ON A NEWTON METHOD FOR THE INVERSE TOEPLITZ EIGENVALUE PROBLEM

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Abstract. Iterative methods for inverse eigenvalue problems involve simultaneous approximation of the matrix being sought and its eigenvectors. This paper revisits one such method for the inverse Toeplitz eigenvalue problems by exploring the eigenstructure of centrosymmetric matrices. All iterations are now taking place on a much smaller subspace. One immediate consequence is that the size of the problem is effectively cut in half and hence the cost of computation is substantially reduced. Another advantage is that eigenvalues with multiplicity up to two are necessarily separated into to disjoint blocks and hence division by zero is unmistakably avoided. Numerical experiment seems to indicate that the domain of convergence is also improved. In addition, a new scheme by using the Wielandt-Hoffman theorem is proposed. This new mechanism makes it possible to handle the case when eigenvalues with multiplicity greater than two are present.

Key words. Inverse Eigenvalue Problem, Toeplitz Matrix, Centrosymmetric Matrix, Tangent Vector, Lift, Wielandt-Hoffman Theorem, Ordering.

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1. Introduction. For decades there has been considerable interest in inverse eigenvalues problems [2, 8, 9]. The inverse Toeplitz eigenvalue problem (ITEP), in particular, has intrigued researchers for years. A symmetric Toeplitz matrix $T = (t_{ij})$ is completely characterized by its first column $r = [r_1, \ldots, r_n]^T$ with the relationship

$$t_{ij} = r_{|i-j|+1}.$$

It is convenient to denote $T = T(r)$. We are mainly interested in real matrices. Thus by an ITEP we mean *to find a vector $r \in R^n$ such that $T(r)$ has a prescribed set of real numbers $\{\lambda_1, \ldots, \lambda_n\}$ as its spectrum.* The ITEP is best known for being intractable on its analytic solvability. It has been a very challenging question whether real symmetric Toeplitz matrices of dimension $n \geq 5$ can have arbitrary real spectra. Some discussion can be found in [7, 10, 13]. Only very recently it is proved by Landau [13] that the eigenvalues of Toeplitz matrices with a special regularity property [13] already attain all possible n-tuple of real numbers. The proof, employing a topological degree argument, inevitably is not constructive. The current paper is concerned with solving the ITEP numerically. We shall present three methods.

At the first glance the ITEP involves exactly n unknowns in n equations. Thus the problem appears to be a well-posed nonlinear algebraic system that can be solved by classical methods. Indeed, a variety of iterative methods have already been proposed. See, for example, [6, 9, 14, 15]. Continuation methods using differential equations approach can also be utilized [4, 5]. Iterative methods necessarily involve simultaneous approximation of the solution matrix and its eigenvectors. The updated eigenvectors

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sometimes are described implicitly as belonging to a symmetric matrix that has the desired eigenvalues and is in some sense close to the current iterate for the solution matrix — The process of obtaining such a symmetric matrix is called "lifting". Continuation methods, in contrast, trace either isospectral flows or orthogonal flows, but not both. Regardless, of most the methods we have known so far the iterations or the flows usually take place in the space $\mathcal{S}(n)$ of real symmetric matrices. The matter of fact is that the set $\mathcal{T}(n)$ of real symmetric Toeplitz matrices is embedded in an even smaller subspace $\mathcal{C}(n)$ of real centrosymmetric matrices of which the spectral properties are well understood [3, 11]. It might be worthwhile, therefore, to explore methods that work within $\mathcal{C}(n)$ when solving the ITEP, and that is the main purpose of this paper.

It should be emphasized that the iterative methods given in [9], if applied to the ITEP, will produce iterates that are Toeplitz and hence centrosymmetric. What lurks behind the computation procedure, however, is the updating of eigenvectors that lifts to non-centrosymmetric matrices. On the other hand, the algorithm proposed by Laurie [14] updates the eigenvectors explicitly and hence make it possible to adopt a strategy of separating the eigenspace. That strategy in a sense has a similar effect of what we are considering in this paper. Our methods, nevertheless, model after those in [9] since a rigorous mathematical proof is readily available.

The eigenstructure of centrosymmetric matrices is quite special [3, 11]. Some of these properties are briefly reviewed in §2. In particular, there is a neat way to divide eigenvalues and the associated eigenvectors into two groups, that will be important in helping our reconstruction. Methods proposed in this paper are variants of the classical Newton method. Based on ideas in [6, 9], we shall discuss the notion of tangent and lift. Because we want to work only in the subspace $\mathcal{C}(n)$, the concept of parity come naturally to the surface. To emphasize the significance of parity, we consider graphically the ITEP for $n = 3$ in §3. In particular, we demonstrate that symmetric Toeplitz matrices *cannot* have arbitrary real spectra with arbitrary parities. Even with Landau's result [13] in mind, to solve the ITEP the eigenvalues must carry appropriately assigned parities. Related to this issue, Delsarte and Genin pointed out [7] that for an ITEP to have a continuous solution as a function of eigenvalues, the eigenvalues arranged in ascending order must alternate in parity. Some examples in §6 also illustrate this point. Let $\mathcal{M}_\mathcal{C} := \mathcal{M}_\mathcal{C}(\lambda_1, \ldots, \lambda_n)$ denote the isospectral subset of $\mathcal{C}(n)$ associated with eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$. By taking advantage of the eigenstructure of $\mathcal{C}(n)$, we describe in §4 a procedure of taking a tangent step from $\mathcal{M}_\mathcal{C}$ to $\mathcal{T}(n)$. This crucial step is analogous to that in the classical Newton method. We then described three ways to lift the intersection of the tangent vector and $\mathcal{T}(n)$ back to $\mathcal{M}_\mathcal{C}$. We shall study how effectively the ITEP is cut in half and how the division by zero can be avoided. We claim that our methods converge quadratically. Convergence theorems involve only some modest modifications of known results [6, 9] and are given in §5. Finally, numerical examples are presented in §6 where we compare these methods and illustrate the convergence.

2. Spectral Properties of $\mathcal{C}(n)$. This section contains a brief overview on spectral properties of $\mathcal{C}(n)$. All these results appear in the paper by Cantoni and Butler [3], and an earlier paper by Andrew [1]. We summarize those that we need for the

sake of convenience. For other properties and more recent studies on centrosymmetric matrices, we refer readers to [11, 17] and the references cited therein.

To describe the result more concisely, we introduce I and J, denoting respectively the identity and the "backward identity" matrices. That is, J has 1's along the secondary (upper-right to lower-left) diagonal and zeros elsewhere. The order of I and J should be clear from the context. We say a vector v is symmetric if $Jv = v$, and skew-symmetric if $Jv = -v$.

Recall that a symmetric and centrosymmetric matrix is one being symmetric with respect to both the main and secondary diagonals. Thus there must be a rich dependence among entries of a centrosymmetric matrix. We exploit these properties in the following theorems.

THEOREM 2.1. *Depending upon if n is even or odd, any $M \in \mathcal{C}(n)$ is of the form:*

$$M = \begin{bmatrix} A & C^T \\ C & JAJ \end{bmatrix}$$

or

$$M = \begin{bmatrix} A & x & C^T \\ x^T & q & x^T J \\ C & Jx & JAJ \end{bmatrix},$$

where $A, C \in R^{\lfloor \frac{n}{2} \times \frac{n}{2} \rfloor}$, $x \in R^{\lfloor \frac{n}{2} \rfloor}$, $q \in R$ and $A = A^T$.

With Theorem 2.1 in mind, we can further reduce a centrosymmetric matrix into blocks by a special orthogonal similar transformation.

THEOREM 2.2. *Let $M \in \mathcal{C}(n)$.*

1. Suppose n is even. Define

$$K := \frac{1}{\sqrt{2}} \begin{bmatrix} I & -J \\ I & J \end{bmatrix}.$$

Then K is orthogonal and

$$KMK^T = \begin{bmatrix} A - JC & 0 \\ 0 & A + JC \end{bmatrix}.$$

2. Suppose n is odd. Define

$$K := \frac{1}{\sqrt{2}} \begin{bmatrix} I & 0 & -J \\ 0 & \sqrt{2} & 0 \\ I & 0 & J \end{bmatrix}.$$

Then K is orthogonal and

$$KMK^T = \begin{bmatrix} A - JC & 0 & 0 \\ 0 & q & \sqrt{2}x^T \\ 0 & \sqrt{2}x & A + JC \end{bmatrix}.$$

We are now able to characterize the eigenstructure of a centrosymmetric matrix as follows.

COROLLARY 2.3. *Let Q denote the matrix of orthonormal eigenvectors of $M \in \mathcal{C}(n)$. Then Q is of the form*

$$Q = K^T Z \tag{1}$$

with

$$Z = \begin{bmatrix} Z_1 & 0 \\ 0 & Z_2 \end{bmatrix}, \tag{2}$$

where Z_1 and Z_2 are orthogonal matrices whose columns are, respectively, eigenvectors of $A-JC$ and, depending upon whether n is even or odd, $A+JC$ or $\begin{bmatrix} q & \sqrt{2}x^T \\ \sqrt{2}x & A+JC \end{bmatrix}$.

Upon multiplying out (1), one can easily attest that for any $M \in \mathcal{C}(n)$ there are exactly $\lfloor \frac{n}{2} \rfloor$ skew-symmetric eigenvectors and $\lceil \frac{n}{2} \rceil$ symmetric eigenvectors. For convenience, we will say that an eigenvalue λ of M is *even* or *odd* in accordance with if λ is associated with a symmetric or skew-symmetric eigenvector. Such a distinction of parity is important because we will see later on that each given eigenvalue in an ITEP must carry a specific parity in order that the problem is solvable. In other words, real symmetric Toeplitz matrices cannot have arbitrary real spectra with arbitrary parity, even if the cardinality of parities has been correctly divided into $\lfloor \frac{n}{2} \rfloor$ and $\lceil \frac{n}{2} \rceil$. This observation is in contrast to the the following theorem which is the converse of Corollary 2.3.

THEOREM 2.4. *Given an arbitrary diagonal matrix $\Lambda = diag\{\lambda_1, \ldots, \lambda_n\}$ and orthogonal matrices $Z_1 \in R^{\lfloor \frac{n}{2} \rfloor \times \lfloor \frac{n}{2} \rfloor}$ and $Z_2 \in R^{\lceil \frac{n}{2} \rceil \times \lceil \frac{n}{2} \rceil}$, the matrix*

$$M := K^T \begin{bmatrix} Z_1 & 0 \\ 0 & Z_2 \end{bmatrix} \Lambda \begin{bmatrix} Z_1 & 0 \\ 0 & Z_2 \end{bmatrix}^T K \tag{3}$$

is centrosymmetric. In this case, $\{\lambda_1, \ldots, \lambda_{\lfloor \frac{n}{2} \rfloor}\}$ are eigenvalues of M with odd parity and $\{\lambda_{\lfloor \frac{n}{2} \rfloor + 1}, \ldots, \lambda_n\}$ are eigenvalues with even parity.

3. An Example of $n = 3$. We find it instructive to consider the geometry of an ITEP when $n = 3$. The subspace $\mathcal{C}(3)$ is of dimension 4. Since we are interested in isospectral subsets of $\mathcal{C}(3)$, we will examine only the cross-section where the trace of a matrix is zero. In this case a matrix $M = [m_{ij}] \in \mathcal{C}(3)$ can be represented by its first column $[m_{11}, m_{12}, m_{13}]^T \in R^3$ since $m_{22} = -2m_{11}$. Note that Toeplitz matrices correspond to those points on the intersection of $\mathcal{C}(3)$ and the plane $m_{11} = 0$.

A direct computation shows that the isospectral subset $\mathcal{M}_{\mathcal{C}}$ of $\mathcal{C}(3)$ with eigenvalues $\{\lambda_1, \lambda_2, \lambda_3\}$ where $\lambda_1 + \lambda_2 + \lambda_3 = 0$ must satisfy the system:

$$\begin{cases} (m_{11} - \frac{\lambda_{\sigma_1}}{4})^2 + \frac{1}{2} m_{12}^2 &= \frac{(\lambda_{\sigma_2} - \lambda_{\sigma_3})^2}{16} \\ m_{13} &= m_{11} - \lambda_{\sigma_1}, \end{cases} \tag{4}$$

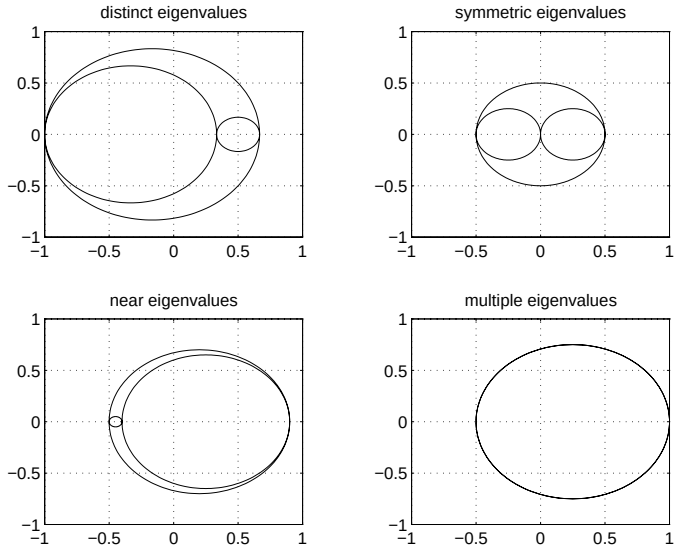


FIG. 1. *Plots of m_{11} versus m_{12} for $\mathcal{M}_{\mathcal{C}}$ in $\mathcal{C}(3)$.*

where σ is a permutation of integers $\{1, 2, 3\}$. The first equation in (4) indicates that the isospectral subset consists of "three ellipses" whereas, by the second equation, these ellipses reside at different planes in R^3. Illustrated in Figure 1 are the typical cases when these ellipses are projected onto the (m_{11}, m_{12})-plane. It is interesting to note that these ellipses must be such that one circumscribes the other two. By counting the m_{12}-intercepts of these ellipses, it is readily proved that when $n = 3$ the ITEP has exactly four real solutions if all given eigenvalues are distinct, and two real solutions if one eigenvalue has multiplicity 2.

The geometry in Figure 1 also illuminates two important ideas.

1. Each of the ellipses represents one parity assignment among eigenvalues. With an inappropriate assignment of parities, the resulting isospectral centrosymmetric matrices may contain no single Toeplitz matrix at all. This is evidenced by the two left subplots in Figure 1 where each case involves one ellipse that does not intersect the m_{12}-axis. This observation confirms our previous claim that real symmetric Toeplitz matrices cannot have arbitrary spectrum with arbitrary parity. The "inappropriateness', however, also depends on the magnitude of eigenvalues. For example, suppose the eigenvalues are arranged in the ascending order. The parity assignment OEE in the upper-left subplot (the ellipse of the median size) of Figure 1 gives rise to two solutions to the ITEP while the same assignment in the lower-left subplot (the ellipse of the smallest size) gives no solution at all. The safeguard appears to be, as was also suggested in [7, 13], that the ordered eigenvalues alternate in parity (the ellipse of the largest size).
2. Recall that centrosymmetric matrices are of the form (3). The geometry in Figure 1 suggests that it is possible to work only within the subspace $\mathcal{C}(n)$ and to bring centrosymmetric matrices closer to a Toeplitz matrix by iteratively

adjusting the two orthogonal matrices Z_1 and Z_2. In doing so, all the matrices generated are isospectral and the parity is left invariant. In other words, we are dealing a refined ITEP where the prescribed spectrum also has a specified parity assignment. We will study in details how this iteration should be done in the next section.

4. A Newton Method in $\mathcal{C}(n)$. In an earlier paper [6] we have introduced a geometry that interprets numerical methods in [9] as Newton methods done by tangent and lift. Similar ideas have been extended successfully to solving inverse singular value problems [6]. Now we apply the same idea again to the ITEP with iterations restricted in the subspace $\mathcal{C}(n)$.

To make this note self-contained we recapitulate the idea as follows. Recall the elementary fact that the new iterate $x^{(\nu+1)}$ of a classical Newton step

$$x^{(\nu+1)} = x^{(\nu)} - (f'(x^{(\nu)}))^{-1} f(x^{(\nu)})$$

for a function $f : R \longrightarrow R$ is precisely the x-intercept of the line which is tangent to the graph of f at $(x^{(\nu)}, f(x^{(\nu)}))$. Our idea is to regard $\mathcal{M}_\mathcal{C}$ as playing the role of the graph of f and the the subspace $\mathcal{T}(n)$ of real symmetric Toeplitz matrices as the role of the x-axis. We develop a procedure that emulates the classical Newton method doing iterations between $\mathcal{M}_\mathcal{C}$ and $\mathcal{T}(n)$. More precisely, we iteratively find a $\mathcal{T}(n)$-intercept of a tangent line from $\mathcal{M}_\mathcal{C}$ and then lift the intercept back to the surface $\mathcal{M}_\mathcal{C}$. We discuss the steps separately below:

4.1. A Tangent Step. Consider the set

$$\mathcal{O}_\mathcal{C}(n) := \{K^T Z | Z = \begin{bmatrix} Z_1 & 0 \\ 0 & Z_2 \end{bmatrix}, Z_1 \in \mathcal{O}(\lfloor \frac{n}{2} \rfloor), Z_2 \in \mathcal{O}(\lceil \frac{n}{2} \rceil)\} \tag{5}$$

where $\mathcal{O}(m)$ stands for the group of all orthogonal matrices in $R^{m\times m}$. It is easy to see that $\mathcal{O}_\mathcal{C}(n)$ is no longer a subgroup of $\mathcal{O}(n)$ but is isomorphic to the manifold $\mathcal{O}(\lfloor \frac{n}{2} \rfloor) \times \mathcal{O}(\lceil \frac{n}{2} \rceil)$. A tangent vector $T_Q(\mathcal{O}_\mathcal{C}(n))$ to $\mathcal{O}_\mathcal{C}(n)$ at $Q = K^T Z$ therefore is of the form

$$T_Q(\mathcal{O}_\mathcal{C}(n)) = K^T \begin{bmatrix} Z_1 S_1 & 0 \\ 0 & Z_2 S_2 \end{bmatrix} = QS \tag{6}$$

with

$$S := \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix}, \tag{7}$$

where S_1 and S_2 are skew-symmetric matrices in $R^{\lfloor \frac{n}{2} \rfloor \times \lfloor \frac{n}{2} \rfloor}$ and $R^{\lceil \frac{n}{2} \rceil \times \lceil \frac{n}{2} \rceil}$, respectively. This realization helps to identify and maintain an "isospectral trajectory" in $\mathcal{C}(n)$ as we will describe now.

Renumbering the eigenvalues if necessary, we assume the parity has been assigned as in Theorem 2.4. To emphasize the parity assignment, we rename the eigenvalues and

denote $\Lambda := \mathrm{diag}\{\phi_1, \ldots, \phi_{\lfloor \frac{n}{2} \rfloor}, \psi_1, \ldots \psi_{\lceil \frac{n}{2} \rceil}\}$. Then $M(t) = Q(t)\Lambda Q(t)^T$ with $Q(t) = K^T Z(t)$ for some one-parameter family $Z(t)$ of orthogonal matrices in the form (2) constitutes an isospectral trajectory with fixed parity assignment in $\mathcal{C}(n)$. Using (6), we further conclude the relationship

$$\begin{aligned} \frac{dM(t)}{dt} &= \frac{dQ(t)}{dt}\Lambda Q(t)^T + Q(t)\Lambda\frac{dQ(t)^T}{dt} \\ &= \left(Q(t)S(t)Q(t)^T\right)M(t) - M(t)\left(Q(t)S(t)Q(t)^T\right) \end{aligned}$$

where $S(t)$ is skew-symmetric matrix in the form (7). Thus any tangent vector $T_M(\mathcal{M}_\mathcal{C})$ to $\mathcal{M}_\mathcal{C}$ at a point $M = Q\Lambda Q^T \in \mathcal{M}_\mathcal{C}$ (about which a local chart can be defined) must be of the form

$$T_M(\mathcal{M}_\mathcal{C}) = \tilde{S}M - M\tilde{S}, \tag{8}$$

where

$$\tilde{S} := QSQ^T \tag{9}$$

is still skew-symmetric.

We are now ready to take a *tangent step* in the Newton method. Given $M^{(\nu)} \in \mathcal{M}_\mathcal{C}$, by Corollary 2.3 there exists a $Q^{(\nu)} = K^T Z^{(\nu)} \in \mathcal{O}_\mathcal{C}(n)$ such that

$${Q^{(\nu)}}^T M^{(\nu)} Q^{(\nu)} = \Lambda. \tag{10}$$

(Note that the parity in Λ is hence determined.) From (8), we know $M^{(\nu)} + \tilde{S}M^{(\nu)} - M^{(\nu)}\tilde{S}$ with any skew-symmetric matrix $\tilde{S}$ in the form (9) represents a tangent vector of $\mathcal{M}_\mathcal{C}$ emanating from $M^{(\nu)}$. To search for an intersection $T(r)$ of such a tangent array with the subspace $\mathcal{T}(n)$, we need to find a skew-symmetric matrix $\tilde{S}^{(\nu)}$ and a vector $r^{(\nu+1)}$ such that

$$M^{(\nu)} + \tilde{S}^{(\nu)}M^{(\nu)} - M^{(\nu)}\tilde{S}^{(\nu)} = T(r^{(\nu+1)}). \tag{11}$$

We now explain how (11) can be solved. Using (9) and (10), we obtain

$$\Lambda + S^{(\nu)}\Lambda - \Lambda S^{(\nu)} = {Q^{(\nu)}}^T T(r^{(\nu+1)})Q^{(\nu)} = {Z^{(\nu)}}^T K T(r^{(\nu+1)})K^T Z^{(\nu)}. \tag{12}$$

Since $T(r^{(\nu+1)}) \in \mathcal{C}(n)$, by Theorem 2.2 we can write

$$KT(r^{(\nu+1)})K^T = \begin{bmatrix} T_1^{(\nu+1)} & 0 \\ 0 & T_2^{(\nu+1)} \end{bmatrix}. \tag{13}$$

Note that every other term in (12) by construction is 2-block diagonal. It is truly remarkable that (12) now breaks down into two disjoint blocks,

$$\Lambda_i + S_i^{(\nu)}\Lambda_i - \Lambda_i S_i^{(\nu)} = {Z_i^{(\nu)}}^T T_i^{(\nu+1)} Z_i^{(\nu)},\ i = 1, 2. \tag{14}$$

Obviously, such a decomposition will substantially reduce the overhead of computation.

Observe that the right-hand side of equation (12) is linear in $r^{(\nu+1)}$. Thus $r^{(\nu+1)}$ can be solved. More precisely, $T(r^{(\nu+1)})$ can be written as

$$T(r^{(\nu+1)}) = \sum_{j=1}^{n} r_j^{(\nu+1)} T(e_j)$$

where e_j stands for the j^{th} column of the identity matrix. It follows that

$$Z^{(\nu)^T} K T(r^{(\nu+1)}) K^T Z^{(\nu)} = \sum_{j=1}^{n} r_j^{(\nu+1)} \begin{bmatrix} Z_1^{(\nu)^T} E_1^{[j]} Z_1^{(\nu)} & 0 \\ 0 & Z_2^{(\nu)^T} E_2^{[j]} Z_2^{(\nu)} \end{bmatrix},$$

where for each $j = 1, \ldots n$,

$$\begin{bmatrix} E_1^{[j]} & 0 \\ 0 & E_2^{[j]} \end{bmatrix} = K T(e_j) K^T \tag{15}$$

is a constant matrix. Equating the n diagonal elements on both sides of (12), we obtain a linear system

$$J^{(\nu)} r^{(\nu+1)} = \lambda, \tag{16}$$

where

$$\lambda := [\phi_1, \ldots, \phi_{\lfloor \frac{n}{2} \rfloor}, \psi_1, \ldots \psi_{\lceil \frac{n}{2} \rceil}]^T \tag{17}$$

and

$$J_{ij}^{(\nu)} := \begin{cases} (Z_1^{(\nu)})_{*i}^T E_1^{[j]} (Z_1^{(\nu)})_{*i}, & \text{if } 1 \le i \le \lfloor \frac{n}{2} \rfloor; \\ \\ (Z_2^{(\nu)})_{*i}^T E_2^{[j]} (Z_2^{(\nu)})_{*i}, & \text{if } \lfloor \frac{n}{2} \rfloor < i \le n \end{cases} \tag{18}$$

for $j = 1, \ldots, n$, with $(Z_k^{(\nu)})_{*i}$ denoting the i^{th} column of the matrix $Z_k^{(\nu)}$. We stress here that each of the vector-matrix-vector multiplications in (18) involves approximately length of $\frac{n}{2}$. Solving (16), we obtain a $\mathcal{T}(n)$-intercept $T(r^{(\nu+1)})$.

4.2. A Lifting Step. The corresponding skew-symmetric matrix $S^{(\nu)}$ does not play a significant role in the tangent step. Once $T(r^{(\nu+1)})$ is determined, $S^{(\nu)}$ (and hence $\tilde{S}^{(\nu)}$) can be obtained by comparing off-diagonal elements on both sides of (12), provided values within each group $\{\phi_1, \ldots, \phi_{\lfloor \frac{n}{2} \rfloor}\}$ and $\{\psi_1, \ldots \psi_{\lceil \frac{n}{2} \rceil}\}$ are distinct. Indeed, we have

$$(S_1^{(\nu)})_{ij} = \frac{(Z_1^{(\nu)})_{*i}^T T_1^{(\nu+1)} (Z_1^{(\nu)})_{*j}}{\phi_i - \phi_j} \tag{19}$$

for $1 \le i < j \le \lfloor \frac{n}{2} \rfloor$ and

$$(S_2^{(\nu)})_{ij} = \frac{(Z_2^{(\nu)})_{*i}^T T_2^{(\nu+1)} (Z_2^{(\nu)})_{*j}}{\psi_i - \psi_j} \tag{20}$$

for $1 \le i < j \le \lceil \frac{n}{2} \rceil$. Note that we do not require eigenvalues $\lambda_1, \ldots, \lambda_n$ to be *mutually* distinct from each other. So the above procedure has another advantage over existing methods. We remind readers a related result [7, Theorem 8] that if an eigenvalue of a Toeplitz matrix has multiplicity greater than one, then the corresponding eigenspace has an orthonormal basis which splits as evenly as possible between the associated symmetric and skew-symmetric eigenvectors. Thus (19) and (20) work fine so long as any of the prescribed eigenvalues $\lambda_1, \ldots, \lambda_n$ has multiplicity *less than or equal to two.* We now described how the matrix $T(r^{(\nu+1)})$ from the "x-axis" $\mathcal{T}(n)$ could be lifted back to the "graph" $\mathcal{M}_{\mathcal{C}}$. We shall discuss three ways to accomplish this task.

4.2.1. Lift by Approximation. If the matrix $S^{(\nu)}$ exists, we may follow existing methods (See [9], for example.) by defining

$$R^{(\nu)} := \left(I + \frac{S^{(\nu)}}{2} \right) \left(I - \frac{S^{(\nu)}}{2} \right)^{-1} \tag{21}$$

and hence the lift

$$M^{(\nu+1)} := Q^{(\nu)} {R^{(\nu)}}^T {Q^{(\nu)}}^T M^{(\nu)} Q^{(\nu)} R^{(\nu)} {Q^{(\nu)}}^T. \tag{22}$$

The motivation for definitions (21) and (22) has been discussed in [6, Section 3]. In practice the matrix $M^{(\nu+1)}$ needs not to be formed explicitly. In our formulation, only the calculation of the orthogonal matrix

$$Z^{(\nu+1)} := Z^{(\nu)} {R^{(\nu)}}^T \tag{23}$$

will be needed in equation (12). We note again that the matrix $R^{(\nu)}$ can be obtained efficiently since matrices on the right-hand side of (21) are 2-block diagonal. Similar comments applies to (23) as well. By now, one Newton step is completed. For later reference, we shall call the above procedure a *lift by approximation.*

4.2.2. Lift by Global Ordering. The lift by approximation fails when the skew-symmetric matrix $S^{(\nu)}$ cannot be defined by either (19) or (20). This is the case where multiple eigenvalues of the same parity exist. We now propose a new lifting mechanism that by-passes the formulation of $S^{(\nu)}$. The idea is to directly look for a matrix $M^{(\nu+1)} \in \mathcal{M}_{\mathcal{C}}$ that is nearest to $T(r^{(\nu+1)})$. The idea is depicted in Figure 2. We formulate the approach as follows.

Let

$$\overline{Z}^{(\nu+1)^T} K T(r^{(\nu+1)}) K^T \overline{Z}^{(\nu+1)} = \begin{bmatrix} \overline{\Lambda}_1^{(\nu+1)} & 0 \\ 0 & \overline{\Lambda}_2^{(\nu+1)} \end{bmatrix} \tag{24}$$

denote the spectral decomposition of $T(r^{(\nu+1)})$. Note that the matrix

$$\overline{Z}^{(\nu+1)} := \begin{bmatrix} \overline{Z}_1^{(\nu+1)} & 0 \\ 0 & \overline{Z}_2^{(\nu+1)} \end{bmatrix} \tag{25}$$

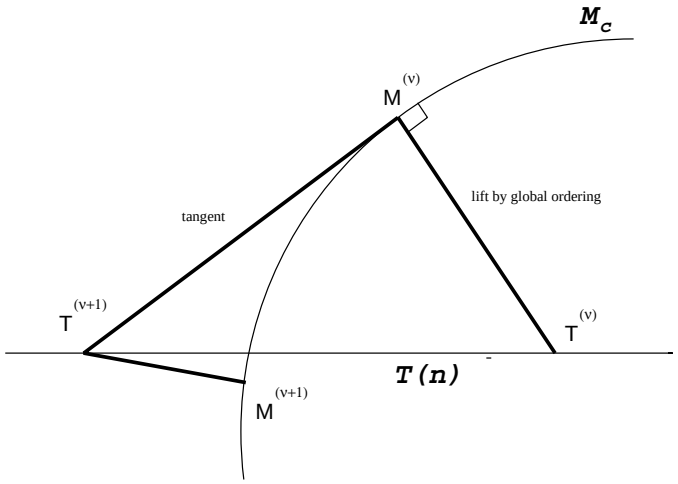


FIG. 2. *Geometry of Lift by Global Ordering.*

can be obtained from eigenvectors of (13). By the Wielandt-Hoffman theorem the nearest matrix $M^{(\nu+1)}$ to $T(r^{(\nu+1)})$ is given by

$$M^{(\nu+1)} := K^T \overline{Z}^{(\nu+1)} \begin{bmatrix} \tilde{\Lambda}_1^{(\nu+1)} & 0 \\ 0 & \tilde{\Lambda}_2^{(\nu+1)} \end{bmatrix} \overline{Z}^{(\nu+1)^T} K \tag{26}$$

where $\tilde{\Lambda}_1^{(\nu+1)}$ and $\tilde{\Lambda}_2^{(\nu+1)}$ are diagonal matrices whose elements as a whole are a rearrangement of $\{\lambda_1, \ldots, \lambda_n\}$ in the same ordering as those in $\overline{\Lambda}_1^{(\nu+1)}$ and $\overline{\Lambda}_2^{(\nu+1)}$. That is, if $\sigma := \sigma^{(\nu+1)}$ is the permutation such that

$$\overline{\lambda}_{\sigma_1}^{(\nu+1)} \leq \overline{\lambda}_{\sigma_2}^{(\nu+1)} \ldots \leq \overline{\lambda}_{\sigma_n}^{(\nu+1)} \tag{27}$$

and ρ is such that

$$\lambda_{\rho_1} \leq \lambda_{\rho_2} \ldots \leq \lambda_{\rho_n}, \tag{28}$$

then

$$\tilde{\Lambda}_1^{(\nu+1)} := \operatorname{diag}\left\{ \lambda_{\rho_{\sigma_1^{-1}}}, \ldots, \lambda_{\rho_{\sigma^{-1}_{\lfloor \frac{n}{2} \rfloor}}} \right\}, \tag{29}$$

$$\tilde{\Lambda}_2^{(\nu+1)} := \operatorname{diag}\left\{ \lambda_{\rho_{\sigma^{-1}_{\lfloor \frac{n}{2} \rfloor + 1}}}, \ldots, \lambda_{\rho_{\sigma_n^{-1}}} \right\} \tag{30}$$

where σ^{-1} denotes the inverse of σ. By taking

$$\Lambda = \Lambda^{(\nu+1)} := \begin{bmatrix} \tilde{\Lambda}_1^{(\nu+1)} & 0 \\ 0 & \tilde{\Lambda}_2^{(\nu+1)} \end{bmatrix}, \tag{31}$$

$$Z^{(\nu+1)} := \overline{Z}^{(\nu+1)}, \tag{32}$$

we are thus at a new starting point for the next tangent step, i.e., solving equation (12). To distinguish the rearrangement $\rho_{\sigma^{-1}}$ from another approach yet to be described, we

shall call the above procedure a *lift by global ordering*. It should be pointed out that the global ordering might be so far-reaching that some elements in $\tilde{\Lambda}_1^{(\nu)}$ are switched into $\tilde{\Lambda}_2^{(\nu+1)}$ and hence the parity assignment will be changed. But more importantly, we see that the skew-symmetric matrix $S^{(\nu)}$ is not needed and hence multiple eigenvalues with same parity can be handled more easily.

4.2.3. Lift by Local Ordering. To avoid both computing $S^{(\nu)}$ and parity switching, we now modify the above procedure to a *lift by local ordering*. Instead of updating $\Lambda^{(\nu+1)}$ according to the global ordering, the diagonal matrix Λ is kept fixed throughout the iteration. But we do permute columns of $\overline{Z}_1^{(\nu+1)}$ and $\overline{Z}_2^{(\nu+1)}$ so that diagonal elements in the corresponding $\overline{\Lambda}_1^{(\nu+1)}$ and $\overline{\Lambda}_2^{(\nu+1)}$ are in the same ordering as those in Λ_1 and Λ_2, respectively. For instance, suppose we have $\phi_1 \leq \ldots \leq \phi_{\lfloor \frac{n}{2} \rfloor}$ and $\psi_1 \leq \ldots \leq \psi_{\lceil \frac{n}{2} \rceil}$ to begin with, then elements in $\overline{\Lambda}_1^{(\nu+1)}$ and $\overline{\Lambda}_2^{(\nu+1)}$ are arranged in ascending order, respectively. The reorganized $\overline{Z}^{(\nu+1)}$ is taken to be $Z^{(\nu+1)}$ for the next step. Using a continuity argument, it is easy to see that upon reaching convergence the lift by global ordering should become stablized and is equivalent to the lift by local ordering.

4.3. Algorithm. For clarity, we set forth in the following a algorithmic summary of the methods we have described. We shall assume that the matrix K in Theorem 2.2 and the basis matrices $E_i^{[j]}$ for $i = 1, 2$ and $j = 1, \ldots n$ defined in (15) are already generated in the pre-processing stage.

ALGORITHM 4.1. *Suppose the target spectrum with specified parity is given by* $\lambda := [\phi_1, \ldots, \phi_{\lfloor \frac{n}{2} \rfloor}, \psi_1, \ldots \psi_{\lceil \frac{n}{2} \rceil}]^T$ *where* ϕ_i *and* ψ_j*, respectively, are arranged in ascending order among themselves. (This practice of rearrangement within parity is referred to below as the "sorted" spectrum.)*

Choose an initial guess $r^{(0)}$*. Obtain the sorted spectrum* $\lambda(r^{(0)})$ *and the corresponding eigenvectors* $Z_i^{(0)}$ *for each of two diagonal blocks in* $KT(r^{(0)})K^T$*.*

For $\nu = 0, 1, 2, \ldots,$

1. *Stop if* $\|\lambda(r^{(\nu)}) - \lambda\|$ *is sufficiently small.*
2. *Form* $J^{(\nu)}$ *(See (18)) and compute* $r^{(\nu+1)}$ *by solving (16).*
3. *Form* $T_i^{(\nu+1)} := \sum_{j=1}^n r_j^{(\nu+1)} E_i^{[j]}$ *for* $i = 1, 2$*.*
4. *If lift by approximation,*
   - (a) *Calculate the sorted spectrum* $\lambda(r^{(\nu+1)})$*.*
   - (b) *Form* $R_i^{(\nu)}$ *for* $i = 1, 2$ *according to (21) where* $S_i^{(\nu)}$ *follows from (19) and (20), respectively.*
   - (c) *Define* $Z_i^{(\nu+1)} := Z_i^{(\nu)} {R_i^{(\nu)}}^T$ *for* $i = 1, 2$*.*

   Elseif lift by global ordering,
   - (a) *Calculate the sorted spectrum* $\lambda(r^{(\nu+1)})$ *and the corresponding eigenvectors* $Z_i^{(\nu+1)}$ *for* $i = 1, 2$*.*
   - (b) *Redefine the target spectrum* λ *according to the "total" ordering in* $\lambda(r^{(\nu+1)})$*.*

   Else lift by local ordering,
   - (a) *Calculate the sorted spectrum* $\lambda(r^{(\nu+1)})$ *and the corresponding eigenvectors* $Z_i^{(\nu+1)}$ *for* $i = 1, 2$*.*

5. Convergence. The numerical method with lift by approximation is exactly the same as Method III proposed by Friedland et al [9] for general inverse eigenvalue problems. The only difference is that the iterations of our method are now restricted to smaller sets. In particular, the lifted eigenvectors now reside in $\mathcal{O}_{\mathcal{C}}(n)$ as opposed to $\mathcal{O}(n)$. The former is a manifold of dimension $\left(\lceil \frac{n}{2} \rceil (\lceil \frac{n}{2} \rceil - 1) + \lfloor \frac{n}{2} \rfloor (\lfloor \frac{n}{2} \rfloor - 1)\right)/2$ whereas the latter is of dimension $n(n-1)/2$. Since it is only the implementation that becomes more perceptive yet the mathematics is the same, we may immediately establish the following theorem.

THEOREM 5.1. *Suppose the ITEP with a prescribed parity has an exact solution T^* where the eigenvalues have multiplicity at most two. Suppose also that the matrix $J^{(\nu)}$ defined in (16) is nonsingular. Then $T(r^{(\nu+1)})$ and $Z^{(\nu+1)}$ (as in (23)) can be defined. Furthermore, suppose $T(r^{(0)})$ is sufficiently close to T^*. Then*

$$\|T^* - T(r^{(\nu+1)})\|_F = O(\|T^* - T(r^{(\nu)})\|_F^2). \tag{33}$$

If λ^* and $\lambda^{(\nu)}$ are eigenvalues of T^* and $T(r^{(\nu)})$, respectively, arranged in the ascending order, then

$$\|\lambda^* - \lambda^{(\nu+1)}\|_2 = O(\|\lambda^* - \lambda^{(\nu)}\|_2^2). \tag{34}$$

Proof. These results follow directly from [9]. A similar proof can also be found in [6]. □

In addition to the advantage that our approach with lift by approximation is computationally more efficient, we want to emphasize another advantage that the method in [9] cannot do, i.e., our method can handle eigenvalues with multiplicity up to two.

Our methods with lift by global or local ordering are new. They have the important advantage of avoiding the computation of $S^{(\nu)}$ and hence allow the eigenvalues to have multiplicity greater than two. On the other hand, we have the following result on its convergence.

THEOREM 5.2. *Suppose the ITEP with a prescribed parity has an exact solution T^*. Suppose also the matrix $J^{(\nu)}$ defined in (16) is nonsingular. Then $T(r^{(\nu+1)})$ and $Z^{(\nu+1)}$ (as in (32) or as the reorganized $\overline{Z}^{(\nu+1)}$ in §4.2.3) are well defined. Furthermore, suppose $T(r^{(0)})$ is sufficiently close to T^*. Then*

$$\|T^* - T(r^{(\nu+1)})\|_F = O(\|T^* - T(r^{(\nu)})\|_F^2), \tag{35}$$

$$\|\lambda^* - \lambda^{(\nu+1)}\|_2 = O(\|\lambda^* - \lambda^{(\nu)}\|_2^2). \tag{36}$$

Proof. Each of these two methods consists of a second order tangent step followed by a first order projection. Our lift by ordering is a projection of shortest distance. No other lifting procedure, including the lift of approximation, can be better. Our methods with lift by ordering, therefore, converges at least quadratically. On the other hand, according to [16, Theorem 2.4], a composition of a linearly convergent method and a

quadratically convergent method would result in at most a quadratically convergent method. The assertions are proved. □

Our numerical results certainly confirm the claimed rate of convergence.

Thus far, our discussion has been centering around the development of a low-cost yet high-efficiency method. Like any other local methods, a good initial value is necessary in order to realize the fast convergence. We conclude our discussion with one remark on the initial value.

Finding a good initial value in general is a very difficult task. For ITEP, fortunately, several good starting strategies are available. For example, it is observed in [14] that the choice $r^{(0)} := [0, 1, 0, \ldots, 0]^T$ usually works well. Our numerical experiment seems to concur this suggestion. As another example, the initial value can be generated, if necessary, from ordinary differential equations that are designed originally for solving the ITEP by numerical integration. A full account of discussion on this differential equation approach can be found in [5] among which we suggest in particular this initial value problem

$$\begin{aligned} \frac{dX}{dt} &= Xk(X) - k(X)X \\ X(0) &= \Lambda \end{aligned} \tag{37}$$

where $k(X) = [k_{ij}(X)]$ is the Toeplitz annihilator matrix defined by

$$k_{ij}(X) := \begin{cases} x_{i+1,j} - x_{i,j-1} & \text{if } 1 \le i < j \le n, \\ 0 & \text{if } 1 \le i = j \le n, \\ x_{i,j-1} - x_{i+1,j} & \text{if } 1 \le j < i \le n. \end{cases} \tag{38}$$

Following the integral curve with high-accuracy integrator, we have always been able to solve the ITEP numerically up to the integration error. We have thus long conjectured that the ITEP is always solvable, only that an asymptotical analysis of the equilibrium points is missing. Combined with the iterative method discussed in this paper, this differential system can now be integrated at a much lower local error requirement just so as to quickly reach within a neighborhood of its stable equilibrium from where the iterative method is turned on to improve the accuracy.

6. Numerical Experiments. In this section we present some of our numerical experiments. Our methods can be regarded as a refinement of existing methods, especially that in [9], at reduced cost. We have already pointed out several main assets of our methods. We feel, therefore, that at this stage it is more important to demonstrate other interesting aspects of our algorithms than to demonstrate their practical efficiency. For the latter, more programming involvement including utilizing some more specialized eigensolver could lead to further substantial savings. For convenience, all numbers are displayed with only five digits although all calculations are done to the machine accuracy.

Example 1. We have pointed out earlier that an ITEP with inappropriate parity assignment will have no solution. Just so that we can perceive what is going on in our

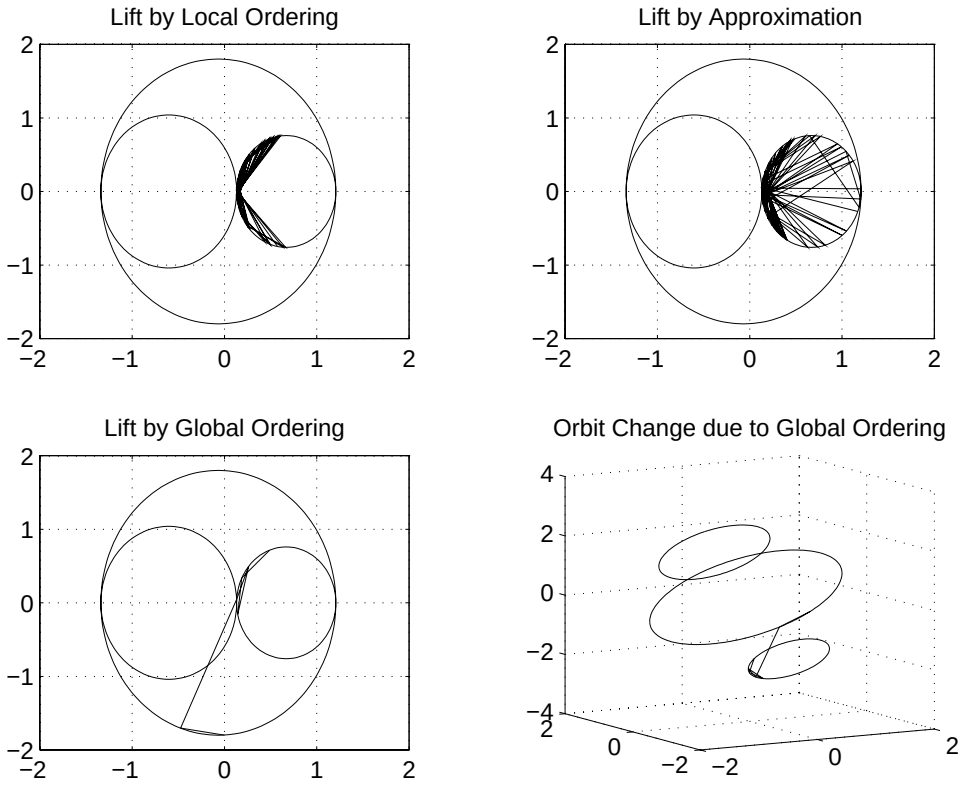


FIG. 3. *Behaviors of Algorithms When Starting with the Wrong Orbit.*

algorithms, we first consider a case of $n = 3$ where we purposefully assign the wrong parity to the eigenvalues. We test the following problem

$$\lambda_1 = -2.4128 \times 10^{+0},\ \lambda_2 = -2.6407 \times 10^{-1},\ \lambda_3 = 2.6769 \times 10^{+0},$$

with λ_3 having odd parity and the other two even. Figure 3 illustrates the behavior of different algorithms. Lifts by local ordering and by approximation keep the iteration staying on the "wrong" orbit, although the shortest distance property in the local ordering method keeps the iterations clustering at the left half of the ellipse. It is most interesting to observe that in the lift by global ordering, the iterations eventually change orbit and then come to convergence where λ_2 should have odd parity. The 3-dimensional subplot in Figure 3 depicts the "jump-off" of the iterations.

Example 2. In this example, we demonstrate the local but quadratic convergence. We first randomly generate a row vector $c^{\#} \in R^4$ from a normal distribution with mean 0 and variance 1. Let $r^{(\#)} := [0, c^{\#}]^T$. The eigenvalues of $T(r^{(\#)})$ are taken as the prescribed eigenvalues with known parity. We then perturb $c^{\#}$ to $c^{(0)}$ by a uniform distribution between -1 and 1 and call $r^{(0)} := [0, c^{(0)}]^T$ as an initial guess.

Table 1 includes $r^{(\#)}$, $r^{(0)}$ and the corresponding limit point $r^{(*)}$ for three test cases. It is interesting to note that

1. The limit point $r^{(*)}$ of a iteration is not necessarily the same as the original vector $r^{(\#)}$ even though to which $r^{(0)}$ is reasonably close.

	Case (a)	Case (b)	Case (c)
$r^{(\#)}$	0	0	0
	-2.0413×10^{-3}	-9.2349×10^{-1}	-3.3671×10^{-1}
	$1.6065\times10^{+0}$	-7.0499×10^{-2}	4.1523×10^{-1}
Original Value	8.4765×10^{-1}	1.4789×10^{-1}	$1.5578\times10^{+0}$
	2.6810×10^{-1}	-5.5709×10^{-1}	$-2.4443\times10^{+0}$
$r^{(0)}$	0	0	0
	-2.8351×10^{-1}	$-1.8024\times10^{+0}$	6.3658×10^{-1}
	9.3953×10^{-1}	7.3881×10^{-1}	4.0318×10^{-1}
Initial Value	8.2068×10^{-1}	1.5694×10^{-1}	$1.0901\times10^{+0}$
	$1.0634\times10^{+0}$	-5.2451×10^{-1}	$-3.2628\times10^{+0}$
$r^{(*)}$	2.0426×10^{-16}	2.2204×10^{-16}	7.4940×10^{-16}
	-2.0413×10^{-3}	-9.2349×10^{-1}	-3.5391×10^{-1}
	$1.6065\times10^{+0}$	-7.0499×10^{-2}	4.3645×10^{-1}
Local Ordering	8.4765×10^{-1}	1.4789×10^{-1}	$1.5244\times10^{+0}$
	2.6810×10^{-1}	-5.5709×10^{-1}	$-2.4655\times10^{+0}$
$r^{(*)}$	8.6831×10^{-16}	0	4.7184×10^{-16}
	-2.0413×10^{-3}	-9.2349×10^{-1}	-3.3671×10^{-1}
	$1.6065\times10^{+0}$	-7.0499×10^{-2}	4.1523×10^{-1}
Approximation	8.4765×10^{-1}	1.4789×10^{-1}	$1.5578\times10^{+0}$
	2.6810×10^{-1}	-5.5709×10^{-1}	$-2.4443\times10^{+0}$
$r^{(*)}$	2.4113×10^{-16}	-1.1102×10^{-16}	6.1062×10^{-16}
	-9.3778×10^{-2}	-9.2646×10^{-1}	3.5391×10^{-1}
	$1.5174\times10^{+0}$	-6.1419×10^{-2}	4.3645×10^{-1}
Global Ordering	9.9597×10^{-1}	1.3518×10^{-1}	$-1.5244\times10^{+0}$
	5.7042×10^{-1}	-5.4694×10^{-1}	$-2.4655\times10^{+0}$

TABLE 1
Initial and Final Values of $r^{(\nu)}$ for Example 2.

Iterations	Local Ordering	Approximation	Global Ordering
0	$1.3847\times10^{+0}$	$1.3847\times10^{+0}$	$1.2194\times10^{+0}$
1	7.1545×10^{-1}	7.1545×10^{-1}	4.2739×10^{-1}
2	2.1982×10^{-2}	6.3866×10^{-2}	1.4179×10^{-2}
3	5.1223×10^{-5}	2.0606×10^{-4}	4.3624×10^{-5}
4	4.4931×10^{-10}	7.1037×10^{-9}	4.7985×10^{-10}
5	1.4729×10^{-15}	2.9671×10^{-15}	1.7659×10^{-15}

TABLE 2
Errors of Eigenvalues for Case (a) in Example 2.

2. The limit points of all three algorithms are not necessarily the same even though they all start from the same $r^{(0)}$.
3. The eigenvalues of $T(r^{(*)})$ do agree with those of $T(r^{\#})$ with the possibility of parity change in the global ordering case.

Table 2 records the 2-norm of the difference between eigenvalues of $T(r^{(*)})$ and $T(r^{(\#)})$ of the three algorithm. Since all three cases behaves similarly, we only report the results from case (a). It is obvious from the table that quadratic convergence indeed occurs in all three methods. We note that in the first few iterations, the errors in the global ordering method should always be less than those in the other two methods because of the shortest distance property.

Example 3. In this example we illustrate quadratic convergence of our methods even when eigenvalues with multiplicity two are present. Again, so that we can comfortably present the data in the running text, we demonstrate the case $n = 5$. We randomly generate four real values and repeat one of them as the multiple eigenvalue. The following is one set of eigenvalues we have tested,

$$-5.8942 \times 10^{-1} \leq -1.8565 \times 10^{-1} \leq -1.8565 \times 10^{-1} \leq 3.7508 \times 10^{-1} \leq 5.8564 \times 10^{-1}.$$

Since the parity is not known, we assume the possibly safest assignment, i.e., the second and the fourth are odd, and the rest are even. With initial guess $r^{(0)}$ given by

$$[0, -1.2367 \times 10^{-1}, 2.3243 \times 10^{-1}, 1.4269 \times 10^{-2}, 5.4264 \times 10^{-1}]^T,$$

all three methods work reasonably well. Both methods by local ordering and approximation converge to

$$[0, -3.0906 \times 10^{-1}, 4.2949 \times 10^{-2}, -6.4816 \times 10^{-2}, -2.3238 \times 10^{-1}]^T$$

while the method by global ordering converges to

$$[-8.2508 \times 10^{-17}, 1.8565 \times 10^{-1}, 1.8565 \times 10^{-1}, 1.8447 \times 10^{-1}, -3.7508 \times 10^{-1}]^T.$$

The logarithmic plot of the 2-norm of errors over the iterations is displayed in Figure 4. It is seen that at the initial stage the method by global ordering struggles for a while,

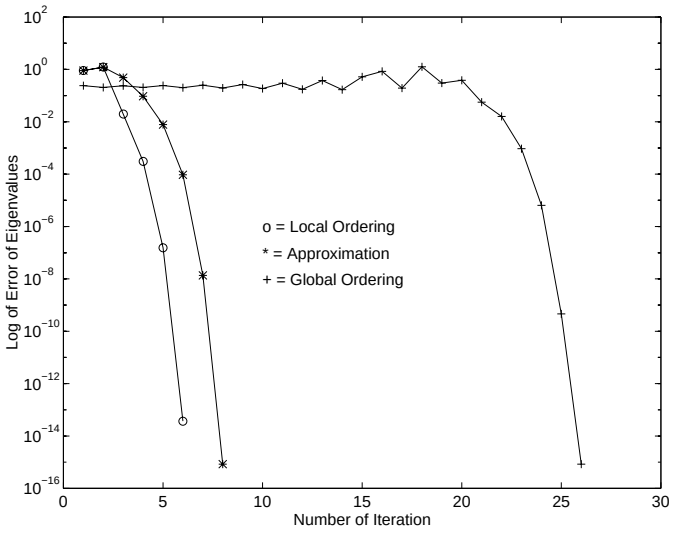


FIG. 4. *Number of Iteration versus Logarithmic Scale of Errors in Example 3.*

but eventually the convergence becomes quadratic. We note that Method III in [9] will not work for a multiply eigenvalue case like this.

Example 4. We continue to challenge our methods by multiple eigenvalues. Suppose the eigenvalues are

$$-8.4328 \times 10^{-1} \leq -1.2863 \times 10^{-1} \leq -1.2863 \times 10^{-1} \leq -1.2863 \times 10^{-1} \leq 1.2292 \times 10^{+0}$$

Since one of the eigenvalue has multiplicity three, we already know the method by approximation fails. It will be interesting to see how the other two methods perform. Again we assume the possibly safest parity assignment, so -1.2863×10^{-1} appears twice as an odd eigenvalue. With $r^{(0)}$ given by

$$[0, 8.6825 \times 10^{-1}, 6.2954 \times 10^{-1}, 7.3622 \times 10^{-1}, 7.2541 \times 10^{-1}]^T,$$

both methods by local and global ordering behave exactly the same. The limit point is given by

$$[2.2204 \times 10^{-16}, 4.2222 \times 10^{-1}, 1.2863 \times 10^{-1}, 4.2222 \times 10^{-1}, 1.2863 \times 10^{-1}]^T.$$

The 2-norm of errors in eigenvalues are

$$2.0327 \times 10^{+0}, 4.0355 \times 10^{-2}, 1.3903 \times 10^{-4}, 3.5477 \times 10^{-9}, 7.8896 \times 10^{-16},$$

which shows quadratic convergence.

Example 5. It is not true that the three methods we proposed *always* perform similarly. Neither is it clear that any of these three method outperforms the others. We illustrate these two points by an example of $n = 20$ with eigenvalues:

$$\begin{array}{rrrrr}
-1.0242\times10^{+1} & -9.6736\times10^{+0} & -5.5608\times10^{+0} & -2.2651\times10^{+0} & 5.5692\times10^{-1} \\
2.1786\times10^{+0} & 3.3867\times10^{+0} & 4.0016\times10^{+0} & 6.3594\times10^{+0} & 8.7090\times10^{+0} \\
-1.0416\times10^{+1} & -9.4352\times10^{+0} & -4.7955\times10^{+0} & -7.7180\times10^{-1} & 6.3996\times10^{-1} \\
2.6374\times10^{+0} & 4.4879\times10^{+0} & 4.7572\times10^{+0} & 6.2222\times10^{+0} & 9.2230\times10^{+0}.
\end{array}$$

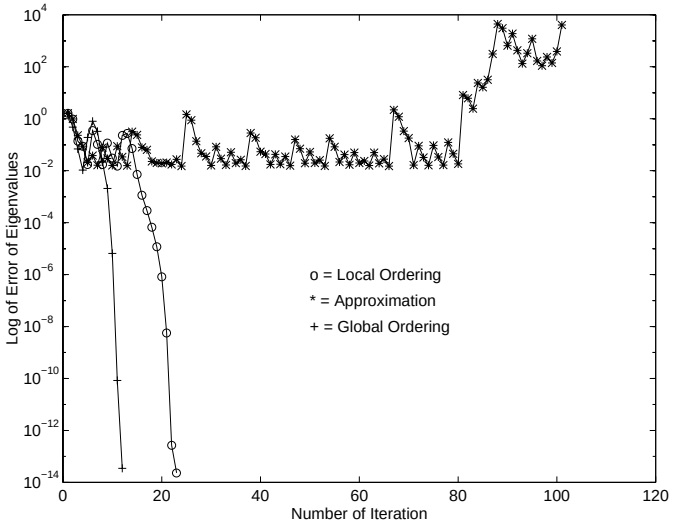


FIG. 5. *Number of Iteration versus Logarithmic Scale of Errors in Example 5 (Irregular parity).*

Consider first the case where the first two rows of eigenvalues above have odd parity and the last two rows have even parity. Note that this parity assignment is not the "safest possible". With initial guess $r^{(0)}$ given by

$$\begin{array}{rrrrr} [0 & 1.9758\times10^{+0} & 9.1234\times10^{-1} & 6.7016\times10^{-1} & 2.4391\times10^{-1} \\ -2.8292\times10^{+0} & 1.8587\times10^{+0} & 7.2979\times10^{-2} & -6.7433\times10^{-1} & -2.0252\times10^{+0} \\ -8.6838\times10^{-2} & 2.3288\times10^{+0} & 3.1950\times10^{-1} & 6.6702\times10^{-1} & -1.5209\times10^{-1} \\ 9.0678\times10^{-1} & -1.5334\times10^{+0} & 9.4695\times10^{-1} & 7.5671\times10^{-1} & -3.7144\times10^{-1}]^T, \end{array}$$

we have observed that the method of global ordering performs best while the method of approximation fails to converge in 100 iterations. The logarithmic plot of errors over iterations is recorded in Figure 5. In contrast, suppose now we arrange the eigenvalues in the ascending order and assign the safest possible parity, i.e., the parity alternates. Then by using the same initial guess $r^{(0)}$, we obtain a behavior as is depicted in Figure 6, showing that even the method by local ordering fails.

Example 6. In this example, we apply our methods to the two numerical examples given in [14]. For comparison, we use the same vector $r^{(0)} = [0, 1, 0, \ldots 0]^T$ as the initial value.

For the case of irregularly clustered eigenvalues $\{1000, 100, 99, 5, 1\}$, our test results using the safest parity assignment are presented in Figure 7. All three methods behave similarly, except that the method by global ordering converges to a Toeplitz matrix whose odd eigenvalues turn out to be $\{1, 100\}$ instead of $\{5, 100\}$.

For the case of larger size problem where eigenvalues are

$$\begin{array}{rrrrrrrrrrrrrrr} 83 & 163 & 182 & 226 & 247 & 283 & 303 & 363 & 456 & 535 & 746 & 796 & 811 & 888 & 900 \\ 110 & 166 & 187 & 239 & 267 & 283 & 339 & 384 & 512 & 557 & 750 & 810 & 837 & 899 & 985, \end{array}$$

we assume the first row of eigenvalues are odd, and the second row are even. Our test results are shown in Figure 8. We find that all methods converge to the same point

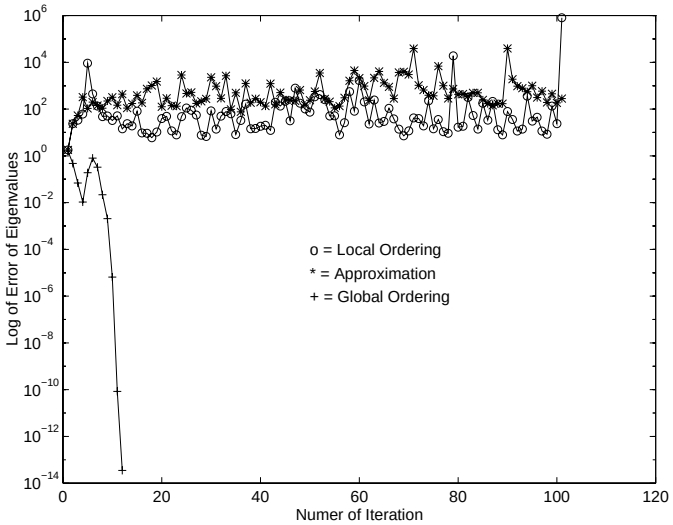


FIG. 6. *Number of Iteration versus Logarithmic Scale of Errors in Example 5 (Safest parity).*

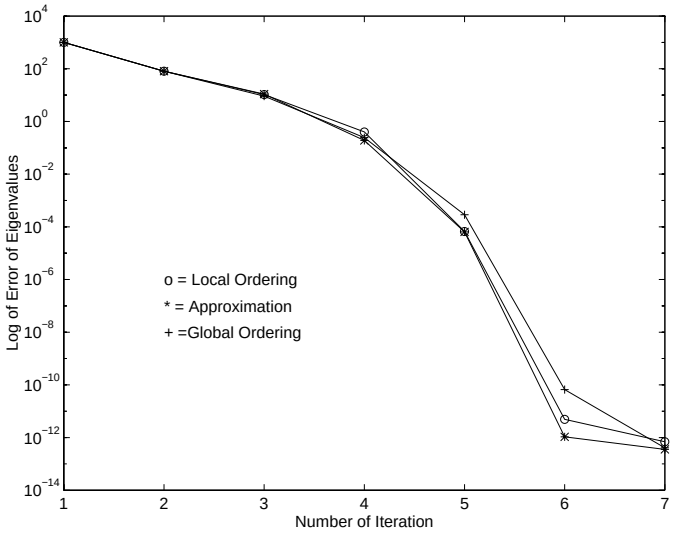


FIG. 7. *Number of Iteration versus Logarithmic Scale of Errors in Laurie's First Example.*

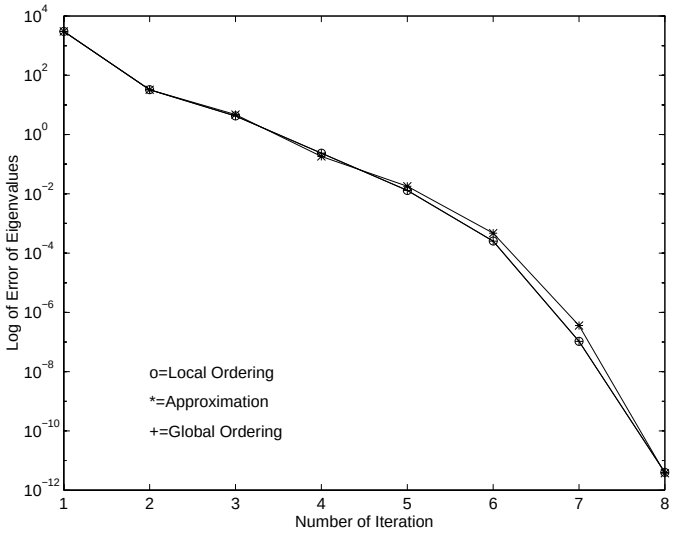


FIG. 8. *Number of Iteration versus Logarithmic Scale of Errors in Laurie's Second Example.*

in seven iterations, and that methods by local ordering and global ordering behave identically.

7. Conclusion. We have investigated the possibility of solving the ITEP within the subspace $\mathcal{C}(n)$ by fast and efficient methods. Three methods that employ the geometrical ideas of tangent and lift are presented. Our methods in a sense are a refinement of a Newton method proposed in [9] with several advantages. One important advancement is that the case of multiple eigenvalues can now be handled. We also have learned that parity assignment of eigenvalues plays an important role in whether an ITEP is solvable. Numerical evidence strongly suggests that our methods have a quadratic rate of convergence.

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