mathcal{M}_s(\Sigma)$ is defined to be

$$X^{(\nu+1)} := R^T X^{(\nu)} S. \tag{64}$$

We note from (46) that

$$X^{(\nu+1)} \approx R^T(e^{H^{(\nu)}} B(c^{(\nu+1)}) e^{-K^{(\nu)}})S \tag{65}$$

is indeed a lifting since $R^T e^{H^{(\nu)}} \approx I_m$ and $e^{-K^{(\nu)}} S \approx I_n$, if $\|H^{(\nu)}\|$ and $\|K^{(\nu)}\|$ are small. Note also that the matrix $X^{(\nu+1)}$ needs not to be formed explicitly in the computation. Rather, only the orthogonal matrices

$$U^{(\nu+1)} := R^T U^{(\nu)} \tag{66}$$

and

$$V^{(\nu+1)} := S^T V^{(\nu)}. \tag{67}$$

are needed in the computation of (51) and (54). The iterative scheme is now completed.

It remains to show that the above scheme converges quadratically. We shall use the induced Frobenius norm in $R^{m\times m} \times R^{n\times n}$ (See (16)) to measure the discrepancy between iterates of $(U^{(\nu)}, V^{(\nu)})$. We first introduce a lemma (See Corollary 3.1 in [10]):

LEMMA 4.1. *Suppose the (ISVP) has an exact solution at* c^*. *Suppose* $B(c^*) = \hat{U}\Sigma\hat{V}^T$ *with* $(\hat{U}, \hat{V}) \in \mathcal{O}(m) \times \mathcal{O}(n)$. *For any* $(U, V) \in \mathcal{O}(m) \times \mathcal{O}(n)$, *let*

$$E := (E_1, E_2) := (U - \hat{U}, V - \hat{V}). \tag{68}$$

denote the error matrix. If the pair of matrices $(H, K) \in R^{m\times m} \times R^{n\times n}$ *are such that* $U\hat{U}^T = e^H$ *and* $V\hat{V}^T = e^K$, *then*

$$||(H, K)|| = O||E||. \tag{69}$$

Proof. The assertion follows immediately from the observation that $U\hat{U}^T = (E_1 + \hat{U})\hat{U}^T = I_m + E_1\hat{U}^T = e^H = I_m + H + O(||H||^2)$ and a similar expression for $V\hat{V}^T$. □

At the ν-th stage, define

$$E^{(\nu)} := (E_1^{(\nu)}, E_2^{(\nu)}) = (U^{(\nu)} - \hat{U}, V^{(\nu)} - \hat{V}). \tag{70}$$

The proof of the next theorem is parallel to that in [10].

THEOREM 4.2. *Suppose all singular values* $\sigma_1^*, \ldots, \sigma_n^*$ *are positive and distinct. Suppose also that the matrix* $J^{(\nu)}$ *defined in (54) is nonsingular. Then the next iteration (66) and (67) can be defined. Furthermore,*

$$||E^{(\nu+1)}|| = O(||E^{(\nu)}||^2). \tag{71}$$

Proof. The matrix ${U^{(\nu)}}^T B(c^*)V^{(\nu)}$, though lying on the manifold $\mathcal{M}_s(\Sigma)$, is not necessarily equal to Σ. If we write

$$\begin{aligned} {U^{(\nu)}}^T B(c^*)V^{(\nu)} &:= e^{-\hat{H}^{(\nu)}}\Sigma e^{\hat{K}^{(\nu)}} \\ &:= ({U^{(\nu)}}^T e^{-H_*^{(\nu)}} U^{(\nu)})\Sigma({V^{(\nu)}}^T e^{K_*^{(\nu)}} V^{(\nu)}) \end{aligned} \tag{72}$$

with

$$H_*^{(\nu)} := U^{(\nu)}\hat{H}^{(\nu)}{U^{(\nu)}}^T, \tag{73}$$

$$K_*^{(\nu)} := V^{(\nu)}\hat{K}^{(\nu)}{V^{(\nu)}}^T, \tag{74}$$

then it is easy to check that

$$e^{H_*^{(\nu)}} = U^{(\nu)}\hat{U}^T, \tag{75}$$

$$e^{K_*^{(\nu)}} = V^{(\nu)}\hat{V}^T. \tag{76}$$

By Lemma 4.1, we know that

$$||(H_*^{(\nu)}, K_*^{(\nu)})|| = O(||E^{(\nu)}||). \tag{77}$$

Because the Frobenius norm is invariant under orthogonal transformations, from (73) and (74) we also have

$$||(\hat{H}^{(\nu)}, \hat{K}^{(\nu)})|| = O(||E^{(\nu)}||). \tag{78}$$

Together with (72), it follows that

$$U^{(\nu)^T} B(c^*) V^{(\nu)} = \Sigma + \Sigma \hat{K}^{(\nu)} - \hat{H}^{(\nu)} \Sigma + O(\|E^{(\nu)}\|^2). \tag{79}$$

Taking the difference between (47) and (79) yields

$$\begin{aligned} & U^{(\nu)^T}(B(c^*) - B(c^{(\nu+1)}))V^{(\nu)} \\ = \; & \Sigma(\hat{K}^{(\nu)} - \tilde{K}^{(\nu)}) - (\hat{H}^{(\nu)} - \tilde{H}^{(\nu)})\Sigma + O(||E^{(\nu)}||^2). \end{aligned} \tag{80}$$

The diagonal equations of (80) give rise to a linear system

$$J^{(\nu)}(c^* - c^{(\nu+1)}) = O(||E^{(\nu)}||^2) \tag{81}$$

where $J^{(\nu)}$ is defined in (54). Thus

$$||c^* - c^{(\nu+1)}|| = O(||E^{(\nu)}||^2). \tag{82}$$

Similarly, from the off-diagonal equations of (80) it is not difficult to see that

$$||\hat{H}^{(\nu)} - \tilde{H}^{(\nu)}|| = O(||E^{(\nu)}||^2), \tag{83}$$

$$||\hat{K}^{(\nu)} - \tilde{K}^{(\nu)}|| = O(||E^{(\nu)}||^2). \tag{84}$$

Because of (78), it must be that

$$||(\tilde{H}^{(\nu)}, \tilde{K}^{(\nu)})|| = O(||E^{(\nu)}||). \tag{85}$$

From (48), (49), (73) and (74), it follows that

$$||H^{(\nu)} - H_*^{(\nu)}|| = O(||E^{(\nu)}||^2), \tag{86}$$

$$||K^{(\nu)} - K_*^{(\nu)}|| = O(||E^{(\nu)}||^2). \tag{87}$$

Observe that

$$\begin{aligned} & E_1^{(\nu+1)} := U^{(\nu+1)} - \hat{U} = R^T U^{(\nu)} - e^{-H_*^{(\nu)}} U^{(\nu)} \\ = \; & [(I - \frac{H^{(\nu)}}{2}) - (I - H_*^{(\nu)} + O(||H_*^{(\nu)}||^2)(I + \frac{H^{(\nu)}}{2})](I + \frac{H^{(\nu)}}{2})^{-1} U^{(\nu)} \\ = \; & [H_*^{(\nu)} - H^{(\nu)} + O(||H_*^{(\nu)} H^{(\nu)}|| + ||H_*^{(\nu)}||^2)](I + \frac{H^{(\nu)}}{2})^{-1} U^{(\nu)}. \end{aligned} \tag{88}$$

So it is clear now that

$$||E_1^{(\nu+1)}|| = O(||E^{(\nu)}||^2). \tag{89}$$

A similar argument works for $E_2^{(\nu+1)}$. Thus (71) is proved. □

We finally remark that one step of the above (Newton) method is a descent direction for the objective function (15). Thus it is possible to combine the objective function with some step-length control to improve the global convergence properties of our iterative method.

5. Multiple Singular Values.

In searching for the $\mathcal{B}$-intercept of a tangent line of $\mathcal{M}_s(\Sigma)$ the definition (57) allows no zero singular values. Similarly, the definitions (60) and (61) require all of the singular values to be distinct. In this section, we consider the case when multiple singular values are present. For clarity, we shall continue to assume that all singular values are positive. To simplify the notation, we shall also assume that only the first singular value σ_1^* is multiple, with multiplicity p.

We observe first that all the formulas (50), (53) and (57) are still well defined. For $1 \le i < j \le p$, however, we can conclude from (58) and (59) only that

$$W_{ij}^{(\nu)} + W_{ji}^{(\nu)} = 0. \tag{90}$$

Instead of determining values for $\tilde{H}_{ij}^{(\nu)}$ and $\tilde{K}_{ij}^{(\nu)}$, the system (90) gives rise to additional $q := \frac{p(p-1)}{2}$ equations for the vector $c^{(\nu+1)}$. That is, multiple singular values gives rise to an overdetermined system for $c^{(\nu+1)}$, a situation analogous to that discussed in [10] for the (IEP). Geometrically, the case implies that maybe no tangent line from $\mathcal{M}_s(\Sigma)$ will intercept the affine subspace $\mathcal{B}$ at all. To remedy this, we follow a strategy of Friedland et al. to modify the (ISVP) as:

> **(ISVP')** Given positive values $\sigma_1^* = \ldots = \sigma_p^* > \sigma_{p+1}^* > \ldots > \sigma_{n-q}^*$, find real values of $c_1, \ldots, c_n$ such that the $n-q$ largest singular values of the matrix $B(c)$ are $\sigma_1^*, \ldots, \sigma_{n-q}^*$.

Now that we have q degrees of freedom in choosing the remaining singular values, we shall use the equation (compare with (47))

$$\hat{\Sigma} + \hat{\Sigma}\tilde{K}^{(\nu)} - \tilde{H}^{(\nu)}\hat{\Sigma} = {U^{(\nu)}}^T B(c^{(\nu+1)}) V^{(\nu)} \tag{91}$$

to find the $\mathcal{B}$-intercept, where

$$\hat{\Sigma} := diag\{\sigma_1^*, \ldots, \sigma_{n-q}^*, \hat{\sigma}_{n-q+1}, \ldots, \hat{\sigma}_n\} \tag{92}$$

and $\hat{\sigma}_{n-q+1}, \ldots, \hat{\sigma}_n$ are free parameters. An algorithm for solving the (ISVP') proceeds as follows:

Given $U^{(\nu)} \in \mathcal{O}(m)$ and $V^{(\nu)} \in \mathcal{O}(n)$,

(a) Solve for $c^{(\nu+1)}$ from the system of equations:

$$\sum_{k=1}^{n} \left({u_i^{(\nu)}}^T B_k v_i^{(\nu)}\right) c_k^{(\nu+1)} = \sigma_i^* - {u_i^{(\nu)}}^T B_0 v_i^{(\nu)}, \text{ for } i = 1, \ldots, n-q \tag{93}$$

$$\begin{aligned} &\sum_{k=1}^{n} \left({u_s^{(\nu)}}^T B_k v_t^{(\nu)} + {u_t^{(\nu)}}^T B_k v_s^{(\nu)}\right) c_k^{(\nu+1)} = \\ &-{u_s^{(\nu)}}^T B_0 v_t^{(\nu)} - {u_t^{(\nu)}}^T B_0 v_s^{(\nu)}, \text{ for } 1 \le s < t \le p. \end{aligned} \tag{94}$$

(b) Define $\hat{\sigma}_k^{(\nu)}$ by

$$\hat{\sigma}_k^{(\nu)} := \begin{cases} \sigma_k^*, & \text{if } 1 \le k \le n-q; \\ {u_k^{(\nu)}}^T B(c^{(\nu+1)}) v_k^{(\nu)}, & \text{if } n-q < k \le n \end{cases} \tag{95}$$

(c) Once $c^{(\nu+1)}$ is determined, calculate $W^{(\nu)}$ according to (51).

(d) Define the skew symmetric matrices $\tilde{K}^{(\nu)}$ and $\tilde{H}^{(\nu)}$ according to the equation (91). For $1 \le i < j \le p$, the equation to be satisfied is

$$W_{ij}^{(\nu)} = \hat{\sigma}_i^{(\nu)} \tilde{K}_{ij}^{(\nu)} - \tilde{H}_{ij}^{(\nu)} \hat{\sigma}_j^{(\nu)}. \tag{96}$$

Thus there are many ways to define $\tilde{K}_{ij}^{(\nu)}$ and $\tilde{H}_{ij}^{(\nu)}$. For example, one may set $\tilde{K}_{ij}^{(\nu)} \equiv 0$ for $1 \le i < j \le p$. In this case, the skew-symmetric matrix $\tilde{K}^{(\nu)}$ is defined by

$$\tilde{K}_{ij}^{(\nu)} := \begin{cases} \frac{\hat{\sigma}_i^{(\nu)} W_{ij}^{(\nu)} + \hat{\sigma}_j^{(\nu)} W_{ji}^{(\nu)}}{(\hat{\sigma}_i^{(\nu)})^2 - (\hat{\sigma}_j^{(\nu)})^2}, & \text{if } 1 \le i < j \le n, \text{ and } p < j; \\ 0, & \text{if } 1 \le i < j \le p \end{cases} \tag{97}$$

and the skew-symmetric symmetric matrix $\tilde{H}^{(\nu)}$ is defined by

$$\tilde{H}_{ij}^{(\nu)} := \begin{cases} -\frac{W_{ij}^{(\nu)}}{\hat{\sigma}_j^{(\nu)}}, & \text{if } 1 \le i < j \le p; \\ -\frac{W_{ij}^{(\nu)}}{\hat{\sigma}_j^{(\nu)}}, & \text{if } n+1 \le i \le m, \text{ and } 1 \le j \le n; \\ \frac{\hat{\sigma}_i^{(\nu)} W_{ji}^{(\nu)} + \hat{\sigma}_j^{(\nu)} W_{ij}^{(\nu)}}{(\hat{\sigma}_i^{(\nu)})^2 - (\hat{\sigma}_j^{(\nu)})^2}, & \text{if } 1 \le i < j \le n, \text{ and } p < j; \\ 0, & \text{if } n+1 \le i \ne j \le m. \end{cases} \tag{98}$$

(e) Once $\tilde{H}^{(\nu)}$ and $\tilde{K}^{(\nu)}$ are determined, proceed with the lifting in the same way as for the (ISVP).

We should point out that no longer we are on a fixed manifold $\mathcal{M}_s(\Sigma)$, since $\hat{\Sigma}$ is changed at each step. We believe a proof of convergence similar to that given by Friedland et al. for the (IEP) can easily be carried out. In particular, we claim that the above algorithm for the case of multiple singular value converges quadratically. Our numerical results should illustrate this point.

6. Zero Singular Value.

A zero singular value indicates rank deficiency. To find a lower rank matrix in a generic affine subspace $\mathcal{B}$ is intuitively a more difficult problem. In this case, it is most likely that the (ISVP) does not have a solution.

To demonstrate what might happen to the algorithm proposed earlier, we consider the simplest case where $\sigma_1^* > \ldots > \sigma_{n-1}^* > \sigma_n^* = 0$. A closer look at equation (52) indicates that except for $\tilde{H}_{in}$ (and $\tilde{H}_{ni}$), $i = n+1, \ldots, m$, all other quantities including $c^{(\nu+1)}$ are well-defined. Furthermore, in order that equation (52) be valid, it is necessary that

$$W_{in}^{(\nu)} = 0 \text{ for } i = n+1, \ldots, m. \tag{99}$$

If condition (99) fails, then it simply says that no tangent line of $\mathcal{M}_s(\Sigma)$ from the current iterate $X^{(\nu)}$ will intersect the affine subspace $\mathcal{B}$. The iteration, therefore, can not be continued.

7. Numerical Experiments.

We have tested both continuous and iterative methods on various types of problems. In this section we report three numerical experiments. For demonstration purpose, we consider the case when $m = 5$ and $n = 4$. The following matrices, generated randomly by MATLAB on a DECstation 5000/200 from a normal distribution with mean 0.0 and variance 1.0, are used as the basis matrices:

$$B_0 = \begin{bmatrix} 1.1650 \times 10^{+0} & 6.2684 \times 10^{-1} & 7.5080 \times 10^{-2} & 3.5161 \times 10^{-1} \\ -6.9651 \times 10^{-1} & 1.6961 \times 10^{+0} & 5.9060 \times 10^{-2} & 1.7971 \times 10^{+0} \\ 2.6407 \times 10^{-1} & 8.7167 \times 10^{-1} & -1.4462 \times 10^{+0} & -7.0117 \times 10^{-1} \\ 1.2460 \times 10^{+0} & -6.3898 \times 10^{-1} & 5.7735 \times 10^{-1} & -3.6003 \times 10^{-1} \\ -1.3558 \times 10^{-1} & -1.3493 \times 10^{+0} & -1.2704 \times 10^{+0} & 9.8457 \times 10^{-1} \end{bmatrix}$$

$$B_1 = \begin{bmatrix} -4.4881 \times 10^{-2} & -7.9894 \times 10^{-1} & -7.6517 \times 10^{-1} & 8.6173 \times 10^{-1} \\ -5.6225 \times 10^{-2} & 5.1348 \times 10^{-1} & 3.9668 \times 10^{-1} & 7.5622 \times 10^{-1} \\ 4.0049 \times 10^{-1} & -1.3414 \times 10^{+0} & 3.7504 \times 10^{-1} & 1.1252 \times 10^{+0} \\ 7.2864 \times 10^{-1} & -2.3775 \times 10^{+0} & -2.7378 \times 10^{-1} & -3.2294 \times 10^{-1} \\ 3.1799 \times 10^{-1} & -5.1117 \times 10^{-1} & -2.0413 \times 10^{-3} & 1.6065 \times 10^{+0} \end{bmatrix}$$

$$B_2 = \begin{bmatrix} 8.4765 \times 10^{-1} & 2.6810 \times 10^{-1} & -9.2349 \times 10^{-1} & -7.0499 \times 10^{-2} \\ 1.4789 \times 10^{-1} & -5.5709 \times 10^{-1} & -3.3671 \times 10^{-1} & 4.1523 \times 10^{-1} \\ 1.5578 \times 10^{+0} & -2.4443 \times 10^{+0} & -1.0982 \times 10^{+0} & 1.1226 \times 10^{+0} \\ 5.8167 \times 10^{-1} & -2.7135 \times 10^{-1} & 4.1419 \times 10^{-1} & -9.7781 \times 10^{-1} \\ -1.0215 \times 10^{+0} & 3.1769 \times 10^{-1} & 1.5161 \times 10^{+0} & 7.4943 \times 10^{-1} \end{bmatrix}$$

$$B_3 = \begin{bmatrix} -5.0770 \times 10^{-1} & 8.8530 \times 10^{-1} & -2.4809 \times 10^{-1} & -7.2625 \times 10^{-1} \\ -4.4504 \times 10^{-1} & -6.1291 \times 10^{-1} & -2.0914 \times 10^{-1} & 5.6215 \times 10^{-1} \\ -1.0639 \times 10^{+0} & 3.5159 \times 10^{-1} & 1.1330 \times 10^{+0} & 1.4999 \times 10^{-1} \\ 7.0314 \times 10^{-1} & -5.2412 \times 10^{-2} & 2.0185 \times 10^{+0} & 9.2416 \times 10^{-1} \\ -1.8141 \times 10^{+0} & 3.4973 \times 10^{-2} & -1.8079 \times 10^{+0} & 1.0282 \times 10^{+0} \end{bmatrix}$$

$$B_4 = \begin{bmatrix} 3.9460 \times 10^{-1} & 6.3941 \times 10^{-1} & 8.7421 \times 10^{-1} & 1.7524 \times 10^{+0} \\ -3.2005 \times 10^{-1} & -1.3741 \times 10^{-1} & 6.1577 \times 10^{-1} & 9.7789 \times 10^{-1} \\ -1.1153 \times 10^{+0} & -5.5002 \times 10^{-1} & 3.9885 \times 10^{-2} & -2.4828 \times 10^{+0} \\ 1.1587 \times 10^{+0} & -1.0263 \times 10^{+0} & 1.1535 \times 10^{+0} & -7.8646 \times 10^{-1} \\ 6.3481 \times 10^{-1} & 8.2041 \times 10^{-1} & -1.7603 \times 10^{-1} & 5.6247 \times 10^{-1} \end{bmatrix}$$

So as to fit the data comfortably in the running text, we display all the numbers with only five digits.

Example 1 (The continuous method)

It is important to note that the system (23) is a differential equation defined on a manifold [11] — the solution $X(t)$ should in theory stay on the manifold $\mathcal{M}_s(\Sigma)$ for all t, provided $X(0) \in \mathcal{M}_s(\Sigma)$. Any numerical method, therefore, ought to maintain the orthogonality of the corresponding $U(t)$ and $V(t)$ in (22).

We can think of at least three approaches to conduct the experiments for the continuous method. The first is to directly apply a suitable initial value problem solver in such a way that the solution trajectory is closely followed. The second is to use ideas similar to [13] or [16] where the trajectory needs not to be followed accurately but a device is added which recovers orthogonality. The third is to design special schemes that are primary for trajectory problems, e.g., [5] ,[12] and the references therein. Apparently, the last two approaches require more theory development and programming involvement whereas the first approach should be quite straightforward. Since our purpose here is to report the behavior of the proposed (23) only, we are less concerned about how efficient the equation is solved. The first approach is used.

The subroutine ODE in [17] is used as the integrator. Both local control parameters ABSERR and RELERR in ODE are set to be 10^{-12}. This criterion is used to control the accuracy in following the solution path of (23). We examine the output values at time interval of 10. Normally, we should expect the loss of one or two digits in the global error. Thus, when the norm of the difference between two consecutive output points becomes less than 10^{-10}, we assume the path has converged to an equilibrium point. The execution is then terminated automatically.

By construction, the path of (23) is guaranteed to converge to a stationary point for the distance between $\mathcal{M}_s(\Sigma)$ and $\mathcal{B}$. Nonetheless, such a point is not necessarily a solution to the (ISVP). In our experiments, we have often run into this situation. Fortunately, the impasse can be circumvented by changing, e.g., a simple plane rotation, to another initial value $X(0)$ that is also in $\mathcal{M}_s(\Sigma)$.

Empirical data for two different sets of singular values are reported in Table 1. In each test, the initial value $X(0)$ is taken simply to be the diagonal matrix with σ^* as its diagonal entries. The value of t indicates how long it takes to meet the convergence criterion mentioned above.

We choose to report, in particular, the second case to illustrate how slow the convergence might be. Although we do not understand the cause completely, we note that the slowness is inherited in the differential equation itself and has nothing to do with the numerical method used. As is mentioned earlier, the topology of $\mathcal{M}_s(\Sigma)$ becomes complicated when multiple singular values are present. We suspect that the much slower convergence is probably due to the cluster of singular values. On the other hand, the status indicator IFLAG in the code ODE has never signaled any abnormal return in our application. Despite of the long integration, it appears that stiffness is not a major concern. For the majority of our other tests, the value of t for convergence seems to incline more toward the lower value reported. It would be interesting to use a statistical method to estimate the expected value of t.

Example 2 (The iterative method)

The iterative algorithm described in section 4 can easily be implemented with the aid of the package MATLAB.

Since the iterative algorithm converges only locally, our numerical experiment is meant solely to examine the quadratic rate of convergence. To make sure that the (ISVP) under testing does have a solution, we first randomly generate a vector $c^\# \in R^4$. Then singular values of the corresponding matrix $B(c^\#)$ are used as the prescribed singular values. We perturb each entry of the vector c by a uniform distribution between -1 and 1 and use the perturbed vector as the initial guess for the iteration.

Table 2 includes the initial guess $c^{(0)}$ and the corresponding limit point c^* for three test cases. It is interesting to note that the limit point c^* of the iteration is not necessary the same as the original vector $c^\#$ to which $c^{(0)}$ is reasonably close. The singular values of $B(c^*)$, however, do agree with those of $B(c^\#)$.

Table 3 indicates how the singular values of $B(c^{(0)})$ differ from those of $B(c^*)$.

The difference between singular values of $B(c^{(\nu)})$ and $B(c^*)$ is measured in the 2-norm. From Table 4 it is obvious that quadratic convergence indeed occurs in practice.

Example 3 (Multiple singular values)

In this example we want to illustrate that the iterative method modified in section 5 for the (ISVP') converges quadratically. Since multiple singular values are present, the construction of a numerical example is not trivial. For demonstration, we continue to use the same basis matrices as in Example 2. We assume the multiplicity is $p = 2$. Instead of doing the same as in Example 2, we fix the prescribed singular values $\sigma^* = (5, 5, 2)^T$ and search by trials the right initial guess of $c^{(0)}$.

Table 5 contains three different initial values of the vector $c^{(0)}$ and the corresponding limit points c^*.

Table 6 contains the singular values of $B(c^{(0)})$ and of $B(c^*)$. We note that the order of singular values in Case (c) has changed. The value 5 is no longer the largest singular value. In fact, unless the initial guess $c^{(0)}$ is close enough to an exact solution c^*, we have no reason to expect that our algorithm, especially (95), will preserve the ordering. Nevertheless, once convergence occurs, then from (91) we know that σ^* must be a subset of the singular values of the final matrix.

Table 7 displays the 2-norm of the vector $\sigma^{(\nu)} - \sigma^*$ throughout the iteration. It is seen that at the initial stage the convergence is slow, but eventually the rate picks up and becomes quadratic. This observation agrees with our prediction.

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	Case (a)
σ^*	$(6.3315, 4.7638, 2.6501, 1.5722)^T$
t	250
$X(\infty)$	$\begin{bmatrix} 6.7675\times10^{-1} & -1.3732\times10^{+0} & -2.3249\times10^{+0} & -9.0641\times10^{-1} \\ -2.8887\times10^{-1} & 2.5478\times10^{+0} & -2.9574\times10^{-1} & 1.5012\times10^{+0} \\ 2.6286\times10^{+0} & -5.4150\times10^{-1} & -1.1678\times10^{+0} & 4.4892\times10^{+0} \\ 6.7451\times10^{-1} & -2.5151\times10^{+0} & -1.4408\times10^{+0} & 1.3932\times10^{-1} \\ -6.5918\times10^{-1} & -3.1698\times10^{+0} & -7.6316\times10^{-1} & 2.4759\times10^{+0} \end{bmatrix}$
	Case (b)
σ^*	$(2.4735, 2.2550, 2.0092, 0.9150)^T$
t	14100
$X(\infty)$	$\begin{bmatrix} 1.3212\times10^{+0} & 7.7597\times10^{-1} & 1.1678\times10^{-1} & -9.0992\times10^{-3} \\ -5.8743\times10^{-1} & 1.5219\times10^{+0} & -1.4853\times10^{-1} & 1.4905\times10^{+0} \\ 5.7452\times10^{-1} & 9.2632\times10^{-1} & -1.8383\times10^{+0} & -6.7451\times10^{-1} \\ 9.4451\times10^{-1} & 1.4559\times10^{-1} & 4.3708\times10^{-1} & -4.2765\times10^{-1} \\ -2.7595\times10^{-1} & -1.2179\times10^{+0} & -8.5478\times10^{-1} & 4.5995\times10^{-1} \end{bmatrix}$

TABLE 1
Empirical data for Example 1

$c^{(\nu)}$	Case (a)	Case (b)	Case (c)
$c_1^{(0)}$	4.4029×10^{-1}	$1.9984\times10^{+0}$	$1.0639\times10^{+0}$
$c_2^{(0)}$	3.9909×10^{-1}	$1.3802\times10^{+0}$	$1.2202\times10^{+0}$
$c_3^{(0)}$	$-1.5330\times10^{+0}$	-6.5989×10^{-1}	-4.4669×10^{-2}
$c_4^{(0)}$	$-1.3434\times10^{+0}$	-9.7495×10^{-1}	6.6642×10^{-2}
c_1^*	-8.8571×10^{-2}	$2.3693\times10^{+0}$	8.8169×10^{-1}
c_2^*	5.1462×10^{-1}	$1.6524\times10^{+0}$	$1.3168\times10^{+0}$
c_3^*	$-1.0132\times10^{+0}$	$-1.3027\times10^{+0}$	-2.5081×10^{-1}
c_4^*	-8.8725×10^{-1}	-9.3529×10^{-1}	-1.5865×10^{-1}

TABLE 2
Initial and final values of $c^{(\nu)}$ for Example 2

$\sigma^{(\nu)}$	Case (a)	Case (b)	Case (c)
$\sigma_1^{(0)}$	$6.8899\times10^{+0}$	$1.0936\times10^{+1}$	$6.8474\times10^{+0}$
$\sigma_2^{(0)}$	$5.1197\times10^{+0}$	$6.5909\times10^{+0}$	$5.7249\times10^{+0}$
$\sigma_3^{(0)}$	$3.9921\times10^{+0}$	$4.4003\times10^{+0}$	$3.2213\times10^{+0}$
$\sigma_4^{(0)}$	$1.4204\times10^{+0}$	8.0052×10^{-1}	$1.4699\times10^{+0}$
σ_1^*	$5.2995\times10^{+0}$	$1.3414\times10^{+1}$	$6.9201\times10^{+0}$
σ_2^*	$3.3937\times10^{+0}$	$8.0732\times10^{+0}$	$5.1246\times10^{+0}$
σ_3^*	$2.2206\times10^{+0}$	$5.0761\times10^{+0}$	$3.3332\times10^{+0}$
σ_4^*	$1.0707\times10^{+0}$	3.8920×10^{-1}	$1.0239\times10^{+0}$

TABLE 3
Singular values of $B(c^{(\nu)})$ for Example 2

Iterations	Case (a)	Case (b)	Case (c)
0	$2.9612 \times 10^{+0}$	$2.9940 \times 10^{+0}$	7.5970×10^{-1}
1	2.0771×10^{-1}	1.4643×10^{-1}	3.1871×10^{-1}
2	1.6456×10^{-2}	2.4630×10^{-2}	5.5975×10^{-2}
3	3.4464×10^{-4}	5.2647×10^{-3}	2.5138×10^{-3}
4	5.6724×10^{-8}	1.0221×10^{-3}	5.7491×10^{-6}
5	1.2113×10^{-14}	6.5434×10^{-5}	1.1412×10^{-11}
6		3.1503×10^{-7}	3.4399×10^{-15}
7		7.2704×10^{-12}	
8		6.5100×10^{-15}	

TABLE 4
Errors of singular values for Example 2

$c^{(\nu)}$	Case (a)	Case (b)	Case (c)
$c_1^{(0)}$	-6.7476×10^{-1}	$-1.1547 \times 10^{+0}$	-2.8000×10^{-1}
$c_2^{(0)}$	-7.3995×10^{-1}	$1.8322 \times 10^{+0}$	$1.6200 \times 10^{+0}$
$c_3^{(0)}$	8.1359×10^{-1}	$1.9587 \times 10^{+0}$	-1.8000×10^{-1}
$c_4^{(0)}$	3.8499×10^{-1}	$1.0081 \times 10^{+0}$	$1.3600 \times 10^{+0}$
c_1^*	$-1.3441 \times 10^{+0}$	$1.1428 \times 10^{+0}$	-7.6956×10^{-1}
c_2^*	-5.7219×10^{-2}	3.6652×10^{-1}	$1.8501 \times 10^{+0}$
c_3^*	4.9023×10^{-1}	3.0984×10^{-1}	$1.7808 \times 10^{+0}$
c_4^*	4.8810×10^{-1}	-2.2213×10^{-1}	9.5787×10^{-1}

TABLE 5
Initial and final values of $c^{(\nu)}$ for Example 3

$\sigma^{(\nu)}$	Case (a)	Case (b)	Case (c)
$\sigma_1^{(0)}$	$5.7416 \times 10^{+0}$	$1.0176 \times 10^{+1}$	$7.7756 \times 10^{+0}$
$\sigma_2^{(0)}$	$5.2465 \times 10^{+0}$	$5.1525 \times 10^{+0}$	$5.1142 \times 10^{+0}$
$\sigma_3^{(0)}$	$2.6168 \times 10^{+0}$	$5.0567 \times 10^{+0}$	$3.8264 \times 10^{+0}$
$\sigma_4^{(0)}$	9.7351×10^{-1}	$2.1606 \times 10^{+0}$	$2.0274 \times 10^{+0}$
σ_1^*	$5.0000 \times 10^{+0}$	$5.0000 \times 10^{+0}$	$9.7369 \times 10^{+0}$
σ_2^*	$5.0000 \times 10^{+0}$	$5.0000 \times 10^{+0}$	$5.0000 \times 10^{+0}$
σ_3^*	$2.0000 \times 10^{+0}$	$2.0000 \times 10^{+0}$	$5.0000 \times 10^{+0}$
σ_4^*	7.2598×10^{-1}	$1.5539 \times 10^{+0}$	$2.0000 \times 10^{+0}$

TABLE 6
Singular values of $B(c^{(\nu)})$ for Example 3

Iterations	Case (a)	Case (b)	Case (c)
0	9.9559×10^{-1}	$6.0128 \times 10^{+0}$	$2.2887 \times 10^{+0}$
1	4.3843×10^{-1}	$4.9779 \times 10^{+1}$	$5.1367 \times 10^{+0}$
2	8.5750×10^{-2}	$7.8916 \times 10^{+1}$	$5.1903 \times 10^{+0}$
3	6.2047×10^{-3}	$1.2682 \times 10^{+1}$	$4.8564 \times 10^{+0}$
4	1.8996×10^{-5}	$4.5087 \times 10^{+0}$	$5.4301 \times 10^{+0}$
5	1.9263×10^{-10}	9.9793×10^{-1}	$4.2116 \times 10^{+0}$
6	3.1086×10^{-15}	1.6460×10^{-1}	$2.8036 \times 10^{+0}$
7		8.2634×10^{-3}	$3.7452 \times 10^{+0}$
8		1.4440×10^{-5}	$2.3434 \times 10^{+0}$
9		9.1313×10^{-11}	8.5750×10^{-2}
10		2.9458×10^{-15}	$5.2509 \times 10^{+0}$
11			$4.4207 \times 10^{+0}$
12			4.9467×10^{-1}
13			2.5058×10^{-1}
14			8.3009×10^{-2}
15			6.7032×10^{-3}
16			9.3119×10^{-5}
17			1.0896×10^{-8}
18			2.5511×10^{-15}

TABLE 7

Errors of singular values for Example 3