SINGULAR VALUE REASSIGNMENT WITH LOW RANK MATRICES
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Abstract. Analogous to the pole assignment problem where eigenvalues of a square matrix are relocated, this paper considers the problem of reassigning singular values of a rectangular matrix by additive low rank matrices. Precise and easy-to-check necessary and sufficient conditions under which the problem is solvable are completely characterized, generalizing some traditional singular value inequalities. The constructive proof makes it possible to compute such a solution numerically.

Key words. singular values, interlacing inequalities, reassignment, low rank updates.

1. Introduction. Given matrices $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$, the state feedback pole assignment problem concerns the finding of a matrix $F \in \mathbb{R}^{m \times n}$ such that the matrix $A + BF$ has a prescribed set of eigenvalues. Such an eigenvalue reassignment problem arises from the application where the state $\mathbf{x}(t) \in \mathbb{R}^n$ of a certain physical system under the dynamic state equation:

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t) \tag{1.1}$$

is to be controlled by the input $\mathbf{u}(t) \in \mathbb{R}^m$. One classical problem in control theory is to select the input $\mathbf{u}(t)$ so that the dynamics of the resulting $\mathbf{x}(t)$ is driven into to a certain desired state. In the state feedback control, the input $\mathbf{u}(t)$ is selected as a linear function of current state $\mathbf{x}(t)$, i.e.,

$$\mathbf{u}(t) = F\mathbf{x}(t). \tag{1.2}$$

In this way, the system (1.1) is changed to a closed-loop dynamical system:

$$\dot{\mathbf{x}}(t) = (A + BF)\mathbf{x}(t). \tag{1.3}$$

A general goal in such a control scheme is to choose the *gain matrix* $F \in \mathbb{R}^{m \times n}$ so as to achieve stability and to speed up response. To accomplish this goal, the problem can be translated into choosing F so as to reassign eigenvalues of the matrix $A + BF$.

The pole assignment problem has been thoroughly studied in the literature. See, for example, [3, 15, 18, 22]. Let $\lambda(M)$ denote the spectrum of a square matrix M. A standard result in this field is that, given any set of n complex numbers $\{\lambda_1, \ldots, \lambda_n\}$ which is closed under complex conjugation, a matrix $F \in \mathbb{R}^{m \times n}$ exists such that $\lambda(A + BF) = \{\lambda_1, \ldots, \lambda_n\}$ if and only if

$$\text{rank}\,[A - \mu I, B] = n, \quad \text{for all } \mu \in \mathbb{C}. \tag{1.4}$$

The condition (1.4) usually is referred to as the pair (A, B) being *controllable*. In the single-input case where $m = 1$, it is further known that the pole assignment problem, if solvable, has a unique solution. On the other hand, it can be proved that

$$\bigcap_{F \in \mathbb{R}^{m \times n}} \lambda(A + BF) = \{\mu \in \lambda(A) \mid \text{rank}\,[A - \mu I, B] < n\},$$

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implying that for a certain peculiar pair (A, B) of matrices the eigenvalues of A cannot be reassigned by any F. This kind of unassignable matrix pair forms a zero measure set.

Analogous to the task of reassigning eigenvalues of a square matrix, it is natural to ask whether the singular values of a rectangular matrix can be arbitrarily reassigned via additive low rank matrices. More specifically, the following special type of inverse singular value problem (ISVPrk) is studied in this paper. For convenience, we use $\sigma(N)$ to denote henceforth the set of singular values of a general matrix N.

> **(ISVPrk)** *Given a matrix $A \in \mathbb{R}^{m \times n}$ ($m \geq n$), an integer $n \geq \ell > 0$ and real numbers $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_n \geq 0$, find a matrix $F \in \mathbb{R}^{m \times n}$ such that $\mathrm{rank}(F) \leq \ell$ and $\sigma(A + F) = \{\beta_1, \beta_2, \cdots, \beta_n\}$.*

The state feedback pole assignment problem is but a special case of the much broader class of inverse eigenvalue problems which has attracted remarkable attention in recent years. See, for example, [2, 7] and the many references cited therein. In contrast, the inverse singular value problems have not received as many studies. Some earlier results that can be considered as inverse problems include the de Oliveira theorem [9] on the principal elements and singular values, the the Weyl-Horn theorem [13, 21] on the relationship between singular values and eigenvalues, and the Sing-Thompson theorem [17, 20] on the majorization between the diagonal elements and singular values. Some related numerical work can be found in [4, 5, 6]. However, we are not aware of any discussion on the problem formulated as the ISVPrk which can also be considered as an analogue of the well-known additive inverse eigenvalue problem.

Although an inverse singular value problem can be recast as a specially structured inverse eigenvalue problem [7], the existing theory does not provide us a clue on when the ISVPrk is solvable. Our main contributions in this paper are twofold: one is that we completely characterize the necessary and sufficient condition under which the above ISVPrk is solvable and the other is that we offer a constructive proof which can be implemented as a numerical means to find the solution.

2. Rank One Update. We shall begin with the case where F is of rank one only. The discussion will be used as a building block to extend to the general case.

Given any fixed column vector $\mathbf{b} \in \mathbb{R}^m$, consider first the case where F is of the form $F = \mathbf{b}\mathbf{f}^\top$ for some undetermined column vector $\mathbf{f} \in \mathbb{R}^n$. Choose the Householder transformation $Q_\mathbf{b} \in \mathbb{R}^{m \times m}$ so that

$$Q_\mathbf{b}^\top \mathbf{b} = \begin{bmatrix} b_0 \\ \mathbf{0} \end{bmatrix}, \tag{2.1}$$

where $b_0 = \|\mathbf{b}\|_2 \in \mathbb{R}$ and $\mathbf{0}$ denotes throughout the paper a zero vector of appropriate size. Denote the product $Q_\mathbf{b}^\top A$ in blocks, i.e.,

$$Q_\mathbf{b}^\top A = \begin{bmatrix} \mathbf{a}_\mathbf{b}^\top \\ A_\mathbf{b} \end{bmatrix},$$

with $\mathbf{a}_\mathbf{b} \in \mathbb{R}^n$ and $A_\mathbf{b} \in \mathbb{R}^{(m-1) \times n}$. Depending on whether $m = n$ or $m > n$, the singular value decomposition of $A_\mathbf{b}$ is of the form,

$$A_\mathbf{b} = U_\mathbf{b} \begin{bmatrix} \gamma_1 & & & & 0 \\ & \gamma_2 & & & 0 \\ & & \ddots & & \vdots \\ & & & \gamma_{n-1} & 0 \end{bmatrix} V_\mathbf{b}^\top \quad \text{or} \quad A_\mathbf{b} = U_\mathbf{b} \begin{bmatrix} \gamma_1 & & & \\ & \gamma_2 & & \\ & & \ddots & \\ & & & \gamma_n \\ \mathbf{0} & & & \mathbf{0} \end{bmatrix} V_\mathbf{b}^\top,$$

respectively. For any $\mathbf{f} \in \mathbb{R}^n$, denote

$$(\mathbf{a}_{\mathbf{b}}^\top + b_0 \mathbf{f}^\top) V_{\mathbf{b}} = [f_1, f_2, \cdots, f_n]. \tag{2.2}$$

Define the matrix $A(\mathbf{f}) \in \mathbb{R}^{(n+1) \times n}$ as a function of $\mathbf{f}$ by

$$A(\mathbf{f}) = \begin{bmatrix} f_1 & f_2 & \cdots & f_{n-1} & f_n \\ \gamma_1 & & & & \\ & \gamma_2 & & & \\ & & \ddots & & \\ & & & \gamma_{n-1} & \\ & & & & \gamma_n \end{bmatrix},$$

where $\gamma_n = 0$ if $m = n$. Then, we know that

$$\sigma(A + \mathbf{bf}^\top) = \{\beta_1, \beta_2, \cdots, \beta_n\} \iff \sigma(A(\mathbf{f})) = \{\beta_1, \beta_2, \cdots, \beta_n\}.$$

In the above, note that for each given $\mathbf{b}$ the matrix $A_{\mathbf{b}}$ is known and hence values of γ_i's are also known. To solve the ISVPrk for the case of $F = \mathbf{bf}^\top$, it suffices to determine the values of $f_1, \ldots, f_n$. Toward that end, we have the following necessary and sufficient condition.

LEMMA 2.1. *Given any fixed $\mathbf{b} \in \mathbb{R}^m$, there exists a vector $\mathbf{f} \in \mathbb{R}^n$ such that*

$$\sigma(A(\mathbf{f})) = \{\beta_1, \beta_2, \cdots, \beta_n\} \tag{2.3}$$

if and only if

$$\beta_i \geq \gamma_i \geq \beta_{i+1}, \quad i = 1, 2, \cdots, n, \tag{2.4}$$

where $\beta_{n+1} := 0$.

Proof. The necessity of the interlacing inequality (2.4) is a well known property of singular value decompositions. See, for example, [11].

To prove the sufficiency, assume that the interlacing inequality (2.4) holds. Observe first that

$$\mathcal{A}_{\mathbf{f}} := A(\mathbf{f})A(\mathbf{f})^\top = \begin{bmatrix} \sum_{i=1}^n f_i^2 & f_1\gamma_1 & f_2\gamma_2 & \cdots & f_{n-1}\gamma_{n-1} & f_n\gamma_n \\ f_1\gamma_1 & \gamma_1^2 & & & & \\ f_2\gamma_2 & & \gamma_2^2 & & & \\ \vdots & & & \ddots & & \\ f_{n-1}\gamma_{n-1} & & & & \gamma_{n-1}^2 & \\ f_n\gamma_n & & & & & \gamma_n^2 \end{bmatrix}$$

is a bordered matrix in $\mathbb{R}^{(n+1) \times (n+1)}$. As such, we now describe a way to construct a vector $\mathbf{f} \in \mathbb{R}^n$ such that (2.3) holds true. Our approach modifies from a technique developed earlier for the Jacobi inverse eigenvalue problems [2]. Because

$$\sigma(A(\mathbf{f})) = \{\beta_1, \cdots, \beta_n\} \iff \lambda(\mathcal{A}_{\mathbf{f}}) = \{\beta_1^2, \cdots, \beta_n^2, 0\},$$

finding a vector $\mathbf{f} \in \mathbb{R}^n$ such that (2.3) holds is equivalent to proving the identity

$$p(\mu) := \mu \prod_{i=1}^n (\mu - \beta_i^2) - \det(\mu I - \mathcal{A}_{\mathbf{f}}) \equiv 0, \tag{2.5}$$

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On one hand, note that

$$p(\mu) = \left(\text{trace}(\mathcal{A}_{\mathbf{f}}) - \sum_{i=1}^{n}\beta_i^2\right)\mu^n + \text{low degree terms in } \mu. \tag{2.6}$$

So $p(\mu)$ is a polynomial of degree at most n in μ. On the other hand, we can expand the determinant of $\mathcal{A}_{\mathbf{f}}$ and thus expect the vector $\mathbf{f}$ to solve the equation

$$\mu\prod_{j=1}^{n}(\mu-\beta_j^2) = (\mu - \sum_{i=1}^{n}f_i^2)\prod_{j=1}^{n}(\mu-\gamma_j^2) - \sum_{i=1}^{n}\left((f_i\gamma_i)^2\prod_{\substack{j=1\\j\neq i}}^{n}(\mu-\gamma_j^2)\right). \tag{2.7}$$

We shall divide our proof into four mutually exclusive cases.

Case 1. Assume that all γ_k, $k = 1,\ldots n$, are distinct and nonzero. For each k, by setting $\mu = \gamma_k^2$ in (2.7), we obtain

$$\gamma_k^2\prod_{j=1}^{n}(\gamma_k^2-\beta_j^2) = -(f_k\gamma_k)^2\prod_{\substack{j=1\\j\neq k}}^{n}(\gamma_k^2-\gamma_j^2).$$

Thus, f_k^2 is uniquely determined [1] by

$$f_k^2 = -\frac{\prod_{j=1}^{n}(\gamma_k^2-\beta_j^2)}{\prod_{\substack{j=1\\j\neq k}}^{n}(\gamma_k^2-\gamma_j^2)}, \quad k = 1,\cdots,n. \tag{2.10}$$

The interlacing property (2.4) guarantees that the right hand side of (2.10) is nonnegative and hence real-valued f_k can be defined. With this choice of $f_1,\cdots,f_n$, we see that $p(\mu)$ has $n+1$ zeros at $\mu = 0, \gamma_1^2, \cdots, \gamma_n^2$ and hence $p(\mu) \equiv 0$.

Case 2. Assume that $\gamma_1 > \cdots > \gamma_t > \gamma_{t+1} = \cdots = \gamma_n = 0$ for some integer t. In this case, the interlacing inequality (2.4) implies that

$$\beta_{t+2} = \cdots = \beta_n = 0,$$

and that the equality (2.7) is reduced to

$$\mu^{n-t}\prod_{j=1}^{t+1}(\mu-\beta_j^2) = \mu^{n-t}\left\{(\mu-\sum_{i=1}^{n}f_i^2)\prod_{j=1}^{t}(\mu-\gamma_j^2) - \sum_{i=1}^{t}\left((f_i\gamma_i)^2\prod_{\substack{j=1\\j\neq i}}^{t}(\mu-\gamma_j^2)\right)\right\}. \tag{2.11}$$

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[1]Incidentally, we have established an interesting equality which would be hard to prove in other context. By the fact that the sum of eigenvalues equals to the trace, we have

$$\sum_{i=1}^{n}f_i^2 + \sum_{i=1}^{n}\gamma_i^2 = \sum_{i=1}^{n}\beta_i^2. \tag{2.8}$$

The equality,

$$\sum_{i=1}^{n}\gamma_i^2 - \sum_{i=1}^{n}\beta_i^2 = \sum_{i=1}^{n}\frac{\prod_{j=1}^{n}(\gamma_i^2-\beta_j^2)}{\prod_{\substack{j=1\\j\neq i}}^{n}(\gamma_i^2-\gamma_j^2)}, \tag{2.9}$$

thus holds for any two sequences of numbers $\{\beta_1,\cdots,\beta_n\}$ and $\{\gamma_1,\cdots,\gamma_n\}$, if all γ_i's are distinct and nonzero, and (2.4) is satisfied.

It follows that

$$\prod_{j=1}^{t+1}(\mu - \beta_j^2) = (\mu - \sum_{i=1}^{n} f_i^2)\prod_{j=1}^{t}(\mu - \gamma_j^2) - \sum_{i=1}^{t}\left((f_i\gamma_i)^2 \prod_{\substack{j=1\\ j\neq i}}^{t}(\lambda - \gamma_j^2)\right). \qquad (2.12)$$

By setting $\mu = \gamma_k^2$, we can determine that

$$f_k^2 = -\frac{\prod_{j=1}^{t+1}(\gamma_k^2 - \beta_j^2)}{\gamma_k^2 \prod_{\substack{j=1\\ j\neq k}}^{t}(\gamma_k^2 - \gamma_j^2)}, \quad k = 1, \cdots, t. \qquad (2.13)$$

To obtain values for the remaining $f_{t+1}, \cdots, f_n$, observe that $\lambda(\mathcal{A}_\mathbf{f}) = \{\beta_1^2, \cdots, \beta_t^2, \beta_{t+1}^2, 0, \cdots, 0\}$. Computing the trace of $\mathcal{A}_\mathbf{f}$ yields the constraint that

$$\sum_{i=1}^{t+1}\beta_i^2 = \sum_{i=1}^{n} f_i^2 + \sum_{i=1}^{n}\gamma_i^2 = \sum_{i=1}^{n} f_i^2 + \sum_{i=1}^{t}\gamma_i^2.$$

Therefore, the only condition imposed upon $f_{t+1}, \cdots, f_n$ is that

$$\sum_{i=t+1}^{n} f_i^2 = \beta_{t+1}^2 + \left(\sum_{i=1}^{t}\beta_i^2 - \sum_{i=1}^{t}\gamma_i^2\right) - \sum_{i=1}^{t} f_i^2,$$

$$= \beta_{t+1}^2 - \sum_{i=1}^{t}\frac{\prod_{j=1}^{t}(\gamma_i^2 - \beta_j^2)}{\prod_{\substack{j=1\\ j\neq i}}^{t}(\gamma_i^2 - \gamma_j^2)} + \sum_{i=1}^{t}\frac{\prod_{j=1}^{t+1}(\gamma_i^2 - \beta_j^2)}{\gamma_i^2 \prod_{\substack{j=1\\ j\neq i}}^{t}(\gamma_i^2 - \gamma_j^2)}$$

$$= \beta_{t+1}^2 \left(1 - \sum_{i=1}^{t}\frac{\prod_{j=1}^{t}(\gamma_i^2 - \beta_j^2)}{\gamma_i^2 \prod_{\substack{j=1\\ j\neq i}}^{t}(\gamma_i^2 - \gamma_j^2)}\right). \qquad (2.14)$$

In the above we have employed the fact derived in (2.9) to arrive at the second equality. So long as $\mathbf{f} \in \mathbb{R}^n$ satisfy (2.14), the polynomial $p(\mu)$ in (2.6) is of degree at most $n-1$ in μ which, by (2.13), has t non-zero roots at $\mu = \gamma_k^2$, $k = 1, \ldots, t$ and one zero root with multiplicity $n - t$. It follows that $p(\mu) \equiv 0$.

Case 3. Assume that the set $\{\gamma_1, \gamma_2, \cdots, \gamma_n\}$ consists of t many distinct non-zero elements. For each $j = 1, \cdots, t$, let s_j denote the first index in the group such that

$$\gamma_{s_j} = \gamma_{s_j+1} = \cdots = \gamma_{s_{j+1}-1},$$

where $s_{t+1} = n + 1$. The interlacing condition (2.4) enforces that

$$\beta_{s_j+1} = \cdots = \beta_{s_{j+1}-1} = \gamma_{s_j}.$$

Furthermore, the equality (2.7) becomes

$$\mu\left\{\prod_{j=1}^{t}(\mu - \beta_{s_j}^2)\right\}\left\{\prod_{j=1}^{t}(\mu - \gamma_{s_j}^2)^{s_{j+1}-s_j-1}\right\} = \left\{\prod_{j=1}^{t}(\mu - \gamma_{s_j}^2)^{s_{j+1}-s_j-1}\right\} \times$$

$$\left\{(\mu - \sum_{i=1}^{n} f_i^2)\prod_{j=1}^{t}(\mu - \gamma_{s_j}^2) - \sum_{i=1}^{t}\left((f_{s_i}^2 + \cdots + f_{s_{i+1}-1}^2)\gamma_{s_i}^2 \prod_{\substack{j=1\\ j\neq i}}^{t}(\mu - \gamma_{s_j}^2)\right)\right\}. \qquad (2.15)$$

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After cancellation, we obtain that

$$\mu\left\{\prod_{j=1}^{t}(\mu-\beta_{s_j}^2)\right\} = \left(\mu - \sum_{j=1}^{t}\sum_{i=s_j}^{s_{j+1}-1} f_i^2\right)\prod_{j=1}^{t}(\mu-\gamma_{s_j}^2)$$

$$-\sum_{i=1}^{t}\left(\left(\sum_{s=s_i}^{s_{i+1}-1} f_s^2\right)\gamma_{s_i}^2 \prod_{\substack{j=1\\j\neq i}}^{t}(\mu-\gamma_{s_j}^2)\right). \tag{2.16}$$

Consequently, by setting $\mu=\gamma_{s_k}^2$, we obtain a constraint where

$$\sum_{s=s_k}^{s_{k+1}-1} f_s^2 = -\frac{\prod_{j=1}^{t}(\gamma_{s_k}^2-\beta_{s_j}^2)}{\prod_{\substack{j=1\\j\neq k}}^{t}(\gamma_{s_k}^2-\gamma_{s_j}^2)}, \quad k=1,\cdots,t. \tag{2.17}$$

Note that formula (2.17) is similarly to formula (2.10) in Case 1. Using a similar argument, it can now be shown that $p(\mu)\equiv 0$ whenever $\mathbf{f}$ satisfies the condition (2.17).

Case 4. Assume that the set $\{\gamma_1,\gamma_2,\cdots,\gamma_n\}$ consists of $t+1$ distinct elements including one zero, that is, assume that

$$\gamma_{s_1} > \gamma_{s_2} > \cdots > \gamma_{s_t} > \gamma_{s_{t+1}} = 0,$$

where the indices s_j's are defined in the same way as those in Case 3. The interlacing inequality (2.4) implies that

$$\beta_{s_j+1} = \cdots = \beta_{s_{j+1}-1} = \gamma_{s_j},$$
$$\beta_{s_{t+1}+1} = \cdots = \beta_n = 0.$$

and the equality (2.7) becomes

$$\mu^{n-s_{t+1}+1}\left\{\prod_{j=1}^{t+1}(\mu-\beta_{s_j}^2)\right\}\left\{\prod_{j=1}^{t}(\mu-\gamma_{s_j}^2)^{s_{j+1}-s_j-1}\right\} = \mu^{n-s_{t+1}+1}\left\{\prod_{j=1}^{t}(\mu-\gamma_{s_j}^2)^{s_{j+1}-s_j-1}\right\} \times$$

$$\left\{(\mu-\sum_{i=1}^{n}f_i^2)\prod_{j=1}^{t}(\mu-\gamma_{s_k}^2) - \sum_{i=1}^{t}\left((f_{s_i}^2+\cdots+f_{s_{i+1}-1}^2)\gamma_{s_i}^2\prod_{\substack{j=1\\j\neq i}}^{t}(\mu-\gamma_{s_j}^2)\right)\right\}, \tag{2.18}$$

which is further reduced to

$$\prod_{j=1}^{t+1}(\mu-\beta_{s_j}^2) = (\mu-\sum_{i=1}^{n}f_i^2)\prod_{j=1}^{t}(\mu-\gamma_{s_j}^2) - \sum_{i=1}^{t}\left((f_{i_k}^2+\cdots+f_{i_{k+1}-1}^2)\gamma_{s_i}^2\prod_{\substack{j=1\\j\neq i}}^{t}(\mu-\gamma_{s_j}^2)\right).$$

So, we have by setting $\mu=\gamma_{s_k}^2$ that

$$\sum_{s=s_k}^{s_{k+1}-1} f_s^2 = -\frac{\prod_{j=1}^{t+1}(\gamma_{s_k}^2-\beta_{s_j}^2)}{\gamma_{s_k}^2\prod_{\substack{j=1\\j\neq k}}^{t}(\gamma_{s_k}^2-\gamma_{s_j}^2)}, \quad k=1,\cdots,t. \tag{2.19}$$

Finally, the constraint imposed upon $\sum_{s=s_t+1}^{n} f_s^2$ can be obtained by the trace condition (2.8). That is, we obtain by using the identity (2.9) that

$$
\begin{aligned}
\sum_{s=s_t+1}^{n} f_s^2 &= \sum_{i=1}^{n} \beta_i^2 - \sum_{i=1}^{n} \gamma_i^2 - \sum_{i=1}^{t} \sum_{s=s_i}^{s_{i+1}-1} f_s^2 \\
&= \beta_{s_{t+1}}^2 + \left(\sum_{i=1}^{t} \beta_{s_i}^2 - \sum_{i=1}^{t} \gamma_{s_i}^2 \right) - \sum_{i=1}^{t} \sum_{s=s_i}^{s_{i+1}-1} f_s^2 \\
&= \beta_{s_{t+1}}^2 - \sum_{i=1}^{t} \frac{\prod_{j=1}^{t}(\gamma_{s_i}^2 - \beta_{s_j}^2)}{\prod_{\substack{j=1 \\ j \neq i}}^{t} (\gamma_{s_i}^2 - \gamma_{s_j}^2)} + \sum_{i=1}^{t} \frac{\prod_{j=1}^{t+1}(\gamma_{s_i}^2 - \beta_{s_j}^2)}{\gamma_{s_i}^2 \prod_{\substack{j=1 \\ j \neq i}}^{t} (\gamma_{s_i}^2 - \gamma_{s_j}^2)} \\
&= \beta_{s_{t+1}}^2 \left(1 - \sum_{i=1}^{t} \frac{\prod_{j=1}^{t}(\gamma_{s_i}^2 - \beta_{s_j}^2)}{\gamma_{s_i}^2 [\prod_{\substack{j=1 \\ j \neq i}}^{t} (\gamma_{s_i}^2 - \gamma_{s_j}^2)]} \right).
\end{aligned} \tag{2.20}
$$

Now using similar arguments as in Cases 1, 2, and 3, we can prove (2.3) so long as $\mathbf{f}$ satisfying both (2.19) and (2.20).

With all four cases considered, the sufficiency of (2.4) is established. The proof is now completed. $\square$

Recall that the values of γ_i's referred to in Lemma 2.1 are determined by the orthogonal transformation $Q_\mathbf{b}$ in (2.1) which, in turn, depends on the vector $\mathbf{b}$. This relationship indicates that if vector $\mathbf{b}$ is changed, then the interlacing inequality (2.4) will also be changed. Recall also that if

$$
\sigma(A) = \{\alpha_1, \cdots, \alpha_n\}, \quad \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n,
$$

then the γ_i's corresponding to any $\mathbf{b}$ must satisfy the inequalities

$$
\alpha_i \geq \gamma_i \geq \alpha_{i+1}, \quad i = 1, \cdots, n, \tag{2.21}
$$

with $\alpha_{n+1} = 0$. We are thus motivated to consider using $\mathbf{b}$ as a parameter to control the singular values. The following result is particularly interesting.

LEMMA 2.2. *Let $A \in \mathbb{R}^{m \times n}$ ($m \geq n$) be given and fixed. Corresponding to any values γ_i, $i = 1, \cdots, n$, satisfying the interlacing inequality (2.21) where $\gamma_n = 0$ if $m = n$, there exists a unit vector $\mathbf{b} \in \mathbb{R}^m$ and an orthogonal matrix $Q_\mathbf{b} \in \mathbb{R}^{m \times m}$ such that*

$$
Q_\mathbf{b}^\top \mathbf{b} = \begin{bmatrix} 1 \\ \mathbf{0} \end{bmatrix},
$$

and

$$
Q_\mathbf{b}^\top A = \begin{bmatrix} \mathbf{a}_\mathbf{b}^\top \\ A_\mathbf{b} \end{bmatrix},
$$

with

$$
\sigma(A_\mathbf{b}) = \begin{cases} \{\gamma_1, \cdots, \gamma_{n-1}\}, & \text{if } m = n, \\ \{\gamma_1, \cdots, \gamma_n\}, & \text{if } m > n. \end{cases} \tag{2.22}
$$

Proof. Depending on whether $m = n$ or $m > n$, define $\tilde{A} \in \mathbb{R}^{m \times n}$ by

$$\tilde{A} = \begin{bmatrix} 0 & 0 & \cdots & & 0 \\ \gamma_1 & & & & \\ & \gamma_2 & & & \\ & & \ddots & & \vdots \\ & & & \gamma_{n-1} & 0 \end{bmatrix} \quad \text{or} \quad \tilde{A} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \gamma_1 & & & \\ & \gamma_2 & & \\ & & \ddots & \\ & & & \gamma_n \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix},$$

where the last row of $\mathbf{0}$'s are used, if $m > n + 1$, to pad the row dimension m. Because of the interlacing inequality (2.21), by Lemma 2.1 with $\tilde{\mathbf{b}} = \begin{bmatrix} 1 \\ \mathbf{0} \end{bmatrix} \in \mathbb{R}^m$, there exists a column vector $\mathbf{c} \in \mathbb{R}^n$ such that

$$\sigma(\tilde{A} + \tilde{\mathbf{b}}\mathbf{c}^\top) = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}.$$

Let the singular value decompositions of A and $\tilde{A} + \tilde{\mathbf{b}}\mathbf{c}^\top$ be denoted as $A = U_1 \Sigma V_1^\top$ and $\tilde{A} + \tilde{\mathbf{b}}\mathbf{c}^\top = U_2 \Sigma V_2^\top$, respectively. Define

$$Q_{\mathbf{b}} := U_1 U_2^\top \quad \text{and} \quad \mathbf{b} := Q_{\mathbf{b}} \tilde{\mathbf{b}}.$$

The partition

$$Q_{\mathbf{b}}^\top A = (\tilde{A} + \tilde{\mathbf{b}}\mathbf{c}^\top)(V_2 V_1^\top) = \begin{bmatrix} \mathbf{a}_{\mathbf{b}}^\top \\ A_{\mathbf{b}} \end{bmatrix}$$

has the properties that

$$\mathbf{a}_{\mathbf{b}} = A^\top \mathbf{b} \quad (= V_1 V_2^\top \mathbf{c})$$

and

$$A_{\mathbf{b}} = \begin{bmatrix} \gamma_1 & & & 0 \\ & \gamma_2 & & \\ & & \ddots & \vdots \\ & & \gamma_{n-1} & 0 \end{bmatrix} V_2 V_1^\top \quad \text{or} \quad \begin{bmatrix} \gamma_1 & & & \\ & \gamma_2 & & \\ & & \ddots & \\ & & & \gamma_n \\ 0 & 0 & \cdots & 0 \end{bmatrix} V_2 V_1^\top.$$

The assertion therefore is proved. $\square$

We are now able to answer ISVPrk completely with the following result.

THEOREM 2.3. *The following three statements are equivalent:*

1. *The ISVPrk with $\ell = 1$ is solvable.*
2. *For each $i = 1, \cdots, n$, there exists a value γ_i satisfying both inequalities*

$$\alpha_i \geq \gamma_i \geq \alpha_{i+1}, \tag{2.23}$$

$$\beta_i \geq \gamma_i \geq \beta_{i+1}, \tag{2.24}$$

   where $\alpha_{n+1} := 0$ and $\beta_{n+1} := 0$.
3. *For each $i = 1, \cdots, n-1$,*

$$\beta_{i+1} \leq \alpha_i \quad \text{and} \quad \alpha_{i+1} \leq \beta_i. \tag{2.25}$$
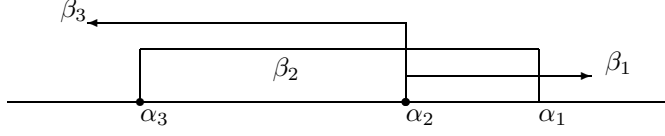
FIG. 2.1. *Feasible range of α_i's and β_i's for the case $n = 3$.*

Proof. By keeping the ordering $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$ and $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_n$, the equivalence of Statements 2 and 3 is obvious. We only need to show the equivalence of Statements 1 and 2.

Assume that the ISVPrk has a rank one solution $F \in \mathbb{R}^{m \times n}$. There exists orthogonal matrix Q_F such that

$$Q_F^\top F = \left[\begin{array}{c} \mathbf{f}^\top \\ \mathbf{O} \end{array} \right],$$

where $\mathbf{f} \in \mathbb{R}^n$ and $\mathbf{O}$ is a zero matrix in $\mathbb{R}^{(m-1) \times n}$. Write

$$Q_F^\top A = \left[\begin{array}{c} \mathbf{a_f}^\top \\ A_F \end{array} \right],$$

where $\mathbf{a_f} \in \mathbb{R}^n$ and $A_F \in \mathbb{R}^{(m-1) \times n}$. Let

$$\sigma(A_F) = \begin{cases} \{\gamma_1, \gamma_2, \cdots, \gamma_{n-1}\} & \text{if } m = n, \\ \{\gamma_1, \gamma_2, \cdots, \gamma_n\}, & \text{if } m > n, \end{cases}$$

with the descending order $\gamma_1 \geq \gamma_2 \geq \cdots$. Then by the fact that A_F is a submatrix of both $Q_F^\top A$ and $Q_F^\top (A+F)$, the singular values of A_F interlace with those of both $Q_F^\top A$ and $Q_F^\top (A+F)$, giving rise to the inequalities (2.23) and (2.24), respectively. We thus have proved that statement 1 implies statement 2.

To show the converse, assume first that the interlacing inequality (2.23) holds. By Lemma 2.2, there exists a unit vector $\mathbf{b} \in \mathbb{R}^m$ and orthogonal matrix $Q_{\mathbf{b}} \in \mathbb{R}^{m \times m}$ such that $Q_{\mathbf{b}}^\top \mathbf{b} = \left[\begin{array}{c} 1 \\ \mathbf{0} \end{array} \right]$, $Q_{\mathbf{b}}^\top A = \left[\begin{array}{c} \mathbf{a_b}^\top \\ A_{\mathbf{b}} \end{array} \right]$, and $\sigma(A_{\mathbf{b}})$ is precisely as described in (2.22). By the interlacing inequality (2.24) and Lemma 2.1, we then obtain a vector $\mathbf{f} \in \mathbb{R}^n$ such that

$$\sigma(A + \mathbf{b}\mathbf{f}^\top) = \{\beta_1, \beta_2, \cdots, \beta_n\}.$$

Therefore, the ISVPrk is solved by defining $F = \mathbf{b}\mathbf{f}^\top$. $\square$

It is worthy to point out that α_i's and β_i's in Theorem 2.3 do not necessarily satisfy any interlacing property. The feasible range for the case $n = 3$ is exemplified in Figure 2.1. Note that β_2 can be any value between α_1 and α_3 so long as $\beta_1 \geq \beta_2 \geq \beta_3$.

We conclude this section with an interesting observation characterizing the class of special matrices A of which the singular values absolutely cannot be reassigned by any rank one matrices.

The following result is a corollary from Theorem 2.3.

COROLLARY 2.4. *Let the multiplicity of all distinct singular values $\alpha_1(A)$, $\cdots$, $\alpha_t(A)$ of A be denoted as $s_1, \cdots, s_t$. Then*

$$\bigcap_{\mathrm{rank}(F) \leq 1} \sigma(A + F) = \{\alpha_k(A) \text{ with algebraic multiplicity } (s_k - 2) \mid s_k > 2, \ 1 \leq k \leq t\}.$$

Hence,

$$\bigcap_{\text{rank}(F)\leq 1} \sigma(A+F) = \emptyset \quad \Longleftrightarrow \quad s_k \leq 2, \ k = 1, \cdots, t.$$

3. Main Result. We are now ready to deal with the general ISVPrk. The following result includes Theorem 2.3 as a special case in that the "gap" between the original singular values α_i's and the desirable singular values β_i's is separated by the rank number ℓ.

THEOREM 3.1. *The ISVPrk is solvable if and only if for each $i = 1, \cdots n - \ell$,*

$$\beta_{i+\ell} \leq \alpha_i, \quad \text{and} \quad \alpha_{i+\ell} \leq \beta_i. \tag{3.1}$$

Proof. We shall prove our result by the mathematical induction. Note that we have already established the case for $\ell = 1$ in Theorem 2.3. Assume that the assertion in Theorem 3.1 is true for $\ell = k$. We want to establish the case $\ell = k + 1 \leq n$.

We first argue the necessity. Any $F \in \mathbb{R}^{m \times n}$ with $\text{rank}(F) \leq k + 1$ can be factorized as

$$F = F_1 + F_2,$$

with $\text{rank}(F_1) \leq k$ and $\text{rank}(F_2) \leq 1$. Denote

$$\sigma(A + F_1) = \{\gamma_1, \gamma_2, \cdots, \gamma_n\}.$$

Then, by assumption, it must be that for each $i = 1, \cdots, n - k$

$$\gamma_{i+k} \leq \alpha_i \quad \text{and} \quad \alpha_{i+k} \leq \gamma_i.$$

In addition, using Theorem 2.3, we have that for each $i = 1, \cdots, n - 1$

$$\beta_{i+1} \leq \gamma_i \quad \text{and} \quad \gamma_{i+1} \leq \beta_i.$$

Together, we see for each $i = 1, \cdots, n - k - 1$ it must be true that

$$\beta_{i+k+1} \leq \alpha_i \quad \text{and} \quad \alpha_{i+k+1} \leq \beta_i.$$

We then argue the sufficiency. Note that

$$\begin{cases} \beta_{k+2} & \leq & \alpha_1 \\ \beta_{k+3} & \leq & \alpha_2 \\ & \vdots & \\ \beta_n & \leq & \alpha_{n-k-1} \end{cases} \quad \text{and} \quad \begin{cases} \alpha_{k+2} & \leq & \beta_1 \\ \alpha_{k+3} & \leq & \beta_2 \\ & \vdots & \\ \alpha_n & \leq & \beta_{n-k-1} \end{cases}.$$

It is a matter of inspection that there exist γ_i, $i = 1, \cdots, n$, with

$$\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_n,$$

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such that

$$
\left\{
\begin{array}{ccccc}
\max\{\beta_1, \alpha_1\} & < & \gamma_1 & & \\
\max\{\alpha_{k+2}, \beta_3\} & \leq & \gamma_2 & \leq & \beta_1 \\
\max\{\alpha_{k+3}, \beta_4\} & \leq & \gamma_3 & \leq & \beta_2 \\
& & \vdots & & \\
\max\{\alpha_n, \beta_{n-k+1}\} & \leq & \gamma_{n-k} & \leq & \beta_{n-k-1} \\
\beta_{n-k} & \leq & \gamma_{n+1-k} & \leq & \beta_{n-k} \\
& & \vdots & & \\
\beta_{k+1} & \leq & \gamma_k & \leq & \beta_{k-1} \\
\beta_{k+2} & \leq & \gamma_{k+1} & \leq & \min\{\beta_k, \alpha_1\} \\
\beta_{k+3} & \leq & \gamma_{k+2} & \leq & \min\{\beta_{k+1}, \alpha_2\} \\
& & \vdots & & \\
\beta_n & \leq & \gamma_{n-1} & \leq & \min\{\beta_{n-2}, \alpha_{n-1-k}\} \\
& & \gamma_n & = & 0
\end{array}
\right. ,
$$

if $n - k < k + 1$, or

$$
\left\{
\begin{array}{ccccc}
\max\{\alpha_1, \beta_1\} & < & \gamma_1 & & \\
\max\{\alpha_{k+2}, \beta_3\} & \leq & \gamma_2 & \leq & \beta_1 \\
\max\{\alpha_{k+3}, \beta_4\} & \leq & \gamma_3 & \leq & \beta_2 \\
& & \vdots & & \\
\max\{\alpha_{2k}, \beta_{k+1}\} & \leq & \gamma_k & \leq & \beta_{k-1} \\
\max\{\alpha_{2k+1}, \beta_{k+2}\} & \leq & \gamma_{k+1} & \leq & \min\{\alpha_1, \beta_k\} \\
\max\{\alpha_{2k+2}, \beta_{k+3}\} & \leq & \gamma_{k+2} & \leq & \min\{\alpha_2, \beta_{k+1}\} \\
& & \vdots & & \\
\max\{\alpha_n, \beta_{k+J+2}\} & \leq & \gamma_{k+J+1} & \leq & \min\{\alpha_{J+1}, \beta_{k+J}\} \\
\beta_{k+J+3} & \leq & \gamma_{k+J+2} & \leq & \min\{\alpha_{J+2}, \beta_{K+J+1}\} \\
\beta_{k+J+4} & \leq & \gamma_{k+J+3} & \leq & \min\{\alpha_{J+3}, \beta_{k+J+2}\} \\
& & \vdots & & \\
\beta_n & \leq & \gamma_{n-1} & \leq & \min\{\alpha_{n-k-1}, \beta_{n-2}\} \\
& & \gamma_n & = & 0.
\end{array}
\right. ,
$$

if $n - k \geq k + 1$ with $J = (n - k) - (k + 1)$. These values of γ_i, $i = 1, \cdots, n$, satisfy

$$
\gamma_{i+k} \leq \alpha_i, \quad \alpha_{i+k} \leq \gamma_i, \quad i = 1, \cdots, n - k,
$$

and

$$
\beta_{i+1} \leq \gamma_i, \quad \gamma_{i+1} \leq \beta_i, \quad i = 1, \cdots, n - 1.
$$

By the inductive assumption, there exists a matrix $F_1 \in \mathbb{R}^{m \times n}$ such that

$$
\operatorname{rank}(F_1) \leq k, \quad \sigma(A + F_1) = \{\gamma_1, \gamma_2, \cdots, \gamma_n\}.
$$

Furthermore, by Theorem 2.3, there exists a matrix $F_2 \in \mathbb{R}^{m \times n}$ such that

$$
\operatorname{rank}(F_2) \leq 1, \quad \sigma((A + F_1) + F_2) = \{\beta_1, \beta_2, \cdots, \beta_n\}.
$$

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Hence, if we define

$$F = F_1 + F_2,$$

then F satisfies

$$\text{rank}(F) \leq \text{rank}(F_1) + \text{rank}(F_2) \leq k + 1, \quad \sigma(A + F) = \{\beta_1, \beta_2, \cdots, \beta_n\}.$$

This completes the proof of sufficiency.

By the mathematical induction principle, Theorem 3.1 is true for any $1 \leq \ell \leq n$. $\square$

The necessary condition of Theorem 3.1 is related to the classical Weyl inequality for singular values of sums of matrices. See, for example, the earlier paper [1] and the seminal books [14] and [19]. However, we believe the fact that the necessary condition is also sufficient is new and is of significant importance. The simplicity of the condition (3.1) is also quite pleasant.

4. Numerical Algorithm. The proofs given above can be implemented as numerical means to compute a solution for the ISVPrk. For clarity, we rehash the procedures in algorithmic format.

ALGORITHM 4.1. *(Singular Value Reassignment with Rank One Update) Given a matrix $A \in \mathbb{R}^{m \times n}$ with $m \geq n$ and a set of nonnegative numbers $\beta_1 \geq \cdots \geq \beta_n$, the following steps check and find vectors $\mathbf{b} \in \mathbb{R}^m$ and $\mathbf{f} \in \mathbb{R}^n$ such that $\sigma(A + \mathbf{b}\mathbf{f}^\top) = \{\beta_1, \cdots, \beta_n\}$.*

1. *Compute the singular value decomposition*

$$A = U_1 \Sigma V_1^\top$$

and denote $\sigma(A) = \{\alpha_1, \cdots, \alpha_n\}$ with $\alpha_1 \geq \cdots \geq \alpha_n$.
2. *For $i = 1, \cdots, n - 1$, check to see if*

$$\beta_{i+1} \leq \alpha_i \quad and \quad \alpha_{i+1} \leq \beta_i.$$

If not, stop.
3. *For $i = 1, \cdots, n - 1$, define*

$$\gamma_i := \frac{\min\{\alpha_i, \beta_i\} + \max\{\alpha_{i+1}, \beta_{i+1}\}}{2}.$$

and

$$\gamma_n := \begin{cases} 0, & if\ m = n, \\ \frac{\min\{\alpha_n, \beta_n\}}{2}, & otherwise. \end{cases}$$

4. *If $\gamma_1 > \cdots > \gamma_n > 0$, define for each $k = 1, \cdots, n$*

$$c_k := \sqrt{-\frac{\prod_{j=1}^n (\gamma_k^2 - \alpha_j^2)}{\prod_{\substack{j=1 \\ j \neq k}}^n (\gamma_k^2 - \gamma_j^2)}};$$

else modify c_k's according to the remaining three cases discussed in Lemma 2.1.
5. *Define*

$$\hat{A} := \begin{cases} [\mathbf{c}^\top; diag(\gamma_1, \cdots, \gamma_{n-1}), zeros(m-1, 1)], & if\ m = n, \\ [\mathbf{c}^\top; diag(\gamma_1, \cdots, \gamma_n)], & if\ m = n + 1, \\ [\mathbf{c}^\top; diag(\gamma_1, \cdots, \gamma_n); zeros(m - n - 1, n)], & otherwise. \end{cases}$$

6. *Compute the singular value decomposition*

$$\hat{A} := U_2 \Sigma V_2^\top.$$

7. *Define*

$$\mathbf{b} := U_1 U_2(1,:)^\top,$$
$$V_\mathbf{b} := V_1 V_2^\top,$$
$$\mathbf{a_b} := A^\top \mathbf{b} \text{ (or } V_\mathbf{b}\mathbf{c}).$$

8. *If $\gamma_1 > \cdots > \gamma_n > 0$, define for each $k = 1, \cdots, n$*

$$\hat{f}_k := \sqrt{-\frac{\prod_{j=1}^n (\gamma_k^2 - \beta_j^2)}{\prod_{\substack{j=1 \\ j \neq k}}^n (\gamma_k^2 - \gamma_j^2)}};$$

else modify $\hat{f}_k$'s according to the remaining three cases discussed in Lemma 2.1.

9. *Define*

$$\mathbf{f} := V_\mathbf{b}\hat{\mathbf{f}} - \mathbf{a_b}.$$

We remark that the selection of midpoints in Step 3 of Algorithm 4.1 is but one safe way to ensure conditions (2.23) and (2.24). Other choices are certainly possible. Also we have to point out that details of a mathematically equivalent but numerically more stable way of computing the ratios in $\mathbf{c}$ and $\hat{\mathbf{f}}$ has been discussed elsewhere in the context of Jacobi inverse eigenvalue problems [8, 12] which we shall not elaborate here. If so desired, one simply needs to replace formulas in Step 4 and Step 8 by the more stable formulation.

Once a rank one update algorithm is available, it is important to realize the entire induction process described in Theorem 3.1 can easily be implemented in any programming language that supports a routine to call itself recursively. See discussions of similar implementations in, for example, [5, 6]. The main feature in the routine should be a single divide and conquer mechanism that can be demonstrated by the pseudo-code listed in Table 4.1.

As the routine `svd_update` is calling itself recursively when `ell` is not one, the rank of the problem is reduced by one successively and the current data are saved internally. This is the "dividing" process. Once `ell` has reached the bottom, that is, when `ell = 1`, the process is reversed and begins to build up matrices `F1` followed by `F2` layer by layer. This is the "conquering" process. The final output `F` is the desired solution to the ISVPrk.

5. Conclusion. We have provided a rigorous theoretic basis for the singular value reassignment problem. A simple yet both necessary and sufficient condition (3.1) completely settles the issue of solvability for the ISVPrk. Our proof is constructive so it can be exploited to provide a possible means for computing the solution numerically. Using the rank one case as the building block, the algorithm features a divide-and-conquer scheme. The numerical procedure as it stands now, particularly Algorithm 4.1, might not be stable when there are close-by singular values. Remedies are available in the literature, but this paper mainly concentrates on the general ideas.

```
function [F]=svd_update(A,alpha,beta,ell);

if ell == 1                                    % The rank one case
   [b,f] = svd_update_rank_1(A,alpha,beta); % Algorithm 4.1
   F = b*f';

else
   k = ell-1;                                  % The general case
   choose gamma(1) >= gamma(2) >= ... >= gamma(n) such that

      gamma(i+k) <= alpha(i); alpha(i+k) <= gamma(i); i = 1, ..., n-k
      beta(i+1) <= gamma(i); gamma(i+1) <= beta(i); i = 1, ..., n-1

   [F1] = svd_update(A,alpha,gamma,k);

   [b,f] = svd_update_rank_1(A+F1,gamma,beta);
   F2 = b*f';

   F = F1+F2;

end
```

TABLE 4.1
A pseudo-MATLAB program for the recursive algorithm.

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