

On a Multivariate Eigenvalue Problem:
I. Algebraic Theory and a Power Method

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Abstract

Multivariate eigenvalue problems for symmetric and positive definite matrices arise from multivariate statistics theory where coefficients are to be determined so that the resulting linear combinations of sets of random variables are maximally correlated. By using the method of Lagrange multipliers such an optimization problem can be reduced to the multivariate eigenvalue problem. For over thirty years, an iterative method proposed by Horst [12] has been used for solving the multivariate eigenvalue problem. Yet the theory of convergence has never been complete. The number of solutions to the multivariate eigenvalue problem also remains unknown. This paper contains two new results. Using the degree theory we first prove a closed form on the cardinality of solutions for the multivariate eigenvalue problem. We then prove a convergence property of Horst's method by forming it as a generalization of the so called power method. The discussion leads to new formulations of numerical methods.

1. Introduction.

Given a symmetric and positive definite matrix $A \in R^{n \times n}$ and a set

$$(1) \quad P := \{n_1, \dots, n_m\}$$

of positive integers with $\sum_{i=1}^m n_i = n$, let A be partitioned into blocks

$$(2) \quad A = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1m} \\ A_{21} & A_{22} & \dots & A_{2m} \\ \vdots & \vdots & & \vdots \\ A_{m1} & A_{m2} & \dots & A_{mm} \end{bmatrix}$$

with $A_{ii} \in R^{n_i \times n_i}$. A *multivariate eigenvalue problem* (MEP) is to find real scalars $\lambda_1, \dots, \lambda_m$ and a real column vector $x \in R^n$ such that equations

$$(3) \quad Ax = \Lambda x$$

$$(4) \quad \|x_i\| = 1, \quad i = 1, \dots, m$$

are satisfied, where Λ is the diagonal matrix

$$(5) \quad \Lambda = \text{diag}\{\lambda_1 I^{[n_1]}, \dots, \lambda_m I^{[n_m]}\}$$

with $I^{[n_i]}$ the identity matrix of size n_i , and $x \in R^n$ is partitioned into blocks

$$(6) \quad x = [x_1^T, \dots, x_m^T]^T$$

with $x_i \in R^{n_i}$.

We first note that a multivariate eigenvalue problem is fundamentally different from the so called multiparameter eigenvalue problem [19]. The latter problem is, given $A_i, B_{is} \in R^{n_i \times n_i}$, to solve the equations

$$(7) \quad (A_i - \sum_{s=1}^m \lambda_s B_{is})x_i = 0, \quad i = 1, \dots, m$$

for $\lambda = [\lambda_1, \dots, \lambda_m]^T \in R^m$ and $x_r \in R^{n_i}$.

Equations (3) and (4) represent a non-linear algebraic system in $n + m$ unknowns. Trivially, if $m = 1$, then (3) is simply a classical symmetric eigenvalue problem. In this case, it is well understood that, counting multiplicity, there are exactly n eigenvalues and that, counting negative signs, there are exactly $2n$ eigenvectors if all eigenvalues are distinct. When $m > 1$, however, the concept of characteristic polynomial is no longer applicable. In fact, very little theory is known concerning the characteristic of a solution to the (MEP).

The first contribution of this paper is that we are able to determine the cardinality of solutions to the (MEP) by using the degree theory. We prove that a total of $\prod_{i=1}^m (2n_i)$ solutions exist for the (MEP) in the generic case. Our method is similar in spirit to that developed in [3] with generalizations. The result apparently is new. As a by-product, the homotopy may also be used as a numerical method to find *all* solutions, if so desired, of the (MEP).

The (MEP) has its origins in the determination of canonical correlation coefficients for multivariate statistics [10, 12, 13]. The history goes back to that Hotelling [10] first

studied the so called maximal correlation problem which later on has been developed into what is known as the canonical correlation analysis. The numerical method proposed then was somewhat awkward and inefficient. Later, Horst [12] proposed an iterative approach to solve the maximal correlation problem. In spite of a seemingly successful numerical experiment, however, no rigorous proof has even been developed to show that the iterative procedure converges and that the limit point obtained gives maximal correlation.

The second contribution of this paper is that we reformulate Horst's iterative algorithm as a generalization of the so called power method [8]. We provide a proof that shows the method does converge monotonically, but only to a *local* maximal correlation. We give an example that shows that Horst's iteration has a good chance of not converging to the *absolute* maximal correlation.

The paper is organized as follows: The statistical background of the (MEP) is reviewed in section 2. Readers who are familiar with the background of the (MEP) or are interested only in the linear algebra may pass over this section entirely. In section 3, we use the homotopy theory to prove the cardinality of solutions to the (MEP). To accomplish this, we establish several auxiliary lemmas which are of interest in their own right. In section 4, we reformulate Horst's algorithm and proves the convergence properties. The key of our success lies in the fact that there are only finitely many solutions to the (MEP).

Many open questions remain to be studied. The power method can be thought of as a Jacobi-like iterative scheme. Thus other iterative techniques used for linear algebraic equations [9], such as the Gauss-Seidel method or the SOR method, can be modified to solve the (MEP). Multivariate shifting is another possible way to find other solutions of the (MEP). Some of these issues are briefly mentioned in the last section. The details of these new formulations will be discussed in a forthcoming paper [6].

2. Statistical Background.

In this section we provide a somewhat detailed statistical background of the (MEP). The discussion should also make this paper more self-contained. However, readers may choose to skip this section entirely without worries of discontinuity.

For clarity, the following rule is used in naming the notation. A calligraphic letter, \mathcal{X} , denotes a scalar random variable, an accented letter, \tilde{X} , denotes an array of random variables while an upper case letter, X or Ω , denotes a matrix.

Given an n -dimensional random variable $\tilde{X} := (\mathcal{X}_1, \dots, \mathcal{X}_n)$ with a certain distribution function, let $[x_{\xi_1}, \dots, x_{\xi_n}]$, $\xi = 1, \dots, k$, denote a random sample of size k of this variable. For convenience, we use

$$(8) \quad X := [x_{\xi_i}]$$

to denote the $k \times n$ sample matrix. As the notion for a random variable can be carried over in a parallel manner to a random sample and vice versa, in what follows we shall not make careful distinction between a random variable \mathcal{X}_i and its corresponding random sample $[x_{1i}, \dots, x_{ki}]^T$.

By shifting if necessary, we may assume without loss of generality that the sample mean $\mu_i := \sum_{\xi=1}^k \frac{x_{\xi i}}{k}$ for each of the random variable \mathcal{X}_i is zero. It follows then the $n \times n$ matrix

$$(9) \quad \Delta := X^T X$$

represents the covariance matrix of the random sample X . Clearly Δ is symmetric. We further assume as a generic case that there is no degenerate component and no linear dependence among the components $\mathcal{X}_1, \dots, \mathcal{X}_n$. It is a well known fact that Δ is positive definite [22].

Corresponding to the same conformation as in (6), let the components of \tilde{X} be divided into mutually disjoint groups

$$(10) \quad \tilde{X} = (\tilde{X}_1, \dots, \tilde{X}_m).$$

Then each \tilde{X}_i is an n_i -dimensional random variable. Let the matrix

$$(11) \quad \Delta = [\Delta_{ij}]$$

be partitioned in the same way as (2). Then Δ_{ii} represents the covariance matrix of the $k \times n_i$ sample block X_i where the sample matrix X is partitioned as

$$(12) \quad X = [X_1, \dots, X_m].$$

In practice it is often desirable to simplify the analysis by combining all n_i components of \tilde{X}_i linearly into a single new variable Z_i . For $i = 1, \dots, m$, let $b_i \in R^{n_i}$ denote the coefficients of linear combinations for variable \tilde{X}_i . Define the $n \times m$ matrix

$$(13) \quad B := \begin{bmatrix} b_1 & 0 & \dots & 0 \\ 0 & b_2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & \dots & b_m \end{bmatrix}.$$

Then the sample matrix X is transformed into the $n \times m$ matrix

$$(14) \quad Z := XB := [Z_1, \dots, Z_m].$$

The covariance matrix corresponding to the new random sample Z is given by

$$(15) \quad \Omega := Z^T Z = B^T \Delta B.$$

Consider the case $m = 2$ as an example. It is often desirable to use Z_1 to predict Z_2 . Thus it is imperative to find b_1 and b_2 so that the correlation coefficient between the two new random samples Z_1 and Z_2 is as large as possible. The covariance matrix of Z is seen to be the 2×2 matrix

$$(16) \quad \Omega = \begin{bmatrix} b_1^T \Delta_{11} b_1 & b_1^T \Delta_{12} b_2 \\ b_2^T \Delta_{21} b_1 & b_2^T \Delta_{22} b_2 \end{bmatrix}.$$

The correlation coefficient ρ to be maximized is

$$(17) \quad \rho = \frac{b_1^T \Delta_{12} b_2}{\sqrt{b_1^T \Delta_{11} b_1} \sqrt{b_2^T \Delta_{22} b_2}}.$$

If the variances of Z_1 and Z_2 are normalized to unity, then to maximize ρ is equivalent to

$$(18) \quad \text{Maximize} \quad b^T \Delta b$$

$$(19) \quad \text{Subject to} \quad b_i^T \Delta_{ii} b_i = 1, \text{ for } i = 1, 2$$

where

$$(20) \quad b := [b_1^T, b_2^T]^T.$$

As each Δ_{ii} is symmetric and positive definite, the Cholesky decomposition

$$(21) \quad \Delta_{ii} = T_i^T T_i$$

exists. Introduce the block diagonal matrix

$$(22) \quad T := \text{diag}\{T_1, T_2\}$$

and define

$$(23) \quad x := Tb := [x_1^T, x_2^T]^T,$$

$$(24) \quad A := T^{-T} \Delta T^{-1}$$

where $T^{-T} = T^{-1T}$. Clearly A is still symmetric and positive definite. The problem (18) and (19) is now transformed into

$$(25) \quad \text{Maximize} \quad x^T Ax$$

$$(26) \quad \text{Subject to} \quad x_i^T x_i = 1, \quad i = 1, 2.$$

Using the method of Lagrange multipliers, we now form the Lagrangian function

$$(27) \quad \phi(x, \lambda_1, \lambda_2) := x^T Ax - \sum_{i=1}^2 \lambda_i (x_i^T x_i - 1)$$

with λ_1 and λ_2 as the Lagrange multipliers. Upon differentiating (27), it is now clear that the maximal correlation problem (18) and (19) for $m = 2$ is reduced to the 2-variate eigenvalue problem (3) and (4).

When $m > 2$, the requirement of maximizing the correlation coefficients between the m random samples Z_1, \dots, Z_m needs to be modified. Intuitively, the more *similar* the vectors Z_1, \dots, Z_m are to each other, the more closely the correlation coefficients will approach unity. Thus it makes sense to require that the sum of the off-diagonal elements of $m \times m$ matrix Ω in (15) be maximized subject to the condition that the diagonal elements of Ω be unity. Apparently the maximal correlation problem for $m > 2$ case can now be formulated in the same way as (18) and (19). We may repeat the same procedure as for the case $m = 2$ to argue that a solution to the maximal correlation problem

$$(28) \quad \text{Maximize} \quad x^T Ax$$

$$(29) \quad \text{Subject to} \quad x_i^T x_i = 1, \quad i = 1, m.$$

is necessarily a solution to the multivariate eigenvalue problem.

Suppose a set of solutions $\{b_1^{(1)}, \dots, b_m^{(1)}\}$ to the maximal correlation problem has been determined. We may wish to find yet another set of solutions $\{b_1^{(2)}, \dots, b_m^{(2)}\}$ so that the resulting composites $Z_1^{(2)}, \dots, Z_m^{(2)}$ are also maximally correlated. Since the second measurement is conducted independently of the first one, it is natural to require that variables within the same *class* are uncorrelated. That is, each $Z_i^{(2)}$ is

correlated zero with the corresponding $Z_i^{(1)}$. This procedure may be repeated until we have obtained $p := \min\{n_1, \dots, n_m\}$ sets of solutions. The zero correlation is required between any two variables in the class $\{Z_i^{(1)}, \dots, Z_i^{(p)}\}$. The reason why such a repeated measurement is needed can be found from many real-world applications [10, 12, 13]. We note that variables from different classes are not subject to any correlation restrictions, although ideally we would like these variables to be zero correlated as well.

The new problem mentioned above was originally studied by Hotelling [10] who mainly focused on the case where $m = 2$. But the numerical method proposed was very inefficient. Horst [12] then developed a direct approach (for $m = 2$) which we found was utilizing techniques of what is now known as the singular value decomposition. It is worthwhile to rewrite the direct method in terms of the current notion as follows:

We first generalize the notation B in (13) to be the $n \times 2p$ matrix

$$(30) \quad B := \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} := \begin{bmatrix} b_1^{(1)} & \dots & b_1^{(p)} & 0 & \dots & 0 \\ 0 & \dots & 0 & b_2^{(1)} & \dots & b_2^{(p)} \end{bmatrix}$$

The random sample Z then becomes an $n \times 2p$ matrix

$$(31) \quad Z := XB := [Z_1^{(1)}, \dots, Z_1^{(p)}, Z_2^{(1)}, \dots, Z_2^{(p)}].$$

Consider the $2p \times 2p$ covariance matrix $\Omega = B^T \Delta B$ of Z . Recall that each Δ_{ii} has a Cholesky decomposition (21). Let

$$(32) \quad T_1^{-T} \Delta_{12} T_2^{-1} = Q_1^T D Q_2$$

be the singular value decomposition [8] of the $n_1 \times n_2$ matrix $T_1^{-T} \Delta_{12} T_2^{-1}$. It follows that

$$(33) \quad \Omega = \begin{bmatrix} B_1^T T_1^T Q_1^T & 0 \\ 0 & B_2^T T_2^T Q_2^T \end{bmatrix} \begin{bmatrix} I^{[n_1]} & D \\ D & I^{[n_2]} \end{bmatrix} \begin{bmatrix} Q_1 T_1 B_1 & 0 \\ 0 & Q_2 T_2 B_2 \end{bmatrix}.$$

If we choose the vectors in B_1 and B_2 to be

$$(34) \quad B_1 = T_1^{-1} Q_1^T I_{n_1 \times p}$$

$$(35) \quad B_2 = T_2^{-1} Q_2^T I_{n_2 \times p}$$

where $I_{s \times t}$ mean the first $s \times t$ submatrix of the identity, than

$$(36) \quad \Omega = \begin{bmatrix} I^{[p]} & \Sigma \\ \Sigma & I^{[p]} \end{bmatrix}$$

where Σ is the diagonal matrix of singular values $\sigma_1 \geq \dots \geq \sigma_p \geq 0$. In other words, if (34) and (35) are satisfied, then σ_i is precisely the highest correlation coefficient between $Z_1^{(i)}$ and $Z_2^{(i)}$ and each of these two variables is uncorrelated to any other $Z_s^{(t)}$ for $s = 1$ or 2 , and $t \neq i$.

For $m > 2$, the coefficient matrix B is generalized to a block diagonal matrix

$$(37) \quad B := \text{diag}\{B_1, \dots, B_m\}$$

where each B_i is an $n_i \times p$ matrix

$$(38) \quad B_i := [b_i^{(1)}, \dots, b_i^{(p)}]$$

and the random sample becomes

$$(39) \quad Z := XB := [Z_1^{(1)}, \dots, Z_1^{(p)}, \dots, Z_m^{(1)}, \dots, Z_m^{(p)}].$$

Unfortunately Horst's direct method cannot be generalized to solve the general m case. The reason is simply because there is no way to transform all $m \times m$ blocks of size $p \times p$ in

$$(40) \quad \Omega = [B_i^T \Delta_{ij} B_j]$$

into diagonal matrices simultaneously when $m > 2$. To remedy this, it was at this point that Horst introduced his iterative method [12, 13] without a proof.

3. Homotopy Method and Cardinality.

In this section we use the degree theory to prove the cardinality of solutions to the (MEP). The theory has been used as a major tool in analysis to prove the existence of solutions for a wide variety of problems. For a complete mathematical treatment of the theory, we refer to the book [16]. The degree theory can often be implemented as a numerical means, known as the homotopy method, to compute a solution of a nonlinear system. Applications of homotopy methods can be found, for example, in [1, 2, 3, 6, 5, 15, 18, 20, 21].

Rewrite the multivariate eigenvalue problem as a nonlinear system:

$$(41) \quad F(x, \Lambda) = 0$$

where $F : R^n \times R^m \longrightarrow R^n \times R^m$ is defined by

$$(42) \quad F(x, \Lambda) := \begin{bmatrix} \Lambda x - Ax \\ \frac{x_1^T x_1 - 1}{2} \\ \vdots \\ \frac{x_m^T x_m - 1}{2} \end{bmatrix}.$$

In general, it is not an easy task to find a solution for (41). It seems perhaps even harder to count the total number of solutions. On the other hand, consider a simple (MEP):

$$(43) \quad \begin{aligned} Dx &= \Lambda x \\ \|x_i\| &= 1, \quad i = 1, \dots, m. \end{aligned}$$

where D is a diagonal matrix with distinct elements $d_1^{(1)}, \dots, d_{n_1}^{(1)}, \dots, d_1^{(m)}, \dots, d_{n_m}^{(m)}$. It is trivial to see that

LEMMA 3.1. *The problem (43) has exactly $\prod_{i=1}^m 2n_i$ solutions. These are, for $i = 1, \dots, m$,*

$$(44) \quad \begin{aligned} \lambda_i &= d_{j_i}^{(i)} \\ x_i &= \pm e_{j_i}^{[n_i]} \end{aligned}$$

where $j_i = 1, \dots, n_i$ and $e_s^{[t]}$ denotes the s^{th} column of the identity matrix $I^{[t]}$.

Our basic idea is to construct a homotopy between (43) and (3) so that no homotopy curve will escape to infinity or turn back. Toward this end, we define a function $H : R^n \times R^m \times R \rightarrow R^n \times R^m$ as follows :

$$(45) \quad H(x, \Lambda, t; D) := \begin{bmatrix} \Lambda x - [D + t(A - D)]x \\ \frac{x_1^T x_1 - 1}{2} \\ \vdots \\ \frac{x_m^T x_m - 1}{2} \end{bmatrix}$$

where D is a diagonal matrix whose elements will be specified later. The main concern is to show that the Jacobian $D_{x, \Lambda, t} H$ of H is of full rank if D is appropriately chosen. In what follows we establish several auxiliary lemmas to help this investigation. The lemmas, which themselves are of interest, concern the spectrum property when a simple matrix M is perturbed by a diagonal matrix.

A matrix M is simple if and only if the algebraic multiplicities and geometric multiplicities of each of its eigenvalues coincide. Symmetric matrices are automatically simple. We begin with a fact that follows from a more general result in [11, Theorem 1.4.9]:

LEMMA 3.2. *Suppose M is a simple matrix in $R^{n \times n}$ and $r \geq 1$ is a positive integer. Then λ is an eigenvalue of M of multiplicity r if and only if λ is an eigenvalue of all $q \times q$ principal submatrices of M whenever $q \geq n - r + 1$.*

For convenience, we denote the diagonally perturbed matrix by

$$(46) \quad \mathcal{M} = \mathcal{M}(D) := M + D$$

with $D = \text{diag}\{d_1, \dots, d_n\}$. We claim that

LEMMA 3.3. *Let $E := \{(d_1, \dots, d_n) \in R^n \mid \mathcal{M} \text{ has multiple eigenvalues}\}$. Then the complement U of E is open, dense and has full Lebesgue measure in R^n .*

Proof. Let

$$(47) \quad E_r := \{(d_1, \dots, d_n) \in R^n \mid \mathcal{M} \text{ has zero eigenvalue of multiplicity } r\}.$$

By Lemma 3.2, there exists an $(n - r) \times (n - r)$ principal submatrix $\hat{\mathcal{M}}$ of \mathcal{M} such that $\hat{\mathcal{M}}$ is nonsingular. Without loss of generality, we may assume $\hat{\mathcal{M}}$ is the leading principal submatrix of \mathcal{M} , i.e., $\hat{\mathcal{M}}$ is indexed by $\{1, \dots, n - r\}$. All principal submatrices of \mathcal{M} with size $\geq n - r + 1$ are singular. In particular, for $i = 1, \dots, r$ the principal submatrix \mathcal{M}_i indexed by $\{1, \dots, n - r, n - r + i\}$ is singular.

Define

$$(48) \quad f_i(d_1, \dots, d_n) := \det(\mathcal{M}_i).$$

Then the value of (d_1, \dots, d_n) which causes \mathcal{M} to have rank $n - r$ must satisfy the system of equations

$$(49) \quad \mathcal{F}(d_1, \dots, d_n) = 0$$

where $\mathcal{F} : R^n \rightarrow R^r$ is defined by $\mathcal{F} := (f_1, \dots, f_r)$.

Observe that

$$(50) \quad \frac{\partial f_i}{\partial d_{n-r+i}} = \det(\hat{\mathcal{M}}) \neq 0$$

and, hence,

$$(51) \quad \frac{\partial \mathcal{F}}{\partial (d_{n-r+1}, \dots, d_n)} = \det(\hat{\mathcal{M}}) I^{[r]}$$

is nonsingular. By the implicit function theorem, we thus know $\{d_{n-r+1}, \dots, d_n\}$ can be rewritten as functions of $\{d_1, \dots, d_{n-r}\}$. Thus E_r is necessarily embedded in a manifold of dimension $n - r$.

Let $e := [1, \dots, 1] \in \mathbb{R}^n$. Obviously, $\langle e \rangle \oplus E_r$ contains all values of (d_1, \dots, d_n) such that $M + D$ has an eigenvalue of multiplicity r . From the fact that

$$(52) \quad E = \langle e \rangle \oplus \bigcup_{r=2}^n E_r,$$

we see that U is open, dense and has full Lebesgue measure in \mathbb{R}^n . \square

By using the resultant theorem, a complex version of the above lemma has been proved in [5, Theorem 2.3]. The new contribution here is that Lemma 3.3 is for real simple matrices perturbed by real diagonal matrices.

We note that (49) is only a necessary condition for \mathcal{M} being rank deficient by r . Conceivably, the set E_r could be much smaller than what the dimension $n - r$ suggests. This is especially so when M is a symmetric matrix as we have the following observation:

Rewrite \mathcal{M} as

$$(53) \quad \mathcal{M} = \begin{bmatrix} \hat{\mathcal{M}} & \hat{\mathcal{P}} \\ \hat{\mathcal{P}}^T & \hat{\mathcal{R}} \end{bmatrix}.$$

Let

$$(54) \quad \hat{\mathcal{L}} = \begin{bmatrix} I^{[n-r]} & 0 \\ -(\hat{\mathcal{M}}^{-1} \hat{\mathcal{P}})^T & I^{[r]} \end{bmatrix}.$$

Then the rank condition of \mathcal{M} is the same as that of

$$(55) \quad \mathcal{N} := \hat{\mathcal{L}} \mathcal{M} \hat{\mathcal{L}}^T = \begin{bmatrix} \hat{\mathcal{M}} & 0 \\ 0 & \hat{\mathcal{R}} - \hat{\mathcal{P}}^T \hat{\mathcal{M}}^{-1} \hat{\mathcal{P}} \end{bmatrix}.$$

In particular, if \mathcal{M} is of rank $n - r$, then the lower right $r \times r$ block of \mathcal{N} must be identically zero, which gives rise to $r(r + 1)/2$ equations in the d_1, \dots, d_n .

The r diagonal elements of $\hat{\mathcal{R}} - \hat{\mathcal{P}}^T \hat{\mathcal{M}}^{-1} \hat{\mathcal{P}}$ are readily solvable for d_{n-r+1}, \dots, d_n in terms of d_1, \dots, d_{n-r} , as is predicted by the implicit function theorem in the proof of Lemma 3.3. The off-diagonal elements impose $r(r - 1)/2$ extra conditions for which d_1, \dots, d_{n-r} must satisfy. For $r = 2$, it is easy to see that this extra condition is not a trivial equation for d_1, \dots, d_{n-2} . Therefore, E_2 should be a manifold of dimension $n - 3$ (rather than $n - 2$). It should be rather convincing (though tedious to prove) that E_r is in general a manifold of codimension $r(r + 1)/2$ as there are a total of $r(r + 1)/2$ equations d_1, \dots, d_n must satisfy. Obviously, E_r is empty if r is too big. Indeed, by the Wilson-Ledermann bound [14], E_r is empty if

$$(56) \quad r > \lceil \frac{2n + 1 - \sqrt{8n + 1}}{2} \rceil.$$

We do not intend to provide here a proof for the exact dimension of each E_r . For the purpose of this paper it suffices to see from the above argument that

LEMMA 3.4. *For any generic and symmetric matrix M , $\dim(E_r) \leq n - 3$ for $r \geq 2$ and, hence, $\dim(E) \leq n - 2$.*

We demonstrate the case $n = 4$ and $r = 2$ as an example. Suppose M is denoted as $M := [m_{ij}]$. The extra equation to be satisfied by d_1 and d_2 is

$$m_{34} = \frac{m_{13}m_{22}m_{14} + m_{14}m_{13}d_2 - m_{12}m_{14}m_{23} + d_1m_{24}m_{23} - m_{12}m_{13}m_{24} + m_{24}m_{23}m_{11}}{m_{11}m_{22} + m_{11}d_2 + d_1m_{22} + d_1d_2 - m_{12}^2}$$

and E_2 is made of points (d_1, d_2, d_3, d_4) where

$$\begin{aligned} d_2 &= \frac{d_1m_{24}m_{23} - d_1m_{22}m_{34} - m_{22}m_{34}m_{11} + m_{13}m_{22}m_{14} + m_{34}m_{12}^2 + m_{24}m_{23}m_{11} - m_{12}m_{14}m_{23} - m_{12}m_{13}m_{24}}{d_1m_{34} + m_{34}m_{11} - m_{14}m_{13}} \\ d_3 &= \frac{d_1m_{23}m_{34} - d_1m_{33}m_{24} + m_{11}m_{23}m_{34} - m_{13}m_{14}m_{23} + m_{24}m_{13}^2 - m_{12}m_{13}m_{34} + m_{12}m_{33}m_{14} - m_{11}m_{33}m_{24}}{d_1m_{24} + m_{24}m_{11} - m_{14}m_{12}} \\ d_4 &= \frac{d_1m_{24}m_{34} - d_1m_{23}m_{44} + m_{11}m_{24}m_{34} - m_{14}m_{13}m_{24} - m_{12}m_{14}m_{34} - m_{11}m_{23}m_{44} + m_{23}m_{14}^2 + m_{12}m_{13}m_{44}}{d_1m_{23} - m_{13}m_{12} + m_{23}m_{11}} \end{aligned}$$

Obviously, E_2 is a 1-dimensional manifold parameterized in d_1 .

We are now ready to establish one of the major results.

LEMMA 3.5. *The set of D such that the matrix $\Lambda - (D + t(A - D))$ is of rank less than $n - m$ for some Λ and some $t \in (0, 1)$ is of measure zero.*

Proof. For convenience, we denote

$$(57) \quad \mathcal{A} = \mathcal{A}(\Lambda, t, D) := \Lambda - (D + t(A - D)).$$

Observe that each of the m diagonal blocks of \mathcal{A} takes the form

$$(58) \quad A_{ii} = \lambda_i I^{[n_i]} - (1 - t)\text{diag}(d_1^{(i)}, \dots, d_{n_i}^{(i)}) - tA_{ii}.$$

If the rank of \mathcal{A} is less than $n - m$, then one of the diagonal blocks of \mathcal{A} must be rank deficient by at least two. Equivalently, this implies that for some $\tau \in (0, \infty)$ the matrix $\tau\text{diag}(d_1^{(i)}, \dots, d_{n_i}^{(i)}) + A_{ii}$ has an eigenvalue with multiplicity at least two. By Lemma 3.4, the union over τ

$$\bigcup_{\tau \in (0, \infty)} \{(d_1^{(i)}, \dots, d_{n_i}^{(i)}) | \tau\text{diag}(d_1^{(i)}, \dots, d_{n_i}^{(i)}) + A_{ii} \text{ has multiple eigenvalues}\}$$

is of dimension at most $n_i - 1$. Thus, over all, the set of D such that $\Lambda - (D + t(A - D))$ is of rank less than $n - m$ for some Λ and some $t \in (0, 1)$ is at most of geometric dimension $n - 1$. \square

Henceforth, we shall assume that D in (45) has been chosen so that $\mathcal{A}(\Lambda, t, D)$ has rank at least $n - m$ for all Λ and all t . We now prove the existence of the homotopy curves.

LEMMA 3.6. *The point $0 \in R^n \times R^m$ is a regular value for H . That is, for each $(x, \Lambda, t) \in R^n \times R^m \times R$ such that $H(x, \Lambda, t) = 0$ the Jacobian matrix $D_{(x, \Lambda, t)}H$ has rank $n + m$.*

Proof. For convenience, we divide the Jacobian matrix $D_{(x,\Lambda,t)}H$ into blocks:

$$(59) \quad \begin{bmatrix} \mathcal{A} & \mathcal{B} & \mathcal{C} \\ \mathcal{B}^T & 0 & 0 \end{bmatrix}$$

where \mathcal{A} is as given in (57),

$$(60) \quad \mathcal{B} := \begin{bmatrix} x_1 & 0 & \dots & 0 \\ 0 & x_2 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & \dots & x_m \end{bmatrix}$$

and

$$(61) \quad \mathcal{C} := (D - A)x.$$

If $H(x, \Lambda, t) = 0$, then

$$(62) \quad \mathcal{A}x = 0,$$

indicating that x is an eigenvector of \mathcal{A} with corresponding eigenvalue zero.

Suppose for a certain $1 \leq i \leq m$, there exists $y_i \in R^n$ such that

$$(63) \quad \mathcal{A}y_i = \begin{bmatrix} 0 \\ \vdots \\ x_i \\ 0 \\ \vdots \end{bmatrix}.$$

Then, by the symmetry of \mathcal{A} , we know

$$(64) \quad \langle \mathcal{A}y_i, x \rangle = \langle y_i, \mathcal{A}x \rangle = 0.$$

On the other hand, we should have

$$(65) \quad \langle \mathcal{A}y_i, x \rangle = \langle x_i, x \rangle = 1.$$

This contradiction implies that none of the m columns of \mathcal{B} can be in the range of \mathcal{A} .

As \mathcal{A} is at least of rank $n - m$ by Lemma 3.5, it is now clear that the $n \times (n + m)$ matrix $[\mathcal{A}, \mathcal{B}]$ is of rank n . It is also clear the rows of $[\mathcal{B}^T, 0]$ are not in the row space of $[\mathcal{A}, \mathcal{B}]$. Thus the assertion is proved. \square

By now, the following theorem is a standard result from the differential topology.

THEOREM 3.7. *The set $\Gamma := \{(x, \Lambda, t) | H(x, \Lambda, t) = 0\}$ is a one dimensional smooth submanifold in $R^n \times R^m \times R$.*

Furthermore, as $D_{(x,\Lambda,t)}H$ is nonsingular, the implicit function theorem asserts that each component of Γ can be characterized as a function of t . Also if $(x, \Lambda, t) \in \Gamma$, then

$$(66) \quad \sum_{i=1}^m \lambda_i^2 = \|\Lambda x\| = \|((1-t)D + tA)x\| \leq m(\|(1-t)D\| + \|tA\|),$$

implying that no homotopy curve can diverge to infinity for $t \in (0, 1)$.

Putting all the above arguments together, we have proved that for $i = 1, \dots, m$, the solution to the initial value problem

$$(67) \quad \frac{d}{dt} \begin{bmatrix} x_1 \\ \vdots \\ x_m \\ \lambda_1 \\ \vdots \\ \lambda_m \end{bmatrix} = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{B}^T & 0 \end{bmatrix}^{-1} \begin{bmatrix} -\mathcal{C} \\ 0 \end{bmatrix}$$

$$(68) \quad x_i(0) = \pm e_{j_i}^{[n_i]}$$

$$(69) \quad \lambda_i(0) = d_{j_i}^{(i)}$$

is a curve in $R^n \times R^m$ that extends from $t = 0$ to $t = 1$. From Lemma 3.1, we thus conclude that

THEOREM 3.8. *For a generic and symmetric matrix A , the (MEP) has exactly $\prod_{i=1}^m 2n_i$ solutions.*

We believe the result of Theorem 3.8 is new. Moreover, it should be pointed out that the positive definiteness of the matrix A is not needed in the proof of Theorem 3.8.

4. Power Method and Convergence.

Horst's algorithm may be reformulated as follows:

ALGORITHM 4.1. (*Horst's Algorithm*)

Given $x^{(0)} = (x_1^{(0)T}, \dots, x_m^{(0)T})^T$ with $\|x_i^{(0)}\| = 1$, do
for $k = 1, 2, \dots$
for $i = 1, \dots, m$

$$(70) \quad y_i^{(k)} := \sum_{j=1}^m A_{ij} x_j^{(k)},$$

$$(71) \quad \lambda_i^{(k)} := \|y_i^{(k)}\|,$$

$$(72) \quad x_i^{(k+1)} := \frac{y_i^{(k)}}{\lambda_i^{(k)}}.$$

end

end

Theoretically it is possible that $\|y_i^{(k)}\| = 0$ and, hence, (72) is not well defined. Such a breakdown, though rarely occurring in practice, can easily be remedied by re-defining $x_i^{(k+1)}$ to be an arbitrary unit vector in R^{n_i} , say $x_i^{(k)}$ itself, and then continuing the iteration.

To our knowledge, Horst's iterative algorithm has never been rigorously proved to converge although some intuitive support as well as some numerical evidence are mentioned in [12, 13]. In this section we prove a convergence property of Algorithm 4.1 for the (MEP).

We find it is more convenient to write the above algorithm in the compact form:

$$(73) \quad Ax^{(k)} = \Lambda^{(k)} x^{(k+1)}$$

where

$$(74) \quad x^{(k)} := [x_1^{(k)T}, \dots, x_m^{(k)T}]^T$$

and

$$(75) \quad \Lambda^{(k)} := \text{diag}\{\lambda_1^{(k)} I^{[n_1]}, \dots, \lambda_m^{(k)} I^{[n_m]}\}.$$

The iterative scheme may be viewed as a generalization of the classical power method. The convergence property of this method, nonetheless, is not nearly obvious. As an example, without the positive definiteness, the method may fail to converge.

This can be seen from the 2×2 matrix $A = \begin{bmatrix} 1 & b \\ b & c \end{bmatrix}$ with $m = 2$. There are exactly four feasible solutions $(\pm 1, \pm 1)$. For any values of b and c satisfying $|b| < 1$, $c < 0$ and $|b| < -c$, the matrix A is not positive definite. In this case the iterations alternate between $(1, 1)$ and $(1, -1)$, or between $(-1, -1)$ and $(-1, 1)$. No convergence occurs. As another example that will be illustrated later, a limit point of the method may depend upon the starting point. The maximal correlation problem may have multiple local solutions even if the matrix A is positive definite.

To study the convergence property of Horst's algorithm for the (MEP), we denote the objective function in (28) by

$$(76) \quad r(x) := x^T A x$$

for $x \in R^n$. We claim that

THEOREM 4.1. *Suppose A is symmetric and positive definite. Then $\{r(x^{(k)})\}$ with $x^{(k)}$ generated by (73) is a monotonically increasing sequence and converges.*

Proof. From (73), we know that

$$(77) \quad r(x^{(k)}) = x^{(k+1)T} \Lambda^{(k)} x^{(k)}.$$

We also know

$$(78) \quad r(x^{(k+1)}) = x^{(k+1)T} A x^{(k+1)} + x^{(k+1)T} \Lambda^{(k)} x^{(k+1)} - x^{(k+1)T} A x^{(k)}.$$

Subtracting (77) from (78), we obtain

$$(79) \quad r(x^{(k+1)}) - r(x^{(k)}) = x^{(k+1)T} \Lambda^{(k)} (x^{(k+1)} - x^{(k)}) + x^{(k+1)T} A (x^{(k+1)} - x^{(k)}).$$

On the other hand, observe that

$$(80) \quad \begin{aligned} & x^{(k)T} \Lambda^{(k)} (x^{(k+1)} - x^{(k)}) + x^{(k)T} A (x^{(k+1)} - x^{(k)}) \\ &= x^{(k)T} A x^{(k)} - x^{(k)T} \Lambda^{(k)} x^{(k)} + x^{(k+1)T} \Lambda^{(k)} x^{(k+1)} - x^{(k)T} A x^{(k)} \\ &= 0 \end{aligned}$$

because $x^{(k)T} \Lambda^{(k)} x^{(k)} = \sum_{i=1}^m \lambda_i^{(k)}$. Subtracting (80) from the right-hand side of (79), we find that

$$(81) \quad r(x^{(k+1)}) - r(x^{(k)}) = (x^{(k+1)} - x^{(k)})^T (A + \Lambda^{(k)}) (x^{(k+1)} - x^{(k)}).$$

As $\Lambda^{(k)}$ is a diagonal matrix with positive elements, the matrix $A + \Lambda^{(k)}$ remains to be symmetric and positive. It follows that

$$(82) \quad r(x^{(k+1)}) \geq r(x^{(k)}).$$

The convergence of the monotone sequence $\{r(x^{(k)})\}$ is obvious since for all x we have

$$(83) \quad \mu_n \|x\|^2 \leq r(x) \leq \mu_1 \|x\|^2$$

where

$$(84) \quad \mu_1 \geq \dots \geq \mu_n$$

are the eigenvalues of A . \square

It is instructive to motivate Theorem 4.1 from another viewpoint. Let $S_i := \{x_i | x_i \in \mathbb{R}^{n_i}, \|x_i\| = 1\}$. The feasible set (29) for the maximal correlation problem may be regarded as the manifold $\prod_{i=1}^m S_i$ embedded in $\prod_{i=1}^m \mathbb{R}^{n_i}$ with product topology. The tangent space $\mathcal{T}_{(x_1, \dots, x_m)} \prod_{i=1}^m S_i$, therefore, is given by $\prod_{i=1}^m \mathcal{T}_{x_i} S_i$. It is easy to show that the gradient $\nabla r(x)$ projected onto $\mathcal{T}_{x_i} S(i)$ is twice of the vector

$$(85) \quad g_i(x) := \sum_{j=1}^m A_{ij} x_j - \langle x_i, \sum_{j=1}^m A_{ij} x_j \rangle x_i.$$

Observe then

$$(86) \quad \begin{aligned} \langle x^{(k+1)} - x^{(k)}, g(x^{(k)}) \rangle &= \sum_{i=1}^m \langle x_i^{(k+1)} - x_i^{(k)}, g_i(x^{(k)}) \rangle \\ &= \sum_{i=1}^m \left(\|y_i^{(k)}\| - \frac{\langle x_i^{(k)}, y_i^{(k)} \rangle}{\|y_i^{(k)}\|} \right) \end{aligned}$$

while each term in (86) is non-negative. That is, the vector $x^{(k+1)} - x^{(k)}$ forms an acute angle with the projected gradient $g(x^{(k)})$ of $r(x^{(k)})$ and, thus, probably (though not necessarily) points to an ascent direction for $r(x)$.

Define the residual

$$(87) \quad \begin{aligned} \delta(x^{(k)}) &:= Ax^{(k)} - \Lambda^{(k)} x^{(k)} \\ &= \Lambda^{(k)} (x^{(k+1)} - x^{(k)}) \end{aligned}$$

where the second equality follows from (73). If the equality in (82) holds for some k , then from (81) it must be $x^{(k+1)} = x^{(k)}$. We find from (87) that $x^{(k)}$ solves (3) exactly. In general we have

THEOREM 4.2. *The residual $\{\delta(x^{(k)})\}$ in the Horst's algorithm converges to zero as $k \rightarrow \infty$.*

Proof. By the definition (71), all $\lambda_i^{(k)}$ remain bounded as $\|x^{(k)}\|^2 \leq m$ for all k . It follows from (81) that there exists a constant $\kappa > 0$ such that

$$(88) \quad r(x^{(k+1)}) - r(x^{(k)}) \geq \kappa \|x^{(k+1)} - x^{(k)}\|^2$$

for all k . The assertion follows from (87) and the fact that $\{r(x^{(k)})\}$ converges. \square

At this point it is not obvious that the sequence $\{x^{(k)}\}$ itself converges. But it is clear the sequence $\{x^{(k)}\}$ does have cluster point(s) due to its boundedness. From Theorem 4.2, it follows every cluster point x^* satisfies (3) with eigenvalues $\lambda_i^* := \|\sum_{j=1}^m A_{ij}x_j^*\|$.

Define

$$(89) \quad \tilde{\lambda}^{(k)} := \min_{1 \leq j \leq m} \lambda_j^{(k)}$$

and

$$(90) \quad \tilde{\lambda} := \liminf \tilde{\lambda}^{(k)}.$$

Consider the case when $\tilde{\lambda} = 0$. Then there exists a subsequence $\{k_j\}$ of positive integers such that

$$(91) \quad \tilde{\lambda}^{(k_j)} < \frac{1}{j}.$$

Any convergent subsequence of the subsequence $\{x^{(k_j)}\}$ must be such that one of the eigenvalues λ_i^* is zero. We do not think this will happen often for a generic A .

The following lemma from real analysis is useful.

LEMMA 4.3. *Let $\{a_k\}$ be a bounded sequence of real numbers with the property $|a_{k+1} - a_k| \rightarrow 0$ as $k \rightarrow \infty$. If there are only finitely many limit points for the sequence, then $\{a_k\}$ converges to a unique limit point.*

Proof. Suppose $\{a_{\alpha_k}\}$ and $\{a_{\beta_k}\}$ are two subsequences of $\{a_k\}$ which converge, respectively, to two distinct limit points, x and y . Let z denote any fixed real number between x and y . For a positive number r let $B_x(r)$ denote the neighborhood $[x - r, x + r]$ of x .

For any $\epsilon > 0$ which is less than $\frac{1}{4} \min\{|x - z|, |y - z|\}$, there exists a large enough integer $K = K(\epsilon)$ such that

$$\begin{aligned} a_{\alpha_k} &\in B_x(\epsilon) \\ a_{\beta_k} &\in B_y(\epsilon) \\ |a_{k+1} - a_k| &< \epsilon \end{aligned}$$

for all $k \geq K$. Infinitely many elements of $\{a_k\}$ must leave $B_x(\epsilon)$ to enter $B_y(\epsilon)$ and vice versa. Thus there exists an index $\gamma \geq K$ such that $a_\gamma \in B_z(\epsilon)$. This shows that z is also a limit point.

Since z is arbitrary, we have shown any number between x and y is a limit point. This contradicts the assumption that there are only finitely many limit points. \square

In the following we prove that Algorithm 4.1 generates a convergent sequence $\{\Lambda^{(k)}, x^{(k)}\}$. We believe this result is new.

First we prove that

THEOREM 4.4. *The sequence $\{\Lambda^{(k)}\}$ converges as k goes to infinity.*

Proof. We have seen already that the sequence $\{\Lambda^{(k)}\}$ is bounded. Suppose $\{\Lambda^{(k_j)}\}$ is a convergent subsequence. Then the corresponding $\{x^{(k_j)}\}$ also has a convergent subsequence. Without causing any ambiguity, we may assume the subsequence $\{(\Lambda^{(k_j)}, x^{(k_j)})\}$ converges. By Theorem 4.2, it follows that the limit point of $\{(\Lambda^{(k_j)}, x^{(k_j)})\}$ is a solution of the (MEP). By Theorem 3.8, there are only finitely many such limit points.

For convenience, we rewrite the matrix A as a column of m blocks, that is, $A = [A_1, \dots, A_m]^T$ where $A_i := [A_{i1}, \dots, A_{im}]$. Then $\lambda_i^{(k)} = \|A_i x^{(k)}\|$. Observe that

$$|\lambda_i^{(k+1)} - \lambda_i^{(k)}| = |\|A_i x^{(k+1)}\| - \|A_i x^{(k)}\|| \leq \|A_i(x^{(k+1)} - x^{(k)})\|.$$

From (88) and Theorem 4.1, it follows that $|\lambda_i^{(k+1)} - \lambda_i^{(k)}| \rightarrow 0$ as $k \rightarrow \infty$. The assertion now follows by applying Lemma 4.3 to the sequence $\{\lambda_i^{(k)}\}$ for each $i = 1, \dots, m$. \square

Then we prove that

THEOREM 4.5. *The sequence $\{x^{(k)}\}$ converges as k goes to infinity.*

Proof. The proof basically is the same as that in Theorem 4.4, as we observe from (88) and Theorem 4.1 that componentwise the difference between $x^{(k+1)}$ and $x^{(k)}$ converges to zero as k goes to infinity. \square

We now illustrate the dependence of the method upon the starting point. Consider the positive definite matrix

$$A = \begin{bmatrix} 4.3299 & 2.3230 & -1.3711 & -0.0084 & -0.7414 \\ 2.3230 & 3.1181 & 1.0959 & 0.1285 & 0.0727 \\ -1.3711 & 1.0959 & 6.4920 & -1.9883 & -0.1878 \\ -0.0084 & 0.1285 & -1.9883 & 2.4591 & 1.8463 \\ -0.7414 & 0.0727 & -0.1878 & 1.8463 & 5.8875 \end{bmatrix}$$

with $m = 2$, $n_1 = 2$ and $n_2 = 3$. It turns out that the sequence $\{x^{(k)}, \lambda_1^{(k)}, \lambda_2^{(k)}\}$ converges to

$$\begin{aligned} x^* &= [0.9357, 0.3528, -0.9341, 0.3508, 0.0667]^T \\ \lambda_1^* &= 6.5186 \\ \lambda_2^* &= 8.2116 \end{aligned}$$

if $x^{(0)} = [0.9777, 0.2098, 0.5066, 0.5069, 0.6975]^T$, and to

$$\begin{aligned} x^{**} &= [0.7166, 0.6975, 0.5654, -0.4327, -0.7022]^T \\ \lambda_1^{**} &= 6.2405 \\ \lambda_2^{**} &= 7.8607 \end{aligned}$$

if $x^{(0)} = [0.7914, 0.6114, 0.4753, 0.2517, -0.8431]^T$. On the other hand, by repeatedly applying the method to randomly generated starting points, it is interesting to find the statistics that approximately 60% of the points give convergence to x^* while all the remaining converge to x^{**} . This ratio stays about the same regardless whether a normal distribution or a uniform distribution is used as the generator. This is a clear indication that out of the 24 solutions to this (MEP) there are *two* local maxima to the maximal correlation problem. The example also illustrates that Horst's algorithm has a substantial possibility of not converging to the absolute maximal correlation.

5. Conclusion and Future Research.

Having demonstrated the convergence of Horst's algorithm for the (MEP), it is clear that there are still many other numerical issues worthy of investigation. For example, let

$$(92) \quad \Gamma := \text{diag}\{\gamma_1 I^{[n_1]}, \dots, \gamma_m I^{[n_m]}\}$$

where each γ_i is a real number. Then $(A - \Gamma)x = \Lambda x$ if and only if $Ax = (\Gamma + \Lambda)x$. In other words, like the power method, *multivariate shifting* is a possible strategy to find other solutions of the (MEP). To which solution does the algorithm converge based on the starting value for $x^{(0)}$ and on the shift parameters is thus an interesting problem. A theoretical problem associated with the multivariate shifting is related to the so called educational testing problem, that is, for what Γ will the matrix $A - \Gamma$ become positive semi-definite.

On the other hand, the scheme involved in (71) is very analogous to the so called Jacobi method used in solving linear equations. It is thus rather natural to formulate the following Gauss-Seidel method for the (MEP):

ALGORITHM 5.1. (*Gauss-Seidel Algorithm*)

Given $x^{(0)} = (x_1^{(0)T}, \dots, x_m^{(0)T})^T$ with $\|x_i^{(0)}\| = 1$, do
for $k = 1, 2, \dots$
for $i = 1, \dots, m$

$$(93) \quad y_i^{(k)} := \sum_{j=1}^{i-1} A_{ij} x_j^{(k+1)} + \sum_{j=i}^m A_{ij} x_j^{(k)}$$

$$(94) \quad \lambda_i^{(k)} := \|y_i^{(k)}\|,$$

$$(95) \quad x_i^{(k+1)} := \frac{y_i^{(k)}}{\lambda_i^{(k)}}.$$

end

end

Let the matrix A be decomposed as

$$(96) \quad A = D + U^T + U$$

where D is the main diagonal part of A and U is the strictly upper triangular part of A . Then Algorithm 5.1 may be written as

$$(97) \quad (D + U)x^{(k)} = (\Lambda^{(k)} - U^T)x^{(k+1)}.$$

Similarly, a SOR algorithm may be formulated as

ALGORITHM 5.2. (*SOR Algorithm*)

Given $x^{(0)} = (x_1^{(0)T}, \dots, x_m^{(0)T})^T$ with $\|x_i^{(0)}\| = 1$, do
for $k = 1, 2, \dots$
for $i = 1, \dots, m$

$$(98) \quad \bar{y}_i^{(k)} := \sum_{j=1}^{i-1} A_{ij} x_j^{(k+1)} + \sum_{j=i}^m A_{ij} x_j^{(k)}$$

$$(99) \quad \xi_i^{(k)} := \|\bar{y}_i^{(k)}\|,$$

$$(100) \quad \bar{z}_i^{(k+1)} := \frac{\bar{y}_i^{(k)}}{\xi_i^{(k)}}.$$

$$(101) \quad y_i^{(k)} := \omega_i \bar{z}_i^{(k+1)} + (1 - \omega_i) x_i^{(k)}$$

$$(102) \quad \lambda_i^{(k)} := \|y_i^{(k)}\|,$$

$$(103) \quad x_i^{(k+1)} := \frac{y_i^{(k)}}{\lambda_i^{(k)}}.$$

end

end

We note that the relaxation parameters ω_i in each block may be chosen differently. We also remark that the scaling in (100) may be done differently, including the possibility of defining $\xi_j^{(k)} := 1$ for all j and all k .

If we define

$$(104) \quad \Xi^{(k)} := \text{diag}\{\xi_1^{(k)}I^{[n_1]}, \dots, \xi_m^{(k)}I^{[n_m]}\}$$

and

$$(105) \quad \Omega := \text{diag}\{\omega_1I^{[n_1]}, \dots, \omega_mI^{[n_m]}\},$$

then Algorithm 5.2 may be written in matrix form:

$$(106) \quad [(I - \Omega)\Xi^{(k)} + \Omega(D + U)]x^{(k)} = (\Xi^{(k)}\Lambda^{(k)} - \Omega U^T)x^{(k+1)}.$$

which includes (97) as a special case.

Apparently a partial list of topics that deserves further research should include proof of convergence, rate of convergence, acceleration of convergence and so on for each of these methods. Work on some of these aspects to the (MEP) will be expounded upon in a forthcoming paper in progress.

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