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HOMOTOPY METHOD FOR GENERAL λ-MATRIX PROBLEMS*

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Abstract. This paper describes a homotopy method used to solve the kth-degree λ-matrix problem $(A_k\lambda^k + A_{k-1}\lambda^{k-1} + \cdots + A_1\lambda + A_0)x = 0$. A special homotopy equation is constructed for the case where all coefficients are general $n \times n$ complex matrices. Smooth curves connecting trivial solutions to desired eigenpairs are shown to exist. The homotopy equations maintain the nonzero structure of the underlying matrices (if there is any) and the curves correspond only to different initial values of the same ordinary differential equation. Therefore, the method might be used to find all isolated eigenpairs for large-scale λ-matrix problems on single-instruction multiple data (SIMD) machines.

Key words. λ-matrix, homotopy continuation method, zeros of polynomial systems

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1. Introduction. Given a kth-degree matrix polynomial

$$P(\lambda) = A_k\lambda^k + A_{k-1}\lambda^{k-1} + \cdots + A_1\lambda + A_0 \tag{1.1}$$

with $A_k, A_{k-1}, \cdots, A_0 \in \mathbb{C}^{n\times n}$, the λ-matrix problem consists of determining scalars λ, called eigenvalues, and corresponding $n \times 1$ nonzero vectors x, called eigenvectors, such that

$$P(\lambda)x = 0 \tag{1.2}$$

is satisfied. Problems of this kind occur in many different application areas. Note that the important regular eigenvalue problem

$$\lambda x = Ax \tag{1.3}$$

and the generalized eigenvalue problem

$$\lambda Bx = Ax \tag{1.4}$$

are just two special linear cases of the general problem (1.2). Various examples of (1.1) in physical applications can be found in [3]–[5] and the references cited therein.

A variety of numerical methods are available for solving the λ-matrix problem. In fact, several review papers have already appeared. Without attempting a complete list, we mention here only those by Gohberg, Lancaster, and Rodman [3], Lancaster [4], [5], Ruhe [12], Scott [13], and Peters and Wilkinson [10]. Roughly, most of the approaches can be classified into three categories:

(1) Solving the linearized problem;
(2) Iterating directly;
(3) Reducing to the canonical form.

Each approach has its strengths and weaknesses. For example, the first approach can make use of the readily available software packages, but it increases the size considerably. The second approach includes subspace and Newton-type iterations. Both iterative pro-

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cesses are plausible in theory. However, concerns over the rate of convergence for the former process and the starting procedure for the latter arise in practice. The third approach involves the problem of finding zeros of one-dimensional polynomials. When the degree increases, this becomes an ill-conditioned problem. Interested readers may find more detailed discussions and references concerning each approach among the review papers mentioned above.

Recently the homotopy method has been applied successfully to find all isolated solutions of the linear algebraic eigenvalue problems. In [1], Chu proposes a homotopy equation for (1.3) when A is real, symmetric, and tridiagonal with nonzero off-diagonal elements. Li and Sauer [7] and Li, Sauer, and Yorke [8] study homotopy methods for (1.3) and (1.4) by using fairly sophisticated concepts from algebraic geometry when both A and B are general matrices. In [2] Chu shows that the equation formed in [1] for tridiagonal symmetric matrices works equally well for general matrices by using elementary algebraic theory. The same idea is also applicable to problem (1.4).

Solving the λ-matrix problem by the homotopy method may be costly because of the task of following the homotopy curves. We feel that with improvements in the curve-tracing techniques (say, a hybrid method) this overhead would be substantially reduced. Recently, Rhee [11] has reported some rather promising results on this subject. On the other hand, the homotopy method may have the following advantages:

(1) All isolated eigenpairs are guaranteed to be reached. The method can even approximate nonisolated eigenpairs.

(2) The homotopy curves correspond only to different initial values of the same ordinary differential equation. Hence, all curves can be followed simultaneously if there are enough processors.

(3) The homotopy equation respects the matrix structure (if there is any) of the original problem.

In this paper we present a general treatment of the homotopy method for solving the general kth-degree λ-matrix problem (1.2). Previous results in regard to the linear algebraic eigenvalue problems should then follow as special cases. Readers should be cautioned, however, that the line of thinking in this paper is fundamentally different from that of previous papers.

This paper is organized as follows. In § 2 we begin with a collection of preliminary observations. All these facts are either easy to prove or well known in the literature. We then use these fundamentally important facts to establish the theory of the homotopy method in § 3. Comments on computational aspects of our method are given in § 4, along with some numerical examples.

2. Preliminaries. In this section we observe some basic facts that will be used in the development of our homotopy method.

Consider an arbitrary λ-matrix

$$P(\lambda; B_k, \cdots, B_0) = B_k\lambda^k + \cdots + B_1\lambda + B_0, \tag{2.1}$$

where $B_k, \cdots, B_0 \in \mathbb{C}^{n\times n}$. When it becomes unambiguous, we shall abbreviate $P(\lambda; B_k, \cdots, B_0)$ as $P(\lambda)$.

We first observe the obvious fact that the determinant of $P(\lambda)$ is a polynomial. Indeed, $\det(P(\lambda)) = (\det(B_k))\lambda^{nk}$ + lower-degree terms. It follows, if we count the multiplicities, that the λ-matrix problem corresponding to (2.1) has exactly nk eigenvalues if the leading coefficient B_k is nonsingular. Such a λ-matrix is said to be regular.

Recall that the resultant $R = R(a_n, \cdots, a_0, b_m, \cdots, b_0)$ of two polynomials

$$f(x) = a_nx^n + \cdots + a_1x + a_0,$$

$$g(x) = b_mx^m + \cdots + b_1x + b_0,$$

with $a_n, \cdots, a_0, b_m, \cdots, b_0 \in \mathbb{C}$, $a_n \neq 0$, and $b_m \neq 0$ is defined to be the determinant of the $(n+m) \times (n+m)$ matrix

$$\begin{bmatrix} a_0, a_1, \cdots, a_n & & \\ & a_0, a_1, \cdots, a_n & \\ & \cdots\cdots\cdots\cdots & \\ & & a_0, a_1, \cdots, a_n \\ b_0, b_1, \cdots, b_m & & \\ & b_0, b_1, \cdots, b_m & \\ & \cdots\cdots\cdots\cdots & \\ & & b_0, b_1, \cdots, b_m \end{bmatrix},$$

which is made of m rows of a's, n rows of b's, and zeros elsewhere. It is well known [14] that f and g have a common nonconstant factor if and only if $R = 0$. Thus a polynomial f has a multiple root if and only if its discriminant, the resultant of f and its derivative f', is zero.

Given $d_i \in \mathbb{C}$, $i = 1, \cdots, n$, let $D = \operatorname{diag}(d_1, \cdots, d_n)$ and

$$p(\lambda) = \det(P(\lambda) - D). \tag{2.2}$$

We claim the following.

LEMMA 2.1. *There exist real numbers* $(d_1, \cdots, d_n)$ *such that* $p(\lambda)$ *has no multiple roots.*

Proof. We prove the lemma by induction on n, the size of $P(\lambda)$. For convenience, we rename the polynomial $p(\lambda)$ as $p_n(\lambda)$.

When $n = 1$, $p_1(\lambda)$ has multiple roots if and only if the discriminant $R_1(d_1)$ of $p_1(\lambda)$ vanishes. Suppose the leading coefficient of $p_1(\lambda)$ is a_k. It is easy to see that $R(d_1)$ is an $(n-1)$th polynomial in d_1 with leading coefficient $(ka_k)^k$. Therefore, $R(d_1)$ can vanish only at finitely many points. There exists a real number d_1 such that $p_1(\lambda)$ has no multiple roots.

Let $d_1, \cdots, d_{n-1}$ be chosen by the induction hypothesis so that $p_{n-1}(\lambda)$, the determinant of the principal $(n-1) \times (n-1)$ minor of $P(\lambda) - D$, has no multiple roots. With these fixed values of $d_1, \cdots, d_{n-1}$, we have

$$p_n(\lambda) = q_n(\lambda) - d_n p_{n-1}(\lambda)$$

where $q_n(\lambda)$ and $p_{n-1}(\lambda)$ do not depend upon the value of d_n. We claim the set of real-valued d_n such that $p_n(\lambda)$ has no multiple roots is dense in $\mathbb{R}$.

Suppose not. Then there would exist an open interval I such that $\lambda(d_n)$ is a multiple root of $p_n(\lambda) = p_n(\lambda; d_n)$. By refining the interval I if necessary, we may assume without loss of generality that $\lambda(d_n)$ is differentiable with respect to d_n. For each $d_n \in I$, we have

$$0 = p_n(\lambda(d_n)) = q_n(\lambda(d_n)) - d_n p_{n-1}(\lambda(d_n)).$$

Upon differentiating with respect to the parameter d_n, we get

$$\begin{aligned} 0 &= q_n'(\lambda(d_n))\lambda'(d_n) - d_n p_{n-1}'(\lambda(d_n))\lambda_n'(d_n) - p_{n-1}(\lambda(d_n)) \\ &= \left[\frac{d}{d\lambda} p_n(\lambda)\right]_{\lambda = \lambda(d_n)} \lambda'(d_n) - p_{n-1}(\lambda(d_n)) \\ &= -p_{n-1}(\lambda(d_n)), \qquad d_n \in I. \end{aligned}$$

The last equality follows from the fact that $\lambda(d_n)$ is a multiple root for $d_n \in I$. Note that $p_{n-1}(\lambda(d_n)) \equiv 0$ for $d_n \in I$ implies $\lambda(d_n) \equiv \lambda_0$ (a constant) for $d_n \in I$, since $p_{n-1}(\lambda)$ is a polynomial. It follows that $p_n(\lambda)$ has a multiple root at $\lambda = \lambda_0$ for all $d_n \in I$. Choose $d_n^{(1)} \neq d_n^{(2)}$ in I. Then

$$\begin{aligned} 0 &= p_n(\lambda_0) = q_n(\lambda_0) - d_n^{(1)} p_{n-1}(\lambda_0), \\ 0 &= p_n(\lambda_0) = q_n(\lambda_0) - d_n^{(2)} p_{n-1}(\lambda_0), \\ 0 &= p_n'(\lambda_0) = q_n'(\lambda_0) - d_n^{(1)} p_{n-1}'(\lambda_0), \\ 0 &= p_n'(\lambda_0) = q_n'(\lambda_0) - d_n^{(2)} p_{n-1}'(\lambda_0). \end{aligned}$$

It follows that $p_{n-1}(\lambda_0) = p'_{n-1}(\lambda_0) = 0$. This contradicts the induction hypothesis that $p_{n-1}(\lambda)$ has no multiple roots. □

The following lemma is an extension of the preceding result.

LEMMA 2.2. *The polynomial $p(\lambda)$ in (2.2) has no multiple roots for $(d_1, \cdots, d_n)$ almost everywhere in $\mathbb{C}^n$ except on a subset of complex codimension 1.*

Proof. The polynomial $p(\lambda)$ has no multiple roots if and only if its discriminant $R(d_1, \cdots, d_n)$ is nonzero. By Lemma 2.1, we know that $R(d_1, \cdots, d_n)$ is not identically zero. Furthermore, since $R(d_1, \cdots, d_n)$ is itself a polynomial in variables $d_1, \cdots, d_n$, it can vanish only on a hypersurface of complex codimension 1 [see 9]. □

It is well known in basic matrix theory that if all eigenvalues of a matrix are distinct, then it has no generalized eigenvectors. In [3] and [6], it is shown that this concept can be extended naturally to matrix polynomials. In particular, the following lemma is equivalent to the statement that there are no generalized eigenvectors [6, eq. 14.3.3] for the λ-matrix $P(\lambda)$. Readers are referred to [3, Chap. 1] and [6, Chap. 14] for more detailed discussions. We simply state the result without proof.

LEMMA 2.3. *Suppose the λ-matrix $P(\lambda)$ has nk distinct eigenvalues $\lambda_1, \cdots, \lambda_{nk}$. Let x_j be a unit eigenvector of $P(\lambda)$ associated with λ_j, i.e., $P(\lambda_j)x_j = 0$. Then $P'(\lambda_j)x_j \notin$ Range $(P(\lambda_j))$, where $P'(\lambda) = (d/d\lambda)P(\lambda) = kB_k\lambda^{k-1} + \cdots + B_1$.*

Henceforth we shall assume that the λ-matrix $P(\lambda)$ has nk distinct eigenvalues. For each eigenpair (x, λ) of $P(\lambda)$, we define $Q = Q(x, \lambda)$ to be the $n \times (n+1)$ complex matrix

$$Q(x,\lambda) = [P(\lambda), P'(\lambda)x]. \tag{2.3}$$

It follows from Lemma 2.3 that Q is of complex rank n.

Recall that a linear transformation from $\mathbb{C}^{n+1}$ to $\mathbb{C}^n$ can be regarded as a linear transformation from $\mathbb{R}^{2n+2}$ to $\mathbb{R}^{2n}$ if each component, say $z = a + ib$, of the complex matrix is replaced by the 2×2 real matrix $[\begin{smallmatrix} a, & -b \\ b, & a \end{smallmatrix}]$. Let $\hat{Q} \in \mathbb{R}^{2n \times (2n+2)}$ denote the real matrix associated with the complex matrix $Q \in \mathbb{C}^{n \times (n+1)}$ defined in (2.3). Suppose each component x_k of the complex vector x is written as $x_k = a_k + ib_k$, $k = 1, \cdots, n$. We define a matrix $M = M(x, \lambda) \in \mathbb{R}^{(2n+1) \times (2n+2)}$ as follows:

$$M(x,\lambda) = \begin{bmatrix} \hat{Q} \\ a_1, b_1, a_2, \cdots, a_n, b_n, 0, 0 \end{bmatrix}. \tag{2.4}$$

Note that the last row of M is orthogonal to all rows of $\hat{Q}$ because $P(\lambda)x = 0$. It follows that the matrix M is of real rank $2n + 1$.

3. Homotopy method. Equipped with the knowledge of the preceding section, we now consider our original λ-matrix problem (1.2).

For simplicity, we shall denote

$$P(\lambda)=A_k\lambda^k+A_{k-1}\lambda^{k-1}+\cdots+A_1\lambda+A_0, \tag{3.1}$$

$$Q(\lambda)=cI\lambda^k-D, \tag{3.2}$$

$$R(\lambda,t,c,D)=(1-t)Q(\lambda)+tP(\lambda) \tag{3.4}$$

where $c\in\mathbb{C}$ and $D=\operatorname{diag}(d_1,\cdots,d_n)\in\mathbb{C}^{n\times n}$ are to be specified later.

Observe that

$$\begin{aligned}R(\lambda,t,c,D)&=(1-t)Q(\lambda)+tP(\lambda)\\&=[(1-t)cI+tA_k]\Lambda^k+tA_{k-1}\lambda^{k-1}+\cdots+tA_1\lambda+[(1-t)D+tA_0]\end{aligned}$$

is still a λ-matrix. It is easy to show [7] that there exists an open dense set U_1 with full measure in $\mathbb{C}$ such that if $c\in U_1$, then $[(1-t)cI+tA_k]$ is nonsingular for all $t\in[0,1)$. Henceforth we shall assume that the scalar c in (3.2) is always chosen from U_1, and abbreviate $R(\lambda,t,c,D)$ as $R(\lambda,t,D)$. We shall also denote

$$\begin{aligned}R_\lambda(\lambda,t)&=\frac{d}{d\lambda}R(\lambda,t,D)\\&=k[(1-t)cI+tA_k]\lambda^{k-1}+(k-1)tA_{k-1}\lambda^{k-2}+\cdots+tA_1.\end{aligned} \tag{3.4}$$

For the λ-matrix problem (1.2), the homotopy function $H:\mathbb{C}^n\times\mathbb{C}\times[0,1)\to\mathbb{C}^n\times\mathbb{R}$ is constructed as follows:

$$H(x,\lambda,t)=\begin{bmatrix}R(\lambda,t,D)x\\(x^*x-1)/2\end{bmatrix} \tag{3.5}$$

where x^* represents the transpose of the complex conjugate of x. We are interested in the set $H^{-1}(0)$. As our main result is we show that $H^{-1}(0)$ is a two-dimensional smooth submanifold in $\mathbb{R}^{2n}\times\mathbb{R}^2\times\mathbb{R}$.

Note first that $H(x,\lambda,1)=0$ corresponds to problem (1.2) with normalized eigenvectors. For $i=1,\cdots,n$, let e_i represent the standard ith unit vector in $\mathbb{R}^n$ and λ_{ij} be the jth complex root of $(d_i/c)^{1/k}$, where $j=1,\cdots,k$. It is obvious that $(e_i,\lambda_{ij},0)\in H^{-1}(0)$. We shall use these nk points $(e_i,\lambda_{ij},0)\in\mathbb{C}^n\times\mathbb{C}\times[0,1)$ as our initial points when constructing homotopy curves connected to the desired solution of (1.2).

The following theorem is the main result.

THEOREM 3.1. *There exists an open dense subset U of full measure in $\mathbb{C}^n$ such that, for $D=\operatorname{diag}(d_1,\cdots,d_n)$ with $(d_1,\cdots,d_n)\in U$ and each initial point $y_{ij}=(e_i,\lambda_{ij},0)$, the connected component $C(y_{ij})$ of y_{ij} in $H^{-1}(0)$, when identified as a subset in $\mathbb{R}^{2n}\times\mathbb{R}^2\times\mathbb{R}$, has the following properties.*

1. *$C(y_{ij})$ is a (real) analytic submanifold in $\mathbb{R}^{2n}\times\mathbb{R}^2\times\mathbb{R}$ with real dimension 2.*
2. *The cross-section of $C(y_{ij})$ with each hyperplane $t\equiv$ constant $\in[0,1)$ is a unit circle centered at $(0,\lambda)\in\mathbb{R}^{2n}\times\mathbb{R}^2$ for some λ.*
3. *The manifolds $C(y_{ij})$ corresponding to different initial points do not intersect for $t\in[0,1)$.*
4. *Each manifold $C(y_{ij})$ is bounded for $t\in[0,1)$.*

Proof. For each fixed $t\in[0,1)$, consider the λ-matrix

$$\bar{R}(\lambda,t,D)=\frac{1}{1-t}R(\lambda,t,D)=(cI\lambda^k-D)+\frac{t}{1-t}P(\lambda).$$

By Lemma 2.2, there exists a hypersurface $\bar{U}(t) \in \mathbb{C}^n$ of complex codimension 1 (real codimension 2) such that if $(d_1, \cdots, d_n) \notin \bar{U}(t)$, then $\det(\bar{R}(\lambda, t, D))$ does not have multiple roots. As t varies in $[0, 1)$, the set $V = \cup_{t \in [0,1)} \bar{U}(t) \subset \mathbb{C}^n$ is of real codimension at most one. Thus the complement U of V in $\mathbb{C}^n$ is open and dense and has full measure.

For $D = \operatorname{diag}(d_1, \cdots, d_n)$ with $(d_1, \cdots, d_n) \in U$, the λ-matrix $R(\lambda, t, D)$ has no multiple eigenvalues. For every $(x, \lambda, t) \in H^{-1}(0)$, it is necessary that $R(\lambda, t, D)x = 0$, i.e., x is an eigenvector of $R(\lambda, t, D)$ associated with the eigenvalue λ. Analogous to (2.3) we now consider the matrix $Q = Q(x, \lambda, t) \in \mathbb{C}^{n \times (n+1)}$, where

$$Q(x, \lambda, t) = [R(\lambda, t, D), R_\lambda(\lambda, t)x] \tag{3.6}$$

and its associated real matrix $M = M(x, \lambda, t) \in \mathbb{R}^{(2n+1) \times (2n+2)}$ is as defined in (2.4). Note that the homotopy function H may be regarded as a mapping from $\mathbb{R}^{2n} \times \mathbb{R}^2 \times \mathbb{R}$ into $\mathbb{R}^{2n} \times \mathbb{R}$ and that $M = (\partial H / \partial(x, \lambda))$, where the derivatives are taken in the real variable sense. By the way the constant $c \in \mathbb{C}$ and the vector $(d_1, \cdots, d_n) \in \mathbb{C}^n$ are selected, we know that the λ-matrix $R(\lambda, t, D)$ has nk distinct eigenvalues for every $t \in [0, 1)$. From the discussion in the preceding section, it follows that M is of full rank.

We may now apply the implicit function theorem to conclude that $H^{-1}(0)$ is a smooth submanifold in $\mathbb{R}^{2n} \times \mathbb{R}^2 \times \mathbb{R}$ with real dimension two. Assertion (1) is proved.

Indeed, given an arbitrary point $(x, \lambda, t) \in H^{-1}(0)$, note that the partial derivatives in forming the matrix M is not taken with respect to t. So a local neighborhood of (x, λ, t) on $H^{-1}(0)$ is diffeomorphic to a two-dimensional neighborhood of t and another suitable real variable from (x, λ). This shows that $H^{-1}(0)$ intersects each hyperplane $t \equiv$ constant $\in [0, 1)$ transversally. If the connected components $C(y_{i_1 j_1})$ and $C(y_{i_2 j_2})$ of two distinct initial points $y_{i_1 j_1}$ and $y_{i_2 j_2}$ ever intersect, then $C(y_{i_1 j_1}) = C(y_{i_2 j_2})$. This is possible only if at the intersection point the two-dimensional surface $C(y_{i_1 j_1})$ "bends" back toward the initial point $y_{i_2 j_2}$. This contradicts the transversal property we have observed. Assertion 3 is proved.

Since $H(x, \lambda, t) = 0$ implies $H(\gamma x, \lambda, t) = 0$ whenever $\gamma \in \mathbb{C}$ and $|\gamma| = 1$, we see that $C(y_{ij})$ indeed is a two-dimensional cylindrical tube whose cross-section with each hyperplane $t \equiv$ constant $\in [0, 1)$ is a unit circle centered at $(0, \lambda) \in \mathbb{R}^{2n} \times \mathbb{R}^2$ for some λ. Assertion 2 is proved.

To prove assertion 4 it remains to show only that on every manifold $C(y_{ij})$ the eigenvalue λ stays bounded for $t \in [0, 1)$. From assertions 2 and 3, it suffices to consider any one-dimensional submanifold on $C(y_{ij})$ that is parameterized by the variable t. Define

$$w = \min_{t \in [0, t_0]} \| [(1-t)cI + tA_k] x(t) \|.$$

Since $(1 - t)cI + tA_k$ is continuous and nonsingular for all $t \in [0, t_0]$, we have $w > 0$. Let $r(t) = |\lambda(t)|$ and $s = \max_{t \in [0,t_0]} \|tA_0 - (1 - t)D\|$. Then, $R(\lambda(t), t, D)x(t) = 0$ implies that

$$\begin{aligned} 0 < wr^k(t) &\leqq \| [(1-t)cI + tA_k] \lambda^k(t) \| \\ &\leqq t \| A_{k-1} \lambda^{k-1}(t) + \cdots + A_1 \lambda(t) \| + \| tA_0 - (1-t)D \| \\ &\leqq \| A_{k-1} \| r^{k-1}(t) + \cdots + \| A_1 \| r(t) + s \end{aligned}$$

since $\|x\| = 1$ and $t < 1$. The solution of this polynomial inequality is obviously bounded. The proof is completed. □

Remarks. 1. It follows from a standard degree argument that each circle of solutions of $P(\lambda)x = 0$ in $\mathbb{C}^{n+1}$ with $\|x\| = 1$ is a limit set of one of the component $C(y_{ij})$.

2. If A_k is nonsingular, the proof of Theorem 3.1 can be easily extended such that $H^{-1}(0)$ is uniformly bounded for $t \in [0, 1]$. If A_k is singular, then some components $C(y_{ij})$ will become unbounded as $t \to 1$. This simply indicates that the λ-matrix problem (1.2) does not have nk eigenvalues.

3. For real symmetric eigenvalue problems $Ax - \lambda x = 0$, the homotopy method can be carried out in real arithmetic. In this case, the zero set $H^{-1}(0)$ consists of smooth curves only. Rhee [11] has shown that the local conditioning of the homotopy curves at $t \in (0, 1)$ is affected by two factors: the separation of eigenvalues of the matrix $D + t(A - D)$ and the closeness of D to A. It is interesting to note that a checking criterion can easily be set up to prevent the ODE solver from jumping from one curve to another in the continuation process.

4. Computations. Theorem 3.1 asserts the existence of nk cylindrical tubes $C(y_{ij})$ starting from the hyperplane $t \equiv 0$. We now show how to extract a path from a tube with the intention that this path could be followed numerically and would lead from $t = 0$ to $t = 1$. According to the proof of Theorem 3.1, assertion 3, we could further require that this path be parameterized by the variable t.

Among the many possible ways to define such a path, we choose to consider the following approach.

Let the homotopy function H be a mapping from $\mathbb{R}^{2n} \times \mathbb{R}^2 \times [0, 1)$ to $\mathbb{R}^{2n} \times \mathbb{R}$ so that (x, λ) is identified as a vector in $\mathbb{R}^{2n} \times \mathbb{R}$ and ix a vector in $\mathbb{R}^{2n}$. We define vector fields $(\dot{x}, \dot{\lambda}, 1)$ on $H^{-1}(0)$ by requiring

$$M(x,\lambda,t)\begin{bmatrix}\dot{x}\\ \dot{\lambda}\end{bmatrix}=\begin{bmatrix}(Q(\lambda)-P(\lambda))x\\ 0\end{bmatrix}, \tag{4.1}$$

$$[ix^T,0]\begin{bmatrix}\dot{x}\\ \dot{\lambda}\end{bmatrix}=0 \tag{4.2}$$

where $M \in \mathbb{R}^{(2n+1)\times(2n+2)}$ is the corresponding real matrix, defined as in (2.4), of the matrix Q in (3.6). Note that (4.1) is a necessary condition for the vector field $(\dot{x}, \dot{\lambda}, 1)$ to be tangent to the surface $H^{-1}(0)$. Equation (4.2) simply means that the vector field is always perpendicular to the circle of the intersection of the hyperplane $t \equiv$ constant and the tube.

The $(2n + 2) \times (2n + 2)$ real matrix

$$\begin{bmatrix}M(x,\lambda,t)\\ ix^T,0\end{bmatrix}=\begin{bmatrix}\hat{Q}(x,\lambda,t)\\ a_1,b_1,\cdots,a_n,b_n,0,0\\ -b_1,a_1,\cdots,-b_n,a_n,0,0\end{bmatrix}$$

is precisely the real representation of the $(n + 1) \times (n + 1)$ complex matrix

$$\begin{bmatrix}R(\lambda,t,D), & R_\lambda(\lambda,t)x\\ x^*, & 0\end{bmatrix}.$$

Therefore, the remaining numerical work is to follow the initial value problem in $\mathbb{C}^n \times \mathbb{C}$:

$$\begin{gathered}\begin{bmatrix}R(\lambda,t,D), & R_\lambda(\lambda,t)x\\ x^*, & 0\end{bmatrix}\begin{bmatrix}dx/dt\\ d\lambda/dt\end{bmatrix}=\begin{bmatrix}(Q(\lambda)-P(\lambda))x\\ 0\end{bmatrix},\\ x(0)=e_i,\quad \lambda(0)=\lambda_{ij}\end{gathered} \tag{4.3}$$

for $i = 1, \cdots, n$ and $j = 1, \cdots, k$.

Remark. Note that the nk homotopy curves we derived are integral curves of the same differential equations subject to different initial values. Since these curves are independent of each other, it is suitable to follow several curves simultaneously on a multiprocessor. Note also that the homotopy function (3.5) does not cause any destruction in the matrix structure of $P(\lambda)$. Combined with a sparse matrix technique, the homotopy method might therefore become attractive for solving large-scale λ-matrix problems.

Remark. In practical computation, it may not be necessary to follow the entire homotopy curve $(x(t), \lambda(t))$ as in (4.3). For example, Rhee [11] proposes an algorithm for real symmetric eigenvalue problems that traces the one-dimensional eigenvalue curves only, whereas the eigenvectors are estimated locally by the Rayleigh quotient iteration. Test results seem to indicate that the overall complexity of the homotopy method for finding all n eigenpairs would be $O(n^2)$ as opposed to $O(n^{2.6})$ of the standard subroutine IMTQL2 in EISPACK. We should point out also that EISPACK is not designed for large-scale matrices, where it can be shown [11] that the conditioning of the eigenvalue curves in the homotopy method is independent of the size of the matrix. With careful coding the homotopy method might be a serious alternative for solving large-scale eigenvalue problems.

We coded the homotopy method (4.3) into an IBM 3081 uniprocessor simply to examine the various behavior of the paths near $t = 1$. With no intention of making this code efficient, we integrated the initial value problem (4.3) by using the subroutine DGEAR found in IMSL. The scalar c and the vector $(d_1, \cdots, d_n)$ were randomly generated. No matrix structure was taken into consideration. The linear equation solver LEQ2C was used to find the vector field in (4.3). The following examples represent a collection of problems we used to experiment with our homotopy method.

Example 1. The following is a symmetric, definite quadratic problem with distinct eigenvalues:

$$P(\lambda)=\begin{bmatrix} -10\lambda^2+\lambda+10, & & & & \\ 2\lambda^2+2\lambda+2, & -11\lambda^2+\lambda+9, & & (\text{symmetry}) & \\ -\lambda^2+\lambda-1, & 2\lambda^2+2\lambda+3, & -12\lambda^2+10, & & \\ \lambda^2+2\lambda+2, & -2\lambda^2+\lambda-1, & -\lambda^2-2\lambda+2, & -10\lambda^2+2\lambda+12, & \\ 3\lambda^2+\lambda-2, & \lambda^2+3\lambda-2, & \lambda^2-2\lambda-1, & 2\lambda^2+3\lambda+1, & -11\lambda^2+3\lambda+10 \end{bmatrix}.$$

The code found all 10 eigenpairs without any difficulty. The eigenvalues (to seven digits) are: (−0.5117619, 0.8799272, 1.465467, −0.7790945, 0.5024152, −1.077167, 0.9365506, −1.004838, 1.956883, −1.271885}.

Example 2. The following is a symmetric, definite quadratic problem with multiple eigenvalues:

$$P(\lambda)=\begin{bmatrix} -\lambda^2-3\lambda+1, & & & \\ \lambda^2-1, & -2\lambda^2-3\lambda+5, & (\text{symmetry}) & \\ -\lambda^2-3\lambda+1, & \lambda^2-1, & -2\lambda^2-5\lambda+2, & \\ -2\lambda^2-6\lambda+2, & 2\lambda^2-2, & -4\lambda, & -9\lambda^2-19\lambda+14 \end{bmatrix}.$$

This problem has double eigenvalues 1 and −2, and simple eigenvalues $-4 \pm \sqrt{19}$ and $-4 \pm \sqrt{18}$. The code had no difficulty in accurately locating all eigenpairs (although it took a slightly extra effort to tackle the multiple eigenvalue cases). The computed eigenvectors associated with multiple eigenvalues were linearly independent.

Example 3. The following is an unsymmetrical cubic problem with singular leading coefficients:

$$P(\lambda) = \begin{bmatrix} 5\lambda^3 + \lambda + 1 & \lambda^3 + 1, & \lambda^3 + 1 \\ \lambda^3 + \lambda, & 5\lambda^3 + \lambda + 1, & \lambda^3 + 1 \\ \lambda^3 + \lambda, & 5\lambda^3 + \lambda, & \lambda^3 + \lambda + 1 \end{bmatrix}.$$

The problem actually has only seven eigenvalues since $\det(P(\lambda)) = 27\lambda^7 + 4\lambda^6 + 9\lambda^5 + 9\lambda^4 + 6\lambda^3 + \lambda^2 + \lambda + 1$. The code had no problem in locating these seven eigenpairs. The eigenvalues (to six digits) are $(0.307991 \pm 0.686745i, -0.453629 \pm 0.460837i, 0.327145 \pm 0.474411i, -0.529683\}$. Two of the nine homotopy curves escaped to infinity (with $\|x\| = 1$ always) as t approaches 1. This deceived the code into giving two large extraneous eigenvalues and their associated eigenvectors.

Example 4. The following is a quadratic problem with high multiplicity of eigenvalues and high degeneracy of eigenvectors:

$$P(\lambda) = \begin{bmatrix} (\lambda-1)(\lambda-4), & 5-2\lambda, & 0 \\ 0, & (\lambda-1)(\lambda-4), & 5-2\lambda \\ 0, & 0, & (\lambda-1)(\lambda-4) \end{bmatrix}.$$

Obviously the eigenvalues of this problem are 1 and 4 only, and each eigenvalue is of multiplicity 3. Furthermore, this problem has only one eigenvector. With local tolerance TOL $= 10^{-6}$ in DGEAR, the code was returned with all six curves being convergent. However, the accuracy was only about 10^{-2}. This was due to a high-order bifurcation occurring at $t = 1$.

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