QUADRATIC INVERSE EIGENVALUE PROBLEM AND ITS APPLICATIONS TO MODEL UPDATING — AN OVERVIEW

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1. Introduction. Modeling is one of the most fundamental tools that we use to simulate the complex world. The goal of modeling is to come up with a representation that is simple enough for mathematical manipulation yet powerful enough for describing, inducing, and reasoning complicated phenomena. When modeling physical systems, the resulting mathematical models are sometimes of a very high order too expensive for simulation. One remedy is the notion of model reduction that assists in approximating very high order mathematical models with lower order models. As is evidenced in this collection, model reduction has been under extensive study and rapid development over the past few years with many physical and engineering applications. On the other hand, precise mathematical models of physical systems are hardly available in practice. Many factors, including inevitable disturbances to the measurement and imperfect characterization of the model, attribute to the inexactitude. Since the model reduction process begets only a partial effect of the original model, it is reasonable to expect that the reduced model might not be consonant with realistic data either. For various reasons, it often becomes necessary to *update* a primitive model to attain consistency with empirical results. This procedure of updating or revising an existing model is another essential ingredient for establishing an effective model. The emphasis of the following discussion is on the model updating of quadratic pencils.

The second order differential system

$$M\ddot{\mathbf{x}} + C\dot{\mathbf{x}} + K\mathbf{x} = f(t), \tag{1.1}$$

where $\mathbf{x} \in \mathbb{R}^n$ and M, C, $K \in \mathbb{R}^{n\times n}$, arises frequently in a wide scope of important applications, including applied mechanics, electrical oscillation, vibro-acoustics, fluid mechanics, signal processing, and finite element discretization of PDEs. In most applications involving (1.1), specifications of the underlying physical system are embedded in the matrix coefficients M, C and K. It is well known that if

$$\mathbf{x}(t) = \mathbf{v}e^{\lambda t}$$

represents a fundamental solution to (1.1), then the scalar λ and the vector $\mathbf{v}$ must solve the *quadratic eigenvalue problem* (QEP)

$$(\lambda^2 M + \lambda C + K)\mathbf{v} = 0. \tag{1.2}$$

That is, characteristic behavior of the system (1.1) usually can be interpreted via the eigenvalues and eigenvectors of the system (1.2). Because of this connection, considerable efforts have be devoted to the QEP in the literature. Readers are referred to the treatise by Tisseur and Meerbergen [25] for a good survey of many applications, mathematical properties, and a variety of numerical techniques for the QEP.

Two aspects of the quadratic pencil associated with the model (1.1) deserve consideration. The process of analyzing and deriving the spectral information and, hence, inducing the dynamical behavior of a system from *a priori* known physical parameters such as mass, length, elasticity, inductance, capacitance, and so on is referred to as a *direct* problem. The *inverse* problem, in

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contrast, is to validate, determine, or estimate the parameters of the system according to its observed or expected behavior. The concern in the direct problem is to express the behavior in terms of the parameters whereas in the inverse problem the concern is to express the parameters in term of the behavior. The inverse problem is just as important as the direct problem in applications. The model updating problem can be regarded as a special case of the inverse eigenvalue problem.

The inverse eigenvalue problem is a diverse area full of research interests and activities. See the newly revised book by Gladwell [17], the review article [5], and the recently completed monograph by Chu and Golub [7] in which more than 460 references are collected. Among current development, the quadratic inverse eigenvalue problem (QIEP) is particularly more important and challenging with many unanswered questions. Depending on the applications, the term QIEP has been used in the literature to mean a rather wide range of diverse formulations. For instance, the QIEP studied by Ram and Elhay in [22] involves only symmetric tridiagonal matrix coefficients where *two* sets of eigenvalues are given. The QIEP studied by Starek and Inman in [24] is associated with nonproportional underdamped systems. Lancaster and Prells [20] considered the QIEP with symmetric and positive semi-definite damping C where *complete* information on eigenvalues and eigenvectors is given and all eigenvalues are simple and non-real. There are also works which utilize notions of feedback control to reassign the eigenstructure [10, 21]. The list goes on and on and can hardly be exhaustive.

In this article, we shall consider the QIEP under one common scenario, that is, the spectral information furnished is obtained from empirical data. In vibration industries, including aerospace, automobile, and manufacturing, through vibration tests where the excitation and the response of the structure at selected points are measured experimentally, there are identification techniques to extract a portion of eigenpair information from the measurements. However, the size of the system can be so large and complicated that it is not always possible to attain knowledge of the entire spectrum. While there is no reasonable analytical tool available to evaluate the entire spectral information, it is simply unwise to use experimental values of high natural frequencies to reconstruct a model. Additionally, it is often demanded, especially in structural design, that certain eigenvectors should also satisfy some specific conditions. A finite-element generated symmetric model therefore needs to be updated using only a few measured eigenvalues and eigenvectors [13, 14]. Furthermore, quantities related to high frequency terms in a *finite model* generally are susceptible to measurement errors due to the finite bandwidth of measuring devices. Spectral information, therefore, should not be used at its full extent. For these reasons, it might be more sensible to consider an inverse eigenvalue problem where only a *portion* of eigenvalues and eigenvectors is prescribed. Under these circumstances, the quadratic model updating problem (MUP) therefore can be formulated as follows:

> **(MUP)** Given a structured quadratic pencil (M_0, C_0, K_0) and a few of its associated eigenpairs $\{(\lambda_i, \mathbf{u}_i)\}_{i=1}^k$ with $k < 2n$, assume that new measured eigenpairs $\{(\sigma_i, \mathbf{y}_i)\}_{i=1}^k$ have been obtained. Update the pencil (M_0, C_0, K_0) to (M, C, K) *of the same structure* such that the subset $\{(\lambda_i, \mathbf{u}_i)\}_{i=1}^k$ is replaced by $\{(\sigma_i, \mathbf{y}_i)\}_{i=1}^k$ as k eigenpairs of (M, C, K).

2. Challenges. The MUP as stated above is of immense practical importance. However, there are considerable difficulties when solving a model updating problem. Many issues remain open for further research. We briefly outline three challenges below. We shall comment on current status of development for facing these challenges in later sections.

Structural Constraint. The structure imposed on a MUP depends inherently on the *connectivity* of the underlying physical system. The typical structure for a general mass-spring system, for example, is that the mass matrix M is diagonal, both the damping matrix C and the stiffness matrix K are symmetric and banded, M is positive definite ($M > 0$) and K is positive semi-definite ($K \geq 0$). As an illustration, the structure corresponding to the four-degree-of-freedom mass-spring system depicted in Figure 2.1 should be of the form where

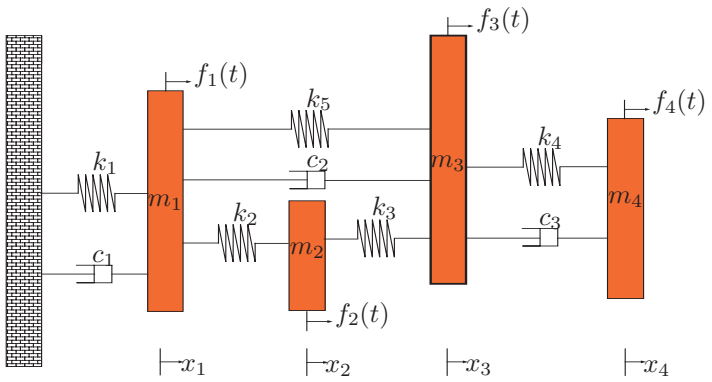

FIG. 2.1. *A four-degree-of-freedom mass-spring system.*

$$M = \begin{bmatrix} m_1 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \\ 0 & 0 & m_3 & 0 \\ 0 & 0 & 0 & m_4 \end{bmatrix}, C = \begin{bmatrix} c_1 + c_2 & 0 & -c_2 & 0 \\ 0 & 0 & 0 & 0 \\ -c_2 & 0 & c_2 + c_3 & -c_3 \\ 0 & 0 & -c_3 & c_3 \end{bmatrix}, K = \begin{bmatrix} k_1 + k_2 + k_3 & -k_2 & -k_5 & 0 \\ -k_2 & k_2 + k_3 & -k_3 & 0 \\ -k_5 & -k_3 & k_3 + k_4 + k_5 & -k_4 \\ 0 & 0 & -k_4 & k_4 \end{bmatrix}.$$

In contrast, the structure associated with an electronic circuit may not be definite or even symmetric. As another illustration, the matrix coefficients in the differential system associated with the RLC network depicted in Figure 2.2 should have the following structure:

$$M = \begin{bmatrix} -L_2 & L_2 & 0 & 0 \\ L_2 & -L_2 & 0 & 0 \\ 0 & 0 & L_3 & 0 \\ 0 & 0 & 0 & L_4 \end{bmatrix}, C = \begin{bmatrix} 0 & R_2 & -R_2 & 0 \\ R_1 + R_4 & 0 & 0 & -R_4 \\ 0 & -R_2 & R_2 + R_3 & 0 \\ -R_4 & 0 & 0 & R_4 \end{bmatrix}, K = \begin{bmatrix} 0 & \frac{1}{C_2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{C_3} & -\frac{1}{C_3} \\ 0 & 0 & -\frac{1}{C_3} & \frac{1}{C_3} + \frac{1}{C_4} \end{bmatrix},$$

For the sake of physical feasibility, the updated model usually is required to inherit the same connectivity as the original model. Since structured problems often results in special interrelationship within its eigenstructure, the observed measurement which often is contaminated with random noise may not be consistent with that innate structure. In other words, the structural constraint often severely limits whether a model could be updated.

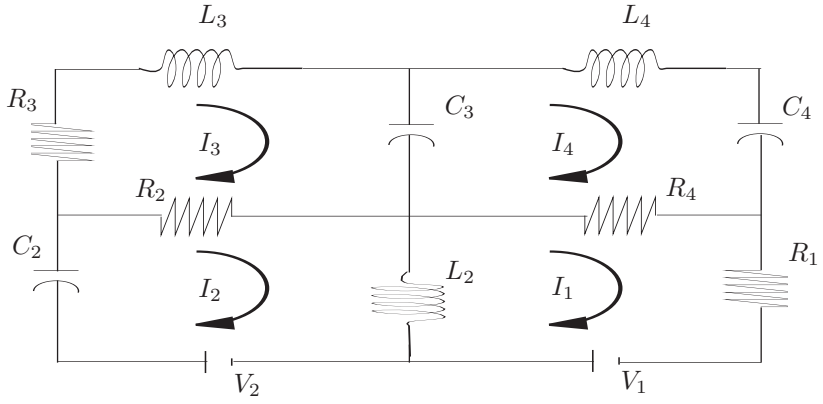

FIG. 2.2. *An RLC electronic network.*

Spurious Eigeninformation. An added challenge, known as the *no spill-over phenomenon* in the engineering literature, is that in updating an existing model it is often desirable that the current

vibration parameters not related to the newly measured parameters should remain invariant. No spill-over is desirable either because these parameters have already been proven to be acceptable in the previous model and engineers do not wish to introduce new vibrations via updating or because engineers simply do not know of any information about these parameters. The MUP under such a circumstance therefore should be formulated as finding the updated model (M, C, K) so that not only he subset $\{(\lambda_i, \mathbf{u}_i)\}_{i=1}^k$ of the original model is replaced by newly measured $\{(\sigma_i, \mathbf{y}_i)\}_{i=1}^k$ as k eigenpairs of (M, C, K), but also the remaining $2n - k$ eigenpairs of (M, C, K), which often are unmeasurable and stay unknown, are the same as those of the original (M_0, C_0, K_0).

Minimal or Robust Modification. The solution to an MUP is not unique. The notion of optimizing the adjustment or the robustness therefore is highly plausible. Earlier work by Friswell, Inman and Pilkey [15] considers model updating by minimal changes of only the damping and the stiffness matrices. The work by Baruch [1], Bermann and Nagy [2], and Wei [26] concentrates only on undamped systems. More recently, the feedback control techniques have also been employed by Nichols and Kautsky [21] and Datta, Elhay, Ram, and Sarkissian [10, 11, 12] to manage the robustness.

Despite much effort, there does not seem to exist adequate theory or techniques thus far that can solve the MUP while addressing the aforementioned concerns. Existing methods have severe computational and engineering limitations, which restrict their usefulness in real applications. The purpose of this article is to provide an overview of this interesting topic with the hope of stimulating further studies toward its solution.

3. Quadratic Inverse Eigenvalue Problem. To answer whether a quadratic pencil can be updated, a more fundamental question is whether a quadratic pencil can have arbitrary k prescribed eigenpairs. For convenience, we adopt the notation that the diagonal matrix $\Lambda \in \mathbb{R}^{k\times k}$ represents the "eigenvalue matrix" of the quadratic pencil (1.2) in the sense that Λ is in real diagonal form with 2×2 blocks along the diagonal replacing the complex-conjugate pairs of eigenvalues originally there. Similarly, let $X \in \mathbb{R}^{n\times k}$ represent the "eigenvector matrix" in the sense that each pair of column vectors associated with a 2×2 block in Λ holds the real and the imaginary part of the original complex eigenvector. For the quadratic pencil (M, C, K) to have eigenstructure (Λ, X), it is clear that the relationship

$$MX\Lambda^2 + CX\Lambda + KX = 0_{n\times k} \tag{3.1}$$

must hold.

3.1. Self-Adjoint Pencils. At first glance, the relationship (3.1) is only a homogeneous linear system of nk algebraic equations. If there are no other constraints, the triplet (M, C, K) constitutes $3n^2$ unknowns. Since k is bounded above by $2n$, the system is well under-determined. It is intuitively true that the system should be solvable in general. The challenge is to characterize the solution in terms of the given (Λ, X). In this section, we discuss how a parametric representation can be obtained for (M, C, K) when these matrix coefficients are required to be symmetric.

To derive the parametric representation, observe that the matrix

$$\Omega := [I_k, \Lambda^\top, \Lambda^{2\top}] \in \mathbb{R}^{k\times 3k} \tag{3.2}$$

has a null space of dimension $2k$. Let columns of the matrix

$$\begin{bmatrix} U \\ T \\ S \end{bmatrix} \in \mathbb{R}^{3k\times 2k},$$

where S, T and U are matrices in $\mathbb{R}^{k\times 2k}$, denote a basis of the null space of Ω. It is clear that once S and T are specified, then

$$U = -\Lambda^\top T - \Lambda^{2\top} S \tag{3.3}$$

is determined. Note that the system (3.1) can be written as

$$\Omega \begin{bmatrix} X^\top K \\ X^\top C \\ X^\top M \end{bmatrix} = 0_{k\times n}, \tag{3.4}$$

implying that there must exist a matrix $\Psi \in \mathbb{R}^{2k\times n}$ such that

$$\begin{bmatrix} U \\ T \\ S \end{bmatrix} \Psi = \begin{bmatrix} X^\top K \\ X^\top C \\ X^\top M \end{bmatrix}. \tag{3.5}$$

Since M, C and K are symmetric, the three matrices

$$A := S\Psi X, \tag{3.6}$$
$$B := T\Psi X, \tag{3.7}$$
$$F := U\Psi X \tag{3.8}$$

must also be symmetric in $\mathbb{R}^{k\times k}$. From (3.3) we know that F is specified once A and B are given. We shall use A and B to characterize the solution (M, C, K) to the QIEP associated with (Λ, X). It is important to note a critical relationship between A and B. Upon substituting (3.3) into (3.8) and using the fact that $F = F^\top$, we find that A and B are related by the equation:

$$\Lambda^\top B - B\Lambda = A\Lambda^2 - \Lambda^{2\top} A. \tag{3.9}$$

That is to say, not all entries in A or B are free. We shall exploit those entries which are free and establish a parametric representation of (M, C, K). Observe that each side of (3.9) represents a skew-symmetric matrix.

We begin with the case when $k = n$ and formulate the following result [8].

THEOREM 3.1. *Given n distinct eigenvalues Λ and n linearly independent eigenvectors X both of which are closed under conjugation, let $A \in \mathbb{R}^{n\times n}$ be an arbitrary symmetric matrix and let B be a solution to the equation (3.9). Then the self-adjoint quadratic pencil with coefficients defined by*

$$M = X^{-\top} A X^{-1}, \tag{3.10}$$
$$C = X^{-\top} B X^{-1}, \tag{3.11}$$
$$K = -X^{-\top}\Lambda^\top (B + \Lambda^\top A) X^{-1}. \tag{3.12}$$

has the prescribed pair (X, Λ) as part of its eigenstructure.

Proof. The proof is straightforward. The relationship (3.5) implies that $M = X^{-\top} S\Psi$ for some $\Psi \in \mathbb{R}^{2n\times n}$. We also know from (3.6) that $A = S\Psi X$. Together, we can express M as $M = X^{-\top} A X^{-1}$. Similar arguments can be applied to C and K. ∎

The choice of A gives rise to $\frac{n(n+1)}{2}$ free parameters. For each given $A \in \mathbb{R}^{n\times n}$, we need to see how B can be determined from the equation (3.9). Without loss of generality, we may assume that Λ is the diagonal matrix with $\ell \times \ell$ blocks,

$$\Lambda = \mathrm{diag}\{\lambda_1^{[2]}, \ldots, \lambda_\nu^{[2]}, \lambda_{\nu+1}, \ldots, \lambda_\ell\}, \tag{3.13}$$

where $\lambda_j^{[2]} = \begin{bmatrix} \alpha_j & \beta_j \\ -\beta_j & \alpha_j \end{bmatrix} \in \mathbb{R}^{2\times 2}$, $\beta_j \neq 0$, if $j = 1, \ldots, \nu$; $\lambda_j \in \mathbb{R}$ if $j = \nu+1, \ldots, \ell$; and $\ell + \nu = n$. Partition B into $\ell \times \ell$ blocks in such a way that, if the (i, j)-block is denoted by B_{ij}, then

$\mathrm{diag}\{B_{11}, \ldots, B_{\ell\ell}\}$ has exactly the same structure as Λ. It is not difficult to see that the (i,j)-block of $\Lambda^\top B - B\Lambda$ is given by

$$\begin{cases} \lambda_i^\top B_{ij} - B_{ij}\lambda_j, & \text{if } \nu+1 \le i,j \le \ell, \\ (\lambda_i^{[2]})^\top B_{ij} - B_{ij}\lambda_j, & \text{if } 1 \le i \le \nu \text{ and } \nu+1 \le j \le \ell, \\ (\lambda_i^{[2]})^\top B_{ij} - B_{ij}(\lambda_j^{[2]}), & \text{if } 1 \le i,j \le \nu. \end{cases}$$

From a comparison with the corresponding blocks in $A\Lambda^2 - \Lambda^{2\top}A$ (cf (3.9), we draw the following conclusion. In the first case, B_{ij} is a scalar and is uniquely determined except that B_{ii} is free. In the second case, B_{ij} is a 2×1 block with all entries being uniquely determined. In the third case, if we write

$$B_{ij} = \begin{bmatrix} x & y \\ y & z \end{bmatrix},$$

then

$$(\lambda_i^{[2]})^\top B_{ij} - B_{ij}(\lambda_j^{[2]}) = \begin{bmatrix} x(\alpha_i - \alpha_j) - y(\beta_i - \beta_j) & -z\beta_i - x\beta_j \\ x\beta_i + y(\alpha_i - \alpha_j) + z\beta_j & y(\beta_i - \beta_j) \end{bmatrix}.$$

It is clear that if $i = j$, then y is free and $x + z = 0$, still giving rise to two degrees of freedom. If $i \neq j$, the all entries of B_{ij} are uniquely determined. We conclude that $A \in \mathbb{R}^{n\times n}$ can be totally arbitrary and B is determined up to n free parameters. We thus have proved the following theorem.

COROLLARY 3.2. *The solutions (M, C, K) to the quadratic inverse eigenvalue problem with eigenstructure (X, Λ) as described in Theorem 3.1 form a subspace of dimensionality $\frac{n(n+3)}{2}$ in the product space $\mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}$.*

It is worth mentioning that if A is selected to be symmetric and positive definite, then so is the leading coefficient M. Indeed, the above construction parameterizes all possible solutions.

We point out in passing that, in contrast to the construction described in Theorem 3.1, Kuo, Lin and Xu [18] have developed independently another parametrization for the solution (M, C, K). Let

$$\Omega := \mathrm{diag}\left\{ \begin{bmatrix} \xi_1 & \eta_1 \\ \eta_1 & -\xi_1 \end{bmatrix}, \ldots, \begin{bmatrix} \xi_\nu & \eta_\nu \\ \eta_\nu & -\xi_\nu \end{bmatrix}, \xi_{\nu+1}, \ldots, \xi_\kappa \right\}.$$

where $\xi_1, \ldots, \xi_\nu, \xi_{\nu+1}, \ldots, \xi_\kappa$ and $\eta_1, \ldots, \eta_\nu$ are arbitrary real numbers. Then the matrices defined by

$$M := \text{an arbitrary symmetric matrix}, \tag{3.14}$$
$$C := -\left(MX\Lambda X^{-1} + (X\Lambda X^{-1})^\top M + X^{-\top}\Omega X^{-1}\right), \tag{3.15}$$
$$K := (X\Lambda X^{-1})^\top M(X\Lambda X^{-1}) + X^{-\top}\Omega\Lambda X^{-1}, \tag{3.16}$$

also solves the QIEP associated with (Λ, X). It can been checked that these two ways of parametrization are equivalent, except that our approach is also able to handle the case $k > n$ which we now explore.

The case $k > n$ is a little bit more involved. It remains true from the relationships (3.5), (3.6), (3.7) and (3.8) that

$$A = S\Psi X = X^\top M X, \tag{3.17}$$
$$B = T\Psi X = X^\top C X, \tag{3.18}$$
$$F = U\Psi X = X^\top K X, \tag{3.19}$$

are symmetric even in the case $k > n$, but we cannot obtain a parametric representation of (M, C, K) from A and B directly because $X \in \mathbb{R}^{n\times k}$ with $k > n$ is no longer an injection transformation. To retrieve (M, C, K), we rewrite the eigenvectors as

$$X = [Z_1, Z_2],$$

where $Z_1 \in \mathbb{R}^{n\times n}$ and $Z_2 \in \mathbb{R}^{n\times(k-n)}$. Then we see that

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^\top & A_{22} \end{bmatrix} = \begin{bmatrix} Z_1^\top M Z_1 & Z_1^\top M Z_2 \\ Z_2^\top M Z_1 & Z_2^\top M Z_2 \end{bmatrix}, \tag{3.20}$$

where A_{ij}, $i, j = 1, 2$, are blocks with appropriate sizes. Thus, instead of using the matrix A as parameter, we should select a symmetric submatrix $A_{11} \in \mathbb{R}^{n\times n}$ arbitrarily and define

$$M = Z_1^{-\top} A_{11} Z_1^{-1}. \tag{3.21}$$

Once $M \in \mathbb{R}^{n\times n}$ is determined, the matrix $A \in \mathbb{R}^{k\times k}$ is completely specified. This selection gives rise to $\frac{n(n+1)}{2}$ degrees of freedom. There is no additional freedom in the choice of A.

With $A \in \mathbb{R}^{k\times k}$ specified, we next want to determine the matrix $B \in \mathbb{R}^{k\times k}$ based on the necessary condition (3.9). Write

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{12}^\top & B_{22} \end{bmatrix} = \begin{bmatrix} Z_1^\top C Z_1 & Z_1^\top C Z_2 \\ Z_2^\top C Z_1 & Z_2^\top C Z_2 \end{bmatrix}. \tag{3.22}$$

Consider the B_{11} block first. Partition the given eigenvalues as

$$\Lambda = \mathrm{diag}\{\Upsilon_1, \Upsilon_2\}$$

where $\Upsilon \in \mathbb{R}^{n\times n}$ and $\Upsilon_2 \in \mathbb{R}^{(k-n)\times(k-n)}$. We note that B_{11} and A_{11} satisfy a relationship

$$\Upsilon_1^\top B_{11} - B_{11}\Upsilon_1 = A_{11}\Upsilon_1^2 - \Upsilon_1^{2\top} A_{11}, \tag{3.23}$$

that is similar to (3.9). The same argument used earlier for the case when $k = n$ can be applied and we conclude that the submatrix B_{11} can be completely determined up to n free parameters. It follows that a symmetric matrix

$$C = Z_1^{-\top} B_{11} Z_1^{-1} \tag{3.24}$$

can be determined and, hence, it appears that the matrix B is completely determined up to n free parameters.

The point is that there are additional limitations on the choice of A and B in the case $k > n$. The very same C defined by (3.24) should also equate the two sides of equation (3.9) for the $(1, 2)$ and $(2, 2)$ blocks, respectively. These blocks involve more than n equations to be satisfied. Thus, B_{11} and consequently A_{11} must be special. We have to go back to modify the selection of A_{11}. In other words, the n free parameters in B_{11} and the matrix A_{11} must be further restricted so that the remaining part of B also satisfies (3.9). To that end, we observe that if we define

$$W := Z_1^{-1} Z_2, \tag{3.25}$$

then it follows that

$$A = \begin{bmatrix} A_{11} & A_{11}W \\ W^\top A_{11} & W^\top A_{11} W \end{bmatrix},$$
$$B = \begin{bmatrix} B_{11} & B_{11}W \\ W^\top B_{11} & W^\top B_{11} W \end{bmatrix}.$$

Expressing the equation (3.9) in block form, we obtain (3.23) and the following two equations:

$$\Upsilon_1^\top B_{11} W - B_{11} W \Upsilon_2 = A_{11} W \Upsilon_2^2 - \Upsilon_1^{2\top} A_{11} W, \tag{3.26}$$

$$\Upsilon_2^\top W^\top B_{11} W - W^\top B_{11} W \Upsilon_1 = W^\top A_{11} W \Upsilon_2^2 - \Upsilon_2^{2\top} W^\top A_{11} W. \tag{3.27}$$

Post-multiplying (3.23) by W and subtracting (3.26), we obtain an equivalent relationship:

$$A_{11}\Upsilon_1^2 W + B_{11}\Upsilon_1 W = A_{11} W \Upsilon_2^2 + B_{11} W \Upsilon_2.$$

It follows that

$$\begin{aligned} W^\top (A_{11} W \Upsilon_2^2 + B_{11} W \Upsilon_2) &= W^\top (A_{11}\Upsilon_1^2 W + B_{11}\Upsilon_1 W) \\ &= (W^\top A_{11}\Upsilon_1^2 + W^\top B_{11}\Upsilon_1) W = (\Upsilon_2^\top W^\top B_{11} + \Upsilon_2^{2\top} W^\top A_{11}) W, \end{aligned}$$

which is precisely (3.27). The final equality follows from taking the transpose of equation (3.26). We have just proved that if we can solve the two equations (3.23) and (3.26), then the third equation (3.27) is automatically solved. We have indicated earlier that any given A_{11} will determine B_{11} through (3.23) up to n free parameters. Thus, it only remains to choose the n free parameters in B_{11} and the $n \times n$ symmetric matrix A_{11} to satisfy the $n(k-n)$ linear equations imposed by (3.26). In total there are

$$\frac{n(n+1)}{2} + n - n(k-n) = \frac{3n(n+1)}{2} - nk$$

degrees of freedom. For nontrivial solutions, it is clear that we need $k < \frac{3(n+1)}{2}$.

Finally, we discuss the case when $k < n$. If less than n eigenpairs (X, Λ) are given, we can solve the inverse eigenvalue problem by embedding this eigeninformation in a larger set of n eigenpairs. In particular, we expand $X \in \mathbb{R}^{n\times k}$ to

$$\widehat{X} := [X, \widetilde{X}] \in \mathbb{R}^{n\times n}, \tag{3.28}$$

where $\widetilde{X} \in \mathbb{R}^{n\times(n-k)}$ is arbitrary under the condition that $\hat{X}$ is nonsingular. Caution should be taken when counting the degrees of freedom. We should consider the columns in $\widetilde{X}$ as being *normalized* since, otherwise, a normalization factor would have been added to the arbitrariness of A. With this normalization in mind, the expansion of eigenvectors involves additional $(n-1)(n-k)$ degrees of freedom. We then expand $\Lambda \in \mathbb{R}^{k\times k}$ to

$$\widehat{\Lambda} := \operatorname{diag}\{\Lambda, \widetilde{\Lambda}\}, \tag{3.29}$$

where $\widehat{\Lambda}$ is a diagonal matrix with distinct eigenvalues. This expansion of eigenvalues gives rise to another $n-k$ degrees of freedom. With $(\widehat{X}, \widehat{\Lambda})$ playing the role of (X, Λ) in Theorem 3.1, we can now construct the coefficient matrices M, C and K according to the formulas (3.10), (3.11) and (3.12), respectively. Recall that A is taken as an arbitrary symmetric matrix in $\mathbb{R}^{n\times n}$ and B, though depending on $\widehat{\Lambda}$ through the relationship (3.9), maintains n degrees of freedom. We conclude that the solutions to the QIEP with $k < n$ form a subspace of dimensionality $\frac{n(n+3)}{2} + n(n-k)$. Note that this embedding approach characterizes the solution (M, C, K) via the parametrization (3.10), (3.11) and (3.12) which in *nonlinear* in terms of A, B, $\widetilde{X}$ and $\widetilde{\Lambda}$.

We end this section with the following summarizing theorem.

THEOREM 3.3. *Assume $1 \le k < \frac{3(n+1)}{2}$. Let (Λ, X) represent k arbitrarily prescribed eigenpairs (Λ, X) which are closed under conjugation. The self-adjoint quadratic inverse eigenvalue problem associated with (Λ, X) is generally solvable. The solutions form a subspace of dimension $\frac{3n(n+1)}{2} - nk$. The maximal allowable number of prescribed eigenpairs is given by (4.2).*

3.2. Structured Pencils. Thus far, the only structure laid upon the QIEP is the symmetry, in which case we have shown its solvability. However, it is important to note that algebraic solvability does not necessarily imply physical feasibility. Physical feasibility means, for example, that the special matrix structure resulting from the underlying connectivity must hold or that the physical parameters must be nonnegative. These additional constraints make the QIEP much more interesting but harder to solve. There does not seem to exist reported research in this direction. We believe that the issue of solvability probably is problem dependent and will have to be analyzed case by case. For demonstration purpose, we shall discuss only one setting in this section.

Consider the serially linked, undamped mass-spring system depicted in Figure 3.1, which can be used to model many other physical systems, including a vibrating beam, a composite pendulum, or a string with beads. The corresponding quadratic pencil $\lambda^2 M + K$ has the structure

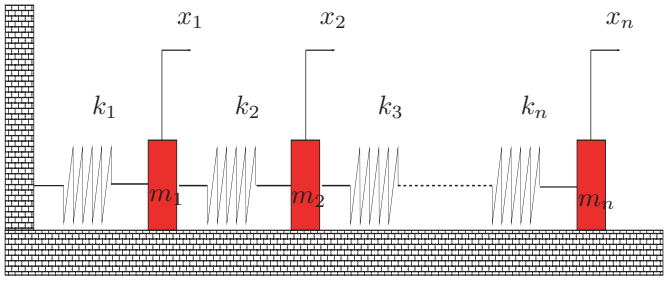

FIG. 3.1. *An undamped mass-spring system.*

$$M = \begin{bmatrix} m_1 & 0 & \dots & 0 \\ 0 & m_2 & & \\ \vdots & & \ddots & \\ 0 & & & m_n \end{bmatrix}, \; K = \begin{bmatrix} k_1+k_2 & -k_2 & 0 & \dots & 0 & 0 \\ -k_2 & k_2+k_3 & -k_3 & & & 0 \\ 0 & -k_3 & k_3+k_4 & & & \\ \vdots & & & \ddots & & \vdots \\ 0 & & & & k_{n-1}+k_n & -k_n \\ 0 & & & & -k_n & k_n \end{bmatrix}. \quad (3.30)$$

The inverse eigenvalue problem would imply to find positive values for the masses $m_1, \dots, m_n$ and spring constants $k_1, \dots, k_n$ from prescribed eigeninformation. A typical approach in the literature has been to recast the quadratic pencil as a linear pencil $\mu I + J$ with a Jacobi matrix $J = M^{-1/2} K M^{-1/2}$. A classical theory has been that *two sets of eigenvalues* can uniquely solve the corresponding Jacobi inverse eigenvalue problem [7, Section 4.2]. What can be said if the system is to be reconstructed from eigenpairs?

Each eigenpair provides n equations. Imposing two eigenpairs generally will lead to the trivial algebraic solution in such a system, unless the prescribed eigenpairs satisfy some additional internal relationship. So we ask the even more fundamental question of constructing the system with one prescribed eigenpair $(i\beta, \mathbf{x})$ where $i = \sqrt{-1}$, $\beta \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$.

Denote $\mathbf{x} = [x_1, \dots, x_n]^\top$ and $x_0 = 0$. It is not difficult to see that the recursive relationship,

$$k_n = \frac{\beta^2 m_n x_n}{x_n - x_{n-1}}, \quad (3.31)$$

$$k_i = \frac{\beta^2 m_i x_i + k_{i+1}(x_{i+1} - x_i)}{x_i - x_{i-1}}, \quad i = n-1, \dots, 1, \quad (3.32)$$

must hold. Our goal is to find a positive approximation for m_i, which then defines a positive value for k_i. The following is a *make-or-break* algorithm for the construction [4].

ALGORITHM 3.1. *Given an arbitrary eigenpair $(i\beta, \mathbf{x})$, assume the normalization $x_n = 1$ and $m_n = 1$. The following steps either construct the masses $m_1, \ldots, m_{n-1}$ and spring constants $k_1, \ldots, k_n$, all positive, for the pencil $\lambda^2 M + K$, or determine that such a system with the prescribed eigenpair does not exist.*

1. *initialization:*
   $s_n = 0.9$; *(mass decreasing factor)*
   $s_p = 1.1$; *(mass increasing factor)*
   $\eta = \beta^2$;
2. *if* $x_{n-1} < 1$,
   $k_n = \frac{\beta^2}{1 - x_{n-1}}$; *(use formula (3.31))*
   else
   return *(inconsistent eigenvector)*
   end
3. *for* i *from* $n-1$ *to* 2, *do*
   (a) $\rho = \frac{\eta}{x_i}$;
   (b) *if* $x_{i-1} < x_i$,
   if $\rho > 0$,
   if $x_i < 0$,
   return *(inconsistent eigenvector)*
   else
   $m_i = 1$; *(any $m_i > 0$ will be fine)*
   end
   else
   if $x_i < 0$,
   $m_i = -\frac{s_n \rho}{\beta^2}$; *(need $0 < m_i < -\frac{\rho}{\beta^2}$)*
   else
   $m_i = -\frac{s_p \rho}{\beta^2}$; *(need $m_i > -\frac{\rho}{\beta^2}$)*
   end
   end
   else
   if $\rho > 0$,
   if $x_i < 0$,
   $m_i = 1$;
   else
   return *(inconsistent eigenvector)*
   end
   else
   if $x_i < 0$,
   $m_i = -\frac{p_n \rho}{\beta^2}$;
   else
   $m_i = -\frac{s_n \rho}{\beta^2}$;
   end
   end
   end
   (c) $\eta = \eta + \beta^2 m_i x_i$;
   (d) $k_i = \frac{\eta}{x_i - x_{i-1}}$; *(use formula (3.32))*

4. finale:
  (a) $rho = \frac{\eta}{x_1}$;
  (b) *if* $\rho > 0$,
    $m_1 = 1$;
  else
    $m_1 = -\frac{s_p \rho}{\beta^2}$;
  end
  (c) $\eta = \eta + \beta^2 m_1 x_1$;
  (d) $k_1 = \frac{\eta}{x_1}$; *(use formula (3.32))*

The above algorithm appears naive since it only checks a few signs, but its simplicity is in fact closely related to the classical Courant Nodal Line Theorem [27]. Roughly speaking, it is known in the literature that critical information about a vibration system can be recovered from places where *nothing happens*. These places are referred to as the nodal lines. Courant's theorem gives a right count of the number of nodal lines. We shall not elaborate the particulars here. Readers are referred to the paper [9, 27] for more detailed discussion. In short, the effect of the algorithm is based on the following result, whose proof can be found in [4]. We believe that the necessary and sufficient conditions on the specified eigenvector is lucid but elegant.

THEOREM 3.4. *A given vector* $\mathbf{x} = [x_1, \ldots, x_n]^\top \in \mathbb{R}^n$ *with distinct entries is an eigenvector of a quadratic pencil* $\lambda^2 M + K$ *with the structure specified in (3.30) if and only if* $x_n(x_n - x_{n-1}) > 0$ *and the signs of the triplets* $(x_{i+1} - x_i, x_i, x_i - x_{i-1})$ *for* $i = 2, \ldots, n-1$ *are not* $(+,+,-)$ *nor* $(-,-,+)$. *Furthermore, if* $\mathbf{x}$ *is feasible and if there are* τ *changes of signs when going through* x_1 *to* x_n, *then* $\mathbf{x}$ *is the* τ*-th eigenvector of the pencil regardless how the masses* m_i *are defined.*

For damped systems and other types of connectivity or RLC configurations, the resulting pencil structure will be different. It is likely that the conditions for solvability will also vary. This is a wide open area for further research.

4. Spill-Over Phenomenon. Recall that a model updating with no spill-over is mathematically equivalent to a QIEP with a *complete set* of prescribed eigenpairs (Λ, X) where we partition Λ and X as

$$\Lambda = \mathrm{diag}\{\Sigma, \Lambda_2\}, \quad X = [Y, X_2], \tag{4.1}$$

with the pair $(\Sigma, Y) \in \mathbb{R}^{k\times k} \times \mathbb{R}^{n\times k}$ representing the portion of eigenstructure that has been modified and (Λ_2, X_2) corresponding to the inert portion of eigenstructure in the original model which should not been changed (and perhaps is even not known). Ideally, we prefer to see no spill-over in the model updating. But can this be achieved? If not, to what extent do we know about the spurious eigenstructure brought in by the updating?

In the case when symmetry is required, we have seen that the additional constraint of symmetry imposes an upper bound on the number k of prescribed eigenpair. The maximal allowable number k_{max} of prescribed eigenpairs is given by

$$k_{max} = \begin{cases} 3\ell + 1, & \text{if } n = 2\ell, \\ 3\ell + 2, & \text{if } n = 2\ell + 1. \end{cases} \tag{4.2}$$

As a consequence, the remaining $2n - k_{max}$ eigenpairs of a quadratic pencil *cannot* be arbitrarily assigned anymore. That is to say, if $n \geq 3$ and if the updating intends to replace k_{max} original eigenpairs by newly measured data, then with probability one the phenomenon of spill-over will occur. The following example from [8] illustrates this.

Consider the case when $n = 3$. A quadratic pencil generally allows six eigenpairs. Suppose that

five of them are prescribed by

$$\Sigma = \mathrm{diag}\{1, 2, 3, 5, 8\}, \quad Y = \begin{bmatrix} 1 & 0 & 0 & 2 & -1 \\ 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & 2 & 2 \end{bmatrix}.$$

This is a case where $k = k_{max} = 5$. By Theorem 3.3, the solution has three degrees of freedom. We find that the solution to the QIEP can be represented as

$$M = \begin{bmatrix} s & -s+4u & u \\ -s+4u & t & -\frac{7}{10}s + \frac{14}{5}u \\ u & -\frac{7}{10}s + \frac{14}{5}u & -\frac{3}{10}u + \frac{7}{10}s \end{bmatrix},$$

$$C = \begin{bmatrix} -9s+10u & 3s-12u & -4u \\ 3s-12u & -\frac{27}{5}s + \frac{108}{5}u - 7t & \frac{7}{2}s - 14u \\ -4u & \frac{7}{2}s - 14u & \frac{34}{5}u - \frac{77}{10}s \end{bmatrix},$$

$$K = -\begin{bmatrix} -8s+10u & 2s-8u & -3u \\ 2s-8u & -\frac{54}{5}s + \frac{216}{5}u - 10t & \frac{21}{5}s - \frac{84}{5}u \\ -3u & \frac{21}{5}s - \frac{84}{5}u & \frac{177}{10}u - \frac{84}{5}s \end{bmatrix}.$$

It can be shown that

$$\det(M) = -\frac{1}{100}(7s - 10u)\left(272u^2 - 136su - 10tu - 10ts + 17s^2\right).$$

Obviously, we can choose s, t and u so that $\det(M) > 0$. Indeed, the three parameters can be chosen to make the other two principal minors of M positive so that M is positive definite. We also find that the sixth eigenvalue is given by

$$\lambda_6 = -2\frac{52u^2 + 37s^2 - 161su + 40st - 35tu}{17s^2 - 136su - 10tu + 272u^2 - 10st}$$

while its corresponding eigenvector is given by

$$\mathbf{x}_6 = \left[\frac{2}{5}\frac{9s - 36u + 5t}{7s - 10u}, 1, \frac{2}{5}\frac{9s - 36u + 5t}{7s - 10u}\right]^\top.$$

It is clear that the sixth eigenvector $\mathbf{x}_6$ cannot be arbitrarily assigned and, hence, no spill-over *cannot* be guaranteed.

On the other hand, suppose $k = n$ eigenpairs have been modified. Then according to the construction specified in Theorem 3.1, we can derive the following result.

THEOREM 4.1. *Given n distinct eigenvalues Σ and n linearly independent eigenvectors Y both of which are closed under conjugation, construct (M, C, K) as in Theorem 3.1 with A and B as parameters. Then the corresponding self-adjoint quadratic pencil can be factorized as*

$$\begin{aligned} \lambda^2 M + \Sigma C + K &= Y^{-\top}\left(\lambda I_n - \Sigma^\top\right)\left(B + (\lambda I_n + \Sigma^\top)A\right)Y^{-1} \\ &= Y^{-\top}\left(B + A(\lambda I_n + \Sigma)\right)(\lambda I_n - \Sigma)\, Y^{-1}. \end{aligned} \tag{4.3}$$

It is interesting to note from Theorem 4.1 that the remaining eigenvalues are the same as the eigenvalues of the linear pencil $\lambda A + B + A\Sigma$. Since the entire matrix A and (diagonal) part of B are

free, there is room to impose additional eigeninformation to the pencil. In [18], for instance, it has been argued that *additional n eigenvalues* could be arbitrarily specified. This fulfills only partially the no spill-over phenomenon. In our context where we require that eigenvalues and eigenvectors are prescribed in pairs, we believe that spill-over phenomenon is inevitable except in the undamped case. In undamped case, note that the quadratic pencil $\lambda^2 M + K$ can be reduced to a linear pencil $\mu M + K$ with $\mu = \lambda^2$. The following result for a self-adjoint linear pencil is proved in [8].

THEOREM 4.2. *A self-adjoint linear pencil $\mu M + K$ can have arbitrary eigenstructure with n distinct eigenvalues and linearly independent eigenvectors. Indeed, given an eigenstructure (Λ, X) in $\mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}$, the solutions (M, K) form a subspace of dimensionality n in the product space $\mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}$ and can be parameterized by the diagonal matrix Γ via the relationships,*

$$M = X^{-\top}\Gamma X^{-1}, \tag{4.4}$$

$$K = -X^{-\top}\Gamma\Lambda X^{-1}. \tag{4.5}$$

5. Least Squares Update. Inverse eigenvalue problems generally are ill-posed. Any measure of sensitivity or robustness of a solution to perturbations must be designed by taking several factors into consideration [21]. One such attempt is to require that the updating is made with minimal changes [15]. The model updating problem can then be formulated as an optimization problem:

$$\text{minimize} \quad \frac{1}{2}\left(\|M - M_0\|_F^2 + \|C - C_0\|_F^2 + \|K - K_0\|_F^2\right), \tag{5.1}$$

$$\text{subject to} \quad MY\Sigma^2 + CY\Sigma + KY = 0, \quad M,\ C \text{ and } K \text{ are all symmetric}, \tag{5.2}$$

where $(\Sigma, Y) \in \mathbb{R}^{k\times k} \times \mathbb{R}^{n\times k}$ are the newly measured eigenpairs. Note that the above formulation is actually a quadratic programming problem for which many techniques are available. See, for example, [16]. In principle, the least squares model updating problem can be handled by standard optimization procedures, provided the feasible set is known to be nonempty.

Of course, advantage can be taken of the special features of the problem so that the quantities needed for numerical computation are calculated with minimal effort. For example, since the problem involves only linear equality constraints, the projected gradient and the projected Hessian can be calculated formally in terms of the null space of the $k \times 3n$ matrix $[X^\top, \Lambda^\top X^\top, \Lambda^{2\top} X^\top]$. Another approach is to utilize the parametric representation of (M, C, K) that we have developed earlier ——*where?*——*Henk* and rewrite the objective function as an unconstrained optimization in terms of the free parameters. Readers are referred to [19] for an implementation that uses a similar but different parametrization specified in [18].

Structured model updating problems can be formulated in a similar way except that the matrices (M, C, K) in (5.2) are restricted to the specified structures. If the matrices are further required to be nonnegative, then we also have bounded constraints. As we have indicated, we can solve currently only a few structured QIEPs, if they are solvable at all, by numerical algorithms. That is to say, when solving a structured least squares model updating problem, a feasible candidate (M, C, K) can be identified only through point-to-point calculation. This would make it very hard to find the optimal solution. Again, we believe that this is an area open for further research.

6. Conclusions. Model updating so as to attain consistent spectral property with empirical data is an essential ingredient for establishing an effective model. In this chapter, we presented an overview on this subject by briefly addressing three important issues involved in model updating: — we have to satisfy the structural constraint for physical feasibility, — we prefer to see that no spurious modes are introduced into the range of the frequency range of interest, and — we want to keep the modifications minimal.

Before we are able to determine whether some updating can be achieved, a more fundamental

question is to solve the quadratic inverse eigenvalue problem when a set of eigenpairs is prescribed. For this problem we are able to provide a parametric representation of the solution to the QIEP if only symmetry is required of the matrices involved. We demonstrated an algorithmic approach for an undamped QIEP when the structure and nonnegativity are to be maintained, but a general solution procedure is not available yet. From inspection of the dimension of the solution space of the QIEP, we conclude that the spill-over phenomenon is unavoidable. We pointed out many open questions that deserve further study.

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