On Computing Minimal Realizable Spectral Radii of Nonnegative Matrices

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SUMMARY

For decades considerable efforts have been exerted to resolve the inverse eigenvalue problem for nonnegative matrices. Yet fundamental issues such as the theory of existence and the practice of computation remain open. Recently it has been proved that, given an arbitrary $(n-1)$-tuple $\mathcal{L} = (\lambda_2, \ldots, \lambda_n) \in \mathbb{C}^{n-1}$ whose components are closed under complex conjugation, there exists a unique positive real number $\mathcal{R}(\mathcal{L})$, called the minimal realizable spectral radius of $\mathcal{L}$, such that the set $\{\lambda_1, \ldots, \lambda_n\}$ is precisely the spectrum of a certain $n \times n$ nonnegative matrix with λ_1 as its spectral radius if and only if $\lambda_1 \geq \mathcal{R}(\mathcal{L})$. Employing any existing necessary conditions as a mode of checking criteria, this paper proposes a simple bisection procedure to approximate the location of $\mathcal{R}(\mathcal{L})$. As an immediate application, it offers a quick numerical way to check whether a given n-tuple could be the spectrum of a certain nonnegative matrix.

KEY WORDS: nonnegative matrices, Perron-Frobenius theorem, inverse eigenvalue problem, minimal realizable spectral radius, bisection method

1. Introduction

The notion of nonnegative matrices has been attracting considerable attention in the field because of its important applications in many areas of sciences. Among the large number of established results, eigenvalue properties of nonnegative matrices, notably the well-known Perron-Frobenius theory, are particularly elegant and momentous. Consequently, the inverse eigenvalue problem for nonnegative matrices has been equally interesting for decades. The nonnegative inverse eigenvalue problem (**NIEP**) concerns the construction of a entry-wise nonnegative matrix $A \in \mathbb{R}^{n \times n}$ with a prescribed set $\{\lambda_1, \ldots, \lambda_n\}$ of eigenvalues. Most of the

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discussions found in the literature for NIEPs center around specifying sufficient or necessary conditions that qualify a given set of values as the spectrum of some nonnegative matrices. A short and incomplete list of references leading to various results in this regard includes articles [1, 3, 7, 8, 13, 6, 15] and books [2, 14]. The inadequacy of the current development is evidenced by the fact that the necessary condition usually is too general while the sufficient condition too specific. Another disparity in the course of studying NIEPs also deserves attention. That is, very few general numerical procedures are available for the construction of nonnegative matrices, even after knowing abstractly the existence of a solution. It appears that the scheme of gradient flow proposed in [5] is the first catholic approach for solving general NIEPs. The method is still imperfect in that it might stagnate at a least-squares solution. A more detailed overview in both aspects of theory and computation for NIEPs as well as other types of structured inverse eigenvalue problems can be found in the recent survey article [4] and the extensive collection of references contained therein.

It is interesting to point out a further refinement in the posing of the NIEP to demonstrate the subtlety involved in the inverse problem. Suppose that the given eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$ are all real. The real-valued nonnegative inverse eigenvalue problem (**RNIEP**) is concerned about which set of values $\{\lambda_1, \ldots, \lambda_n\}$ can occur as the spectrum of a nonnegative matrix. The symmetric nonnegative inverse eigenvalue problem (**SNIEP**) is concerned about which set can occur as the spectrum of a symmetric nonnegative matrix. It turns out that there exist real numbers $\{\lambda_1, \ldots, \lambda_n\}$ that solve the RNIEP but *not* the SNIEP [9, 11].

To demonstrate the complexity involved in the solvability of an NIEP, also to facilitate later reference in this paper, we mention below two results that in some sense provide the most broad-based necessary criteria. We shall use these criteria to qualify or reject approximations being computed. We hasten to stress even at this early stage that in the implementation of our algorithm any necessary conditions can be inserted as additional or independent modules to filter out undesirable situations.

Recall that the *moments* of a given matrix A are defined to be the sequence of numbers $s_k = \operatorname{trace}(A^k)$. Obviously, if $\sigma(A) = \{\lambda_1, \ldots, \lambda_n\}$, then

$$s_k = \sum_{i=1}^{n} \lambda_i^k.$$

For nonnegative matrices, the moments are always nonnegative. The following necessary condition can easily be proved by using the Hölder inequality [13].

Theorem 1.1. *Suppose $\{\lambda_1, \ldots, \lambda_n\}$ are eigenvalues of an $n \times n$ nonnegative matrix. Then the inequalities*

$$s_k^m \leq n^{m-1} s_{km} \tag{1}$$

hold for all $k, m = 1, 2, \ldots$.

The inequalities in (1) are sharp because equalities hold in each of them for the identity matrix. Note also that by taking $k = 1$, the inequalities in Theorem 1.1 imply the obvious necessary condition that for all $m = 1, 2, \ldots$,

$$s_m \geq \frac{s_1^m}{n^{m-1}} \geq 0, \tag{2}$$

provided $s_1 \geq 0$.

If we further limit the inverse problem to positive matrices, i.e., every entry exceeds zero, it turns out that the eigenvalues can be "completely" characterized as follows [3, page 313].

Theorem 1.2. *The set $\{\lambda_1, \ldots, \lambda_n\} \subset \mathbb{C}$ with $\lambda_1 = \max_{1 \leq i \leq n} |\lambda_i|$ is the nonzero spectrum of a positive matrix of size $m \geq n$ if and only if the following three conditions hold:*

1. $\lambda_1 > |\lambda_i|$ *for all* $i > 1$,
2. $s_k > 0$ *for all* $k = 1, 2, \ldots,$ *and*
3. $\prod_{i=1}^{n}(t - \lambda_i)$ *has real coefficients in* t.

Although Theorem 1.2 gives rise to the necessary and sufficient conditions, be aware that the size m of the underlying positive matrix has not been specified in the statement. This setting is somewhat different from the usual posing of NIEPs where the size of the matrix is the same as the cardinality of the prescribed spectrum.

The approach adopted in this paper is motivated by the following notion: In many practices, satisfying the necessary conditions has been used as an *exclusion* procedure to eliminate unwanted situations. For instance, finding critical points to satisfy the first order optimality condition usually is the first step in most optimization techniques. Clearly, other conditions need to be brought in to further articulate whether a critical point is a maximizer or minimizer. In a similar manner, we shall use necessary conditions to exclude eigenvalues that are not suitable for NIEPs. As it stands now, however, neither of the above criteria is practical for computation. There are infinitely many inequalities involved. Indeed, most other existing criteria can be of little help for such a task either. Thus the issue of solvability have long remained an open question.

In this paper, we offer yet another avenue to tackle this challenging problem. Our idea is based on the Perron-Frobenius theorem asserting that every nonnegative matrix necessarily has one positive eigenvalue λ_1, called the *Perron root*, which is equal to the spectral radius of the matrix. Given $\{\lambda_1, \ldots, \lambda_n\} \subset \mathbb{C}$ that is closed under complex conjugation, we therefore seek conditions on λ_1 in terms of $\{\lambda_2, \ldots, \lambda_n\}$ so that $\{\lambda_1, \ldots, \lambda_n\}$ is the spectrum of some nonnegative matrix. It turns out that the notion of minimal realizable spectral radius suits precisely for that purpose. Our contribution in this paper is to offer a simple numeral means to approximately locate the minimal realizable spectral radius. Though the significance of this approximation is correlated to the necessary conditions used to establish it, the computation is much more effective than using the necessary conditions themselves. Furthermore, our numerical experiments seem to suggest that the approximation in general is quite robust and close to the true minimal realizable spectral radius.

2. Minimal Realizable Spectral Radius

The notion of minimal realizable spectral radius for nonnegative matrices was first introduced in a 1991 M.S. Thesis at Peking University, which later appeared in [10]. Following the notation fashioned by Xu [17, Chapter 5], we briefly outline this simple yet elegant idea in this section. This discussion naturally leads to a computational means.

For convenience, we shall denote the spectrum of a matrix A by $\sigma(A)$ and the spectral radius by $\rho(A)$. Let

$$\mathcal{E}_{n-1} := \{\mathcal{L} = \{\lambda_2, \ldots, \lambda_n\} \subset \mathbb{C} \mid \lambda_i \in \mathcal{L} \text{ if and only if } \bar{\lambda}_i \in \mathcal{L}\} \tag{3}$$

denote the set of all $(n-1)$-tuples that are closed under complex conjugation. We say that the set $\{\lambda_1; \mathcal{L}\}$ with $\mathcal{L} = \{\lambda_2, \ldots, \lambda_n\} \in \mathcal{E}_{n-1}$ is *realizable* if and only if there exists an $n \times n$ nonnegative matrix A such that $\sigma(A) = \{\lambda_1, \ldots, \lambda_n\}$ and $\rho(A) = \lambda_1$.

Given any prescribed set $\{\lambda_1, \ldots, \lambda_n\}$ as possible eigenvalues in the formulation of an NIEP, the Perron-Frobenius theorem makes it easy to identify the Perron root, say, λ_1. The question of whether the underlying NIEP is solvable, therefore, is reduced to whether the remaining $\mathcal{L} = \{\lambda_2, \ldots, \lambda_n\}$ can be realizable via its association with λ_1. To address this question, we make a slightly different inquiry: Given any $\mathcal{L} = \{\lambda_2, \ldots, \lambda_n\} \in \mathcal{E}_{n-1}$, can it be associated with *some* λ_1 to become realizable? If so, how to characterize this λ_1? The following observations made in [10, 17] provide critical insights into these issues.

First, it is true that any given $\mathcal{L} \in \mathcal{E}_{n-1}$ is realizable via some $\lambda_1 > 0$. Indeed, assume that $\mathcal{L} = \{\lambda_2, \ldots, \lambda_n\}$ where $\lambda_{2k} = \bar{\lambda}_{2k+1} = \mu_k + i\nu_k$, $k = 1, \ldots, s$, are complex-valued and $\lambda_{2s+2}, \ldots \lambda_n$ are real-valued. Then it is straightforward to see that the product

$$P \begin{pmatrix} 2n\delta & 2\delta & 2\delta & \ldots & & & 2\delta & \ldots & 2\delta \\ 0 & \mu_1 & \nu_1 & & & & 0 & \ldots & 0 \\ 0 & -\nu_1 & \mu_1 & & & & & & \\ \vdots & & & \ddots & & & & & \vdots \\ & & & & \mu_s & \nu_s & & & 0 \\ & & & & -\nu_s & \mu_s & 0 & & 0 \\ & & & & & & \lambda_{2s+2} & & 0 \\ & & & & & & & \ddots & \\ 0 & 0 & & & & & 0 & \ldots & \lambda_n \end{pmatrix} P^{-1}, \tag{4}$$

with $\delta = \delta(\mathcal{L}) := \max_{2 \le i \le n} |\lambda_i|$ and $P = [\mathbf{1}, \mathbf{e}_2, \ldots, \mathbf{e}_n]$ where $\mathbf{1} = [1, \ldots, 1]^T$ and $\mathbf{e}_j$ is the standard j-th unit vector, is a positive matrix with spectrum $\{2n\delta; \lambda_2, \ldots, \lambda_n\}$.

Secondly, it is clear that the map $R : \mathcal{E}_{n-1} \longrightarrow \mathbb{R}$ via

$$R(\mathcal{L}) := \inf\{\lambda_1 \in \mathbb{R} | \{\lambda_1; \mathcal{L}\} \text{ is realizable}\} \tag{5}$$

is well-defined. Furthermore, from the above example, we see that

$$\delta(\mathcal{L}) \le R(\mathcal{L}) \le 2n\delta(\mathcal{L}). \tag{6}$$

The quantity $R(\mathcal{L})$, called the *minimal realizable radius* of $\mathcal{L}$, plays a decisive role in the solvability of an NIEP because of the following result [10, 17].

Theorem 2.1. *A given set $\{\lambda_1, \ldots, \lambda_n\}$ with $\lambda_1 \ge |\lambda_i|$, $i = 2, \ldots, n$, and $\{\lambda_2, \ldots, \lambda_n\} \in \mathcal{E}_{n-1}$ is precisely the spectrum of a certain $n \times n$ nonnegative matrix with λ_1 as its spectral radius if and only if $\lambda_1 \ge \mathcal{R}(\mathcal{L})$. The nonnegativity can be replaced by strict positivity if the inequality is replaced by strick inequality.*

This necessary and sufficient criterium $\lambda_1 \ge R(\{\lambda_2, \ldots, \lambda_n\})$ is all we need to determine whether the NIEP with a prescribed set of eigenvalue $\{\lambda_1, \ldots, \lambda_n\}$ is solvable, provided the minimal realizable radius $R(\{\lambda_2, \ldots, \lambda_n\})$ is known. It thus become an interesting and worthy task to compute $R(\mathcal{L})$ for any given $\mathcal{L}$. So far as we know, no effort has been taken to carry out this work.

3. Stochastic Inverse Spectrum Problem

The minimal realizable spectral radius has an immediate application to the stochastic inverse spectrum problem [14]. Recall that the set Θ_n of points in the complex plane that are eigenvalues of an arbitrary $n \times n$ stochastic matrices has been completely characterized in a lengthy treatise by Karpelevič [12]. The main points of the fairly involved Karpelevič theorem can be summarized as follows [14, Theorem 1.8].

Theorem 3.1. *The region Θ_n is contained in the unit disk and is symmetric with respect to the real axis. It intersects the unit circle at points $e^{2\pi i a/b}$ where a and b range over all integers such that $0 \leq a < b \leq n$. The boundary of Θ_n consists of curvilinear arcs connecting these points in circular order. For $n > 3$, any λ on these arcs must satisfy one of these equations*

$$\begin{aligned} \lambda^q(\lambda^p - t)^r &= (1-t)^r, \\ (\lambda^b - t)^d &= (1-t)^d \lambda^q, \end{aligned}$$

where $0 \leq t \leq 1$, and b, d, p, q, r are natural integers determined according to certain specific rules (explicitly given in [12, 14]).

Despite its complexity, the Karpelevič theorem is able to characterize only one complex value a time. The theorem does not provide further insights into when two or more points in Θ_n are eigenvalues of the *same* stochastic matrix. For example, each element of the set $\mathcal{L} = \{-0.90, 0.88 + 0.10i, -.88 - 0.10i\}$ belongs to Θ_4, but these points cannot belong to the same spectrum of any 4×4 stochastic matrix. Determining necessary and sufficient conditions for a complex n-tuple to be spectrum of a row stochastic matrix is a hard problem. One classical approach in this direction is to construct a generic solution for the NIEP first. The nonnegative matrix can then be transformed into a stochastic matrix via the fact that if the maximal eigenvector of a nonnegative matrix A associated with the Perron root, say, λ_1, is positive, then the matrix $D^{-1}\lambda_1^{-1}AD$ is row stochastic [4, 14]. Such an assumption on the positivity of the eigenvector associated with the Perron root λ_1 is in fact not needed, as was observed in the following result [17, Lemma 5.3.2].

Theorem 3.2. *If the set $(\lambda_1; \mathcal{L})$ is realizable, then the set $(1; \mathcal{L}/\lambda_1)$ is the spectrum of a row stochastic matrix.*

Furthermore, with Theorem 2.1, we can extend the Karpelevič theorem by quickly determining whether any given $n-1$ points in the domain Θ_n simultaneously belong to the spectrum of a certain stochastic matrix or not.

Corollary 3.1. *Given $(n-1)$-tuple $\mathcal{L} \in \mathcal{E}_{n-1}$ whose elements are from within Θ_n, the set $\{1; \mathcal{L}\}$ is the spectrum of a certain row stochastic matrix if and only if $\mathcal{R}(\mathcal{L}) \leq 1$.*

Once we have identified a set of $n-1$ points in Θ_n to be realizable by a stochastic matrix, we can identify a "segment" of sets in Θ_n, each of which is realizable by some other stochastic matrices.

Corollary 3.2. *If the set $\{1; \mathcal{L}\}$ is the spectrum of a row stochastic matrix, then every set $\{1; \alpha\mathcal{L}\}$, $1 \leq \alpha \leq \frac{1}{\mathcal{R}(\mathcal{L})}$, is also the spectrum of a certain row stochastic matrix.*

4. Bisection Method

In this section we outline our ideas for the computation of $\mathcal{R}(\mathcal{L})$, followed by an illustrative algorithm. Our notion is quite straightforward, but it offers an effective way to judge whether an NIEP is solvable.

For convenience, denote

$$\omega_k = \omega_k(\mathcal{L}) := \sum_{i=2}^{n} \lambda_i^k, \quad k = 1, 2, \dots \tag{7}$$

for each given $\mathcal{L}$. To estimate $R(\mathcal{L})$, it suffices to consider the sequence of (moment) functions $s_k : [\delta, 2n\delta] \longrightarrow \mathbb{R}$, $k = 1, 2, \dots$, where

$$s_k(t) := t^k + \omega_k. \tag{8}$$

We propose a bisection strategy by using the necessary conditions (1) and (2) to adjust the variable t. In principle, we seek after the minimal value of t so that $s_k(t)$ satisfies all necessary conditions. Note that this is a one-dimensional computation.

It should be pointed out right away that even if *all* necessary conditions are satisfied, we can at most assert that the estimate we obtained serves as a *lower bound* of $\mathcal{R}(\mathcal{L})$. Any λ_1 that is less than the estimated $\mathcal{R}(\mathcal{L})$, therefore, is assured to be unfeasible. For practicality, we shall check in fact only a predestinated set $\mathcal{N}$ of *finitely many* necessary conditions in the form of either (1) or (2).

Beginning with the interval $[\delta, 2n\delta]$, we take the midpoint $t = (2n+1)\delta/2$. If $s_k(t)$ violates any of these necessary conditions in $\mathcal{N}$, then it must be that $t < R(\mathcal{L})$. In this case, the left endpoint could be adjusted safely since we know that $\mathcal{R}(\mathcal{L}) \in [t, 2n\delta]$. We then proceed to the next midpoint and repeat the procedure. On the other hand, if all the given conditions in $\mathcal{N}$ are satisfied, then we have to face two choices: one is to increase the level of complexity in $\mathcal{N}$ by adding more necessary conditions and repeat the procedure; the other is to *presume* that $\mathcal{R}(\mathcal{L}) < t$, adjust the right endpoint, and proceed to work with the next interval $[\delta, t]$. The latter strategy obviously has the danger of greatly underestimate the location of $\mathcal{R}(\mathcal{L})$.

The following algorithm demonstrates one of the tactics in selecting $\mathcal{N}$. We use the parameter ℓ to prepare the "level" of table lookup for values of ω_k and the parameter d to decide the "depth" of checkup on necessary conditions.

Algorithm 4.1. *Given $\mathcal{L} \in \mathcal{E}_{n-1}$ and a tolerance $\epsilon > 0$, let $\delta = \delta(\mathcal{L}) = \max_{2\le i\le n} |\lambda_i|$. The following iterations converge to a lower estimate of $\mathcal{R}(\mathcal{L})$.*

1. *Define $L := \delta$ and $R := 2n\delta$.*
2. *Select an integer level parameter ℓ and an integer depth parameter d.*
3. *Generate $\omega_k(\mathcal{L})$ for $k = 1, 2, \dots, \ell!$.*
4. *While $(R - L)/2 > \epsilon$, do*
   - *(a) $t = (R + L)/2$;*
   - *(b) for $k = 1 : \ell!/d$*

          if $t^k + \omega_k \le 0$, go to (4d).
          else
              for $m = 2 : \lfloor \ell!/k \rfloor$

if $(t^k + \omega_k)^m > n^{m-1}(t^{km} + \omega_{km})$, *go to (4d).*

end

end

end

(c) *go to (4f). (No left-end is reset. Might need more depth.)*
(d) *set* $L = t$.
(e) *go to (4).*
(f) *set* $R = t$.

We iterate again that the decision made at Step (4c) to cut back the right endpoint has the danger of underestimating $\mathcal{R}(\mathcal{L})$. We also emphasize that Step (4b) is fully flexible in that any additional necessary conditions can easily be included, if necessary, in $\mathcal{N}$ to provide a mechanism of multi-layer filtering before we have to make a decision at (4c).

The advantage of determining $\mathcal{R}(\mathcal{L})$ is quite obvious. For each given $\mathcal{L}$, it settles the issue of feasibility of all potential $\{\lambda_1; \mathcal{L}\}$. Once the spectrum $\{\lambda_1; \mathcal{L}\}$ is determined to be feasible, what remains to be done is to construct such a nonnegative matrix with the prescribed spectrum. We have mentioned earlier that techniques for the construction of nonnegative matrices are still in need of further research.

5. Numerical Experiment

In this section we demonstrate by numerical examples how Algorithm 4.1 would perform in estimating $\mathcal{R}(\mathcal{L})$. We choose $\epsilon = 10^{-12}$, $\ell = 4$ and $d = 1$. The level parameter and the depth parameter can certainly be enlarged to increase the complexity of $\mathcal{N}$ and possibly the degree of confidence of the computed result.

Example 1. We begin with a pathological example demonstrating the case when the algorithm fails! The algorithm returns $\mathcal{R}(\{\sqrt{2}, i, -i\}) = \sqrt{2}$, suggesting that $\{\sqrt{2}, \sqrt{2}, i, -i\}$ should be the spectrum of a 4×4 nonnegative matrix. By the Perron-Frobenius theorem, such a matrix must be reducible. It follows that $\{\sqrt{2}, i, -i\}$ must be spectrum of a 3×3 nonnegative matrix, which is impossible because the necessary condition (1) is clearly violated. Indeed, the algorithm returns $\mathcal{R}(\{i, -i\}) = \sqrt{3}$, assuring that $\{\sqrt{2}, i, -i\}$ cannot be spectrum of any 3×3 nonnegative matrix.

The problem with the wrongly calculated minimal realizable radius $\mathcal{R}(\{\sqrt{2}, i, -i\}) = \sqrt{2}$ is because the set $\{\sqrt{2}, \sqrt{2}, i, -i\}$ happens to satisfy necessary conditions (1) and (2) for all k and m. That is, the necessary conditions we have used in (4b) are never enough to capture a correct upper bound. This extreme example shows that the checking mechanism (4b) as it is now fails, regardless of the values of level ℓ and d. Remedies might come only if some other necessary conditions independent of (1) and (2) are implemented in Step (4b) as additional filters.

Example 2. For 3×3 NIEPs, the solutions are known in closed form. For instance, it is known that with $\nu > 0$, the exact minimal realizable radius is given by $\mathcal{R}(\{\mu + i\nu, \mu - i\nu\}) = \max\{-2\mu, \mu + \sqrt{3}\nu\}$. We test our algorithm against this closed form solution.

The Algorithm 4.1 is run on a grid of size 0.05 over the range $-5 \leq \mu \leq 5$ and $0 < \nu \leq 5$. We compare the computed minimal realizable spectral radii with the theoretical ones point by point on the grid. The maximal error is approximately 5.9×10^{-11}, clearly indicating that our algorithm works reasonably well.

Example 3. A well-known result by Suleǐmanova asserts that if $\mathcal{L} = \{\lambda_2, \dots, \lambda_n\} \subset \mathbb{R}$ and if all $\lambda_i \in \mathcal{L}$ are non-positive, then $\mathcal{R}(\mathcal{L}) = -\lambda_2 - \dots - \lambda_n$ [16]. This knowledge provides us a chance to test our algorithm against problems of relatively large sizes.

We generate 100 sets of random numbers with sizes ranging from 2 to 101. The random numbers come from a uniform distribution over the interval $[-10, 0]$. Using each set as the prescribed $\mathcal{L}$, we apply our algorithm and compare the computed minimal realizable spectral radii with the Suleimanova bound. Errors in Figure 1 clearly indicates that our algorithm correctly (up to the stopping criterium) computes the minimal realizable spectral radius in each case.

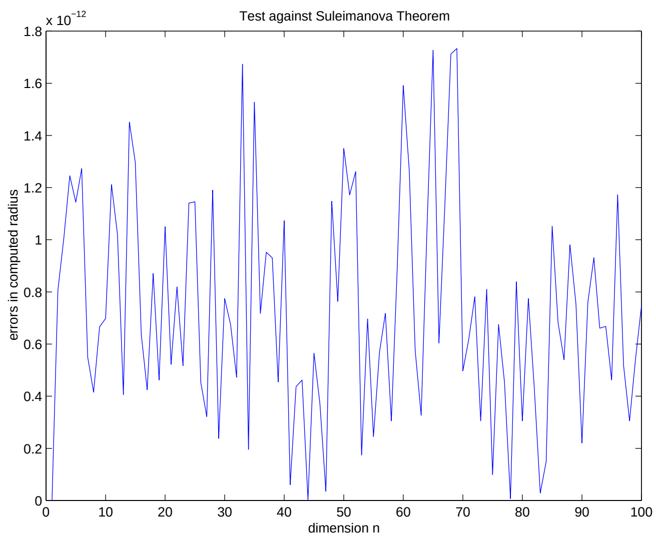

Figure 1. Errors in predicting the Suleimanova bound.

Example 4. Consider the example by Johnson el al in [11] where it is shown that the spectrum with $\lambda_1 = \sqrt[3]{51} + \epsilon$, $\lambda_2 = \lambda_3 = \lambda_4 = 1$, $\lambda_5 = \lambda_6 = -3$, and $\lambda_7 = \dots = \lambda_m = 0$ provides an example that solves the RNIEP but not the SNIEP. We apply our algorithm to $\mathcal{L}_m = \{1, 1, 1, -3, -3, 0, \dots, 0\}$ with m ranging from 7 to 250. We are curious to know how Johnson's λ_1 would differ from the computed $\mathcal{R}(\mathcal{L}_m)$.

We plot the difference $\mathcal{R}(\mathcal{L}_m) - \sqrt[3]{51}$ and obtain an interesting relationship depicted in Figure 2. The result affirms that $\mathcal{R}(\mathcal{L}_m) < \lambda_1$ and suggests that $\mathcal{R}(\mathcal{L}_m)$ converges to the critical value of $\sqrt[3]{51}$ as m goes to infinity. It also seems to provide a bound on the size of ϵ needed to substantiate Johnson's RNIEP result.

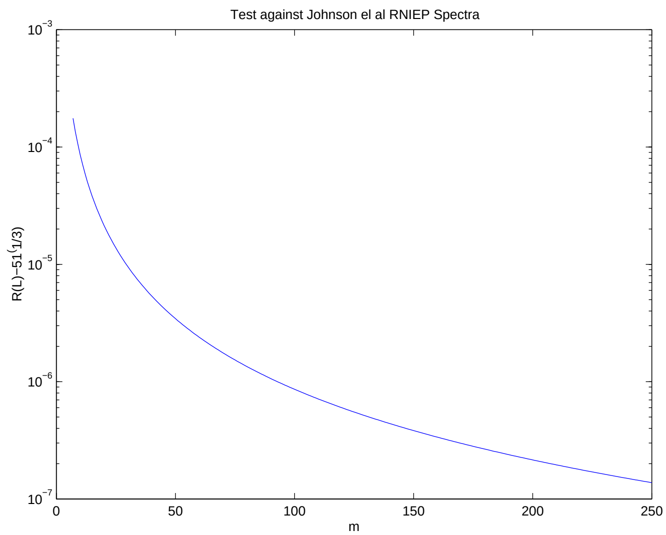

Figure 2. Computed minimal realizable spectral radius for the Johnson el al spectra.

6. Conclusion

The minimal realizable spectral radius of any given $\mathcal{L} = \{\lambda_2, \ldots, \lambda_n\}$, closed under complex conjugation, serves as the sole critical threshold for λ_1 in determining whether $\{\lambda_1, \ldots, \lambda_n\}$ is the spectrum of an $n \times n$ nonnegative matrix. We offer a simple bisection idea to estimate $\mathcal{R}(\mathcal{L})$ numerically. The more necessary conditions we can implement into the filtering mechanism $\mathcal{N}$, the more accurate we can estimate the location of $\mathcal{R}(\mathcal{L})$ which, in turn, offers an effective way to determine whether an NIEP is solvable. Numerical experiments seem to suggest that, except in the extreme case, this simple algorithm works reasonably well.

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