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Updating quadratic models with no spillover effect on unmeasured spectral data

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Abstract

Model updating concerns the modification of an existing but inaccurate model with measured data. For models characterized by quadratic pencils, the measured data usually involve incomplete knowledge of natural frequencies, mode shapes, or other spectral information. In conducting the updating, it is often desirable to match only the part of observed data without tampering with the other part of unmeasured or unknown eigenstructure inherent in the original model. Such an updating, if possible, is said to have no spillover. Model updating with no spillover has been a very challenging task in applications. This paper provides a complete theory on when such an updating with no spillover is possible.

1. Introduction

Modelling is one of the most fundamental tools that we use to simulate the complex world. The goal of modelling is to come up with a representation that is simple enough for mathematical manipulation yet powerful enough for describing, inducing and reasoning complicated phenomena. Nonetheless, precise mathematical models of physical systems are rarely available in practice. Many factors, including inevitable disturbances to the measurement and imperfect characterization of the model, contribute to the inexactitude. For various reasons, it often becomes necessary to *update* a primitive model to attain consistency with empirical results. This procedure of updating or revising an existing model is an essential

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ingredient for establishing an effective model. The emphasis of this paper is on the updating of a self-adjoint quadratic model in the form,

$$Q(\lambda) := \lambda^2 M + \lambda C + K, \tag{1.1}$$

where M, C and $K \in \mathbb{R}^{n\times n}$ are symmetric with M being positive definite and K positive semi-definite. The quadratic matrix polynomial $Q(\lambda)$ is generally known as a quadratic pencil.

Self-adjoint quadratic pencils arise in many areas of important applications. Indeed, when modelling physical properties in applied mechanics, electrical oscillation, vibro-acoustics, fluid mechanics, signal processing, or discretizing PDEs by finite elements, one often has to deal with a second-order differential system

$$M\ddot{\mathbf{v}} + C\dot{\mathbf{v}} + K\mathbf{v} = f(t), \tag{1.2}$$

where specifications of the underlying physical system are embedded in the matrix coefficients M, C and K. It is well known that if

$$\mathbf{v}(t) = \mathbf{x}\,\mathrm{e}^{\lambda t}$$

represents a fundamental solution to (1.2), then the scalar λ and the vector $\mathbf{x}$ must solve the *quadratic eigenvalue problem* (QEP)

$$(\lambda^2 M + \lambda C + K)\mathbf{x} = 0. \tag{1.3}$$

The scalar $\lambda \in \mathbb{C}$ and $x \in \mathbb{C}^n$ are called, respectively, the eigenvalue and the eigenvector corresponding to λ. Because of the connection that the bearing of the dynamical system (1.2) usually can be interpreted via the eigenvalues and eigenvectors of the algebraic system (1.3), considerable efforts have been devoted to the QEP in the literature. A good survey of many applications, mathematical properties, and a variety of numerical techniques for the QEP can be found in the treatise by Tisseur and Meerbergen [25]. For convenience, we shall refer to the triplet (M, C, K) interchangeably as a quadratic pencil.

Two aspects of the quadratic pencil associated with the model (1.2) deserve attention. The *direct problem* involves analysing and deriving the spectral information and, hence, inducing the dynamical behaviour of a system from *a priori* known physical parameters such as mass, length, elasticity, inductance, capacitance, and so on. The *inverse* problem involves validating, determining, or estimating the parameters of the system according to its observed or expected behaviour. The direct problem concerns manifesting the behaviour in terms of the parameters whereas the inverse problem concerns expressing the parameters in terms of the behaviour. Both problems are of significant importance in applications. We have seen in the above that the QEP is a direct problem. Its counterpart, known as a *quadratic inverse eigenvalue problem* (QIEP) can be formulated as follows:

> (QIEP) Construct a nontrivial quadratic pencil $Q(\lambda) = \lambda^2 M + \lambda C + K$ so that its matrix coefficients (M, C, K) are of a specified structure and $Q(\lambda)$ has a specified set $\{(\lambda_i, \mathbf{x}_i)\}_{i=1}^{\kappa}$ as its eigenpairs.

Since we are only interested in real matrices, it is natural to expect that the prescribed eigenpairs are closed under complex conjugation. Without loss of generality, we shall denote the κ prescribed eigenpairs in the matrix form (Λ, X) where $\Lambda \in \mathbb{R}^{\kappa\times\kappa}$ is a block diagonal with at most 2×2 blocks along the diagonal wherever a complex-conjugate pair of eigenvalues appear in the prescribed spectrum and $X \in \mathbb{R}^{n\times\kappa}$ represents the 'eigenvector matrix' in the sense that each pair of column vectors associated with a 2×2 block in Λ retains the real and the imaginary part, respectively, of the original complex eigenvector. In this way, we may identify the given eigenpairs (Λ, X) as an element in $\mathbb{R}^{\kappa} \times \mathbb{R}^{n\times\kappa}$. The QIEP therefore amounts to solving the algebraic equation

$$MX\Lambda^2 + CX\Lambda + KX = 0 \tag{1.4}$$

for the matrices M, C and K subject to some structural constraints.

By a model updating for the quadratic pencil (1.1), we mean to replace a portion of its original eigenstructure by some newly measured eigeninformation. Among current developments for the quadratic model updating, one challenge that is of practical importance is to update the model while maintaining current vibration parameters not related to the newly measured parameters invariant. We state the model updating problem as follows:

(MUP) Given a quadratic pencil (M_0, C_0, K_0) and a few of its associated eigenpairs $\{(\lambda_j, \mathbf{x}_j)\}_{j=1}^k$ with $k < n$, assume that new eigenpairs $\{(\sigma_j, \mathbf{y}_j)\}_{j=1}^k$ have been measured. Update the quadratic pencil (M_0, C_0, K_0) to a new quadratic pencil (M, C, K) such that

(i) the newly measured $\{(\sigma_j, \mathbf{y}_j)\}_{j=1}^k$ form k eigenpairs of the new model (M, C, K);
(ii) the remaining $2n - k$ eigenpairs of (M, C, K) are kept the same as those of the original (M_0, C_0, K_0).

The second condition above is known as the *no spillover phenomenon* [9] to the unmeasured or unknown eigenstructure. We stress that, though similar in spirit, this spillover described here is not to be confused with the spillover effect in the context of control of flexible systems [1] where a reduced-order model including some lower modes is used to determine a control law and the controller based on such a model interacts with the residual modes which cannot synthesize the modal coordinates exactly and the 'unmodelled' mode might lead to some excitement and instability. No spillover is required in the updating process either because these parameters are proven to be acceptable in the previous model and engineers do not wish to introduce new vibrations via updating or, more importantly, because engineers simply do not know any information about these parameters. It is sensible to consider the MUP as a special QIEP with the no spillover portion as the prescribed eigenstructure. However, keep in mind that it is highly desirable to construct the update (M, C, K) without knowledge of the remaining $2n - k$ eigeninformation.

Model updating problems emerged in the 90s as an important tool for the design, construction and maintenance of mechanical systems [14, 21, 22]. The application intends to correct errors in a finite element model by incorporating the measured modal data into the analytical finite element model, producing an adjusted model on the mass, damping and stiffness whose resulting behaviour closely matches the experimental data. Over the years, a number of approaches has been proposed. We briefly review some of them below.

For undamped systems, i.e., $C = 0$, various techniques have been discussed by Baruch [2], Baruch and Bar-Itzak [4], Bermann [5], Bermann and Nagy [7] and Wei [26–28]. For damped systems, under the assumption of proportional damping, which seems to be sufficient where damping levels are lower than 10% of being critical [15], identification techniques have been developed by Pilkey [23] to estimate the damping matrices. For 'strong' damped systems, the theory and computation were first proposed by Friswell, Inman and Pilkey [15, 23]. Along a similar vein but employing the ideas in [2, 4] to minimize changes between the analytical and updated model subject to the spectral constraints, Kuo, Lin and Xu [19] have recently proposed a direct method which seems more efficient and reliable. Another line of thought is to update with symmetric low-rank correction of damping and stiffness matrices [13, 18, 22, 29, 30]. All these existing methods can reproduce the given set of measured data while keeping updated matrices symmetry, but cannot guarantee that the remaining eigenvalues and eigenvectors of the QEP are invariant after the update.

On the other hand, one can consider the MUP from a control point of view. It is sometimes desirable, such as averting some immediate danger, to alter the dynamical behaviour of a certain physical system quickly and temporarily by making minimal changes in its parameters while keeping the structure properties intact as much as possible. The resulting mathematical problem, known as the partial pole-assignment problem in control theory [20], is often solved by using feedback control techniques. Advances in this area include studies by Srinathkumar [24], Datta, Elhay, Ram and Sarkissian [11–13], and Lin and Wang [16]. The difficulty is that the feedback used in the second-order control system leads to a nonsymmetric system. Recently an iterative scheme was suggested in [8] to reassign one eigenvalue at a time preserving both symmetry and no spillover in the process. The trouble is that the algorithm can break down prematurely and cannot guarantee that all desirable eigenvalues are updated.

Our main contribution in this paper is that we offer a complete theory on the solvability of the MUP. We believe that our necessary and sufficient condition is new in the field and should give considerable insight into the important model updating problem.

2. Preliminaries

In a previous study [9], we have shown that the QIEP with no damping, i.e., $C = 0$, can be solved with *any* number of arbitrarily assigned eigenpairs. In this case, updating with no spillover is entirely possible for undamped quadratic pencils. In contrast, the QIEP with damping can be solved with up to $k_{\max}$ arbitrarily assigned eigenpairs where the maximal allowable number $k_{\max}$ is given by

$$k_{\max} = \begin{cases} 3\ell + 1, & \text{if} \quad n = 2\ell, \\ 3\ell + 2, & \text{if} \quad n = 2\ell + 1. \end{cases} \tag{2.1}$$

More specifically, we have proved the following theorem concerning the general solvability.

Theorem 2.1. *Given any positive integer $\kappa \leqslant k_{\max}$, let (Λ, X) represent κ arbitrarily prescribed eigenpairs which are closed under complex conjugation. Then*

(i) the self-adjoint QIEP associated with (Λ, X) is always solvable;
(ii) for almost all κ prescribed eigenpairs (Λ, X), the solutions to the corresponding self-adjoint QIEP form a subspace of dimensionality $\frac{3n(n+1)}{2} - n\kappa$.

If more than $k_{\max}$ eigenpairs are prescribed, examples can be established to show that the QIEP has no solution. Since the MUP can be considered as a QIEP with $2n - k$ eigenpairs fixed (though maybe unknown) and k eigenpairs specified, this theorem seems to suggest the MUP as unsolvable in general. In particular, let the eigenvectors and eigenvalues of the original system (M_0, C_0, K_0) be partitioned, in real-value form as we have described before, as $[X_1, Z] \in \mathbb{R}^{n\times 2n}$ and $\operatorname{diag}\{\Lambda_1, \Upsilon\} \in \mathbb{R}^{2n\times 2n}$, respectively, where the portion $(\Lambda_1, X_1) \in \mathbb{R}^{k\times k} \times \mathbb{R}^{n\times k}$ is to be updated by a newly measured eigenpair (Σ, Y). Updating with no spillover means finding symmetric matrices ΔM, ΔC and ΔK such that the equations,

$$(M_0 + \Delta M)Z\Upsilon^2 + (C_0 + \Delta C)Z\Upsilon + (K_0 + \Delta K)Z = 0, \tag{2.2}$$

$$(M_0 + \Delta M)Y\Sigma^2 + (C_0 + \Delta C)Y\Sigma + (K_0 + \Delta K)Y = 0, \tag{2.3}$$

are satisfied simultaneously. By (2.2), it is necessary that the incremental pencil,

$$\Delta Q(\lambda) := \lambda^2\Delta M + \lambda\Delta C + \Delta K, \tag{2.4}$$

has the $\kappa = 2n - k$ eigenpairs (Υ, Z) as part of its eigenstructure. It seems plausible to conclude that if $k < 2n - k_{\max}$, that is, if too few eigenpairs (Λ_1, X_1) of the original pencil are to be updated, then the QIEP for $\Delta Q(\lambda)$ is over-determined and can have only trivial solution. In other words, it appears that spillover for the damped quadratic pencil generally is unavoidable. This notion, if true, would be quite disappointing because in practice it is often the case that only the first few low frequencies and modes are measurable. It must be emphasized, however, that in our setting for the MUP the original triplet (M_0, C_0, K_0) is itself a nontrivial solution to the QIEP associated with (Υ, Z). The QIEP for $\Delta Q(\lambda)$, though over-determined, does have nontrivial solutions and is not the generic QIEP described in theorem 2.1. The question is how to characterize the general solution $(\Delta M, \Delta C, \Delta K)$ for (2.2) so as to further specify conditions on (Σ, Y) for (2.3). The purpose of this paper is to address the solvability of the MUP with a more thorough analysis.

Without loss of generality, we shall adopt the following notation and make some basic assumptions throughout the discussion.

A1. Assume that all eigenvalues of the original pencil $Q_0(\lambda) := \lambda^2 M_0 + \lambda C_0 + K_0$ are simple.

A2. Assume that the number k of eigenpairs to be updated is less than n.

A3. Assume that the original (complete) eigenstructure $(\Lambda, X) \in \mathbb{R}^{2n\times 2n} \times \mathbb{R}^{n\times 2n}$ can be partitioned into *three* parts of sizes as indicated,

$$\Lambda = \operatorname{diag}\{\underbrace{\Lambda_1}_{k\times k}, \overbrace{\underbrace{\Lambda_2}_{(n-k)\times(n-k)}, \underbrace{\Lambda_3}_{n\times n}}^{\Upsilon}\}, \qquad X = [\underbrace{X_1}_{n\times k}, \overbrace{\underbrace{X_2}_{n\times(n-k)}, \underbrace{X_3}_{n\times n}}^{Z}],$$

and that Λ_1 is invertible.

A4. Assume further that each block is closed under conjugation and, hence, we can write

$$\Lambda_1 = \operatorname{diag}\{\lambda_1^{[2]}, \ldots, \lambda_{\ell_1}^{[2]}, \lambda_{2\ell_1+1}, \ldots, \lambda_k\},$$
$$\Lambda_2 = \operatorname{diag}\{\lambda_{k+1}^{[2]}, \ldots, \lambda_{k+\ell_2}^{[2]}, \lambda_{k+2\ell_2+1}, \ldots, \lambda_n\},$$
$$\Lambda_3 = \operatorname{diag}\{\lambda_{n+1}^{[2]}, \ldots, \lambda_{n+\ell_3}^{[2]}, \lambda_{n+2\ell_3+1}, \ldots, \lambda_{2n}\},$$

with $\lambda_j \in \mathbb{R}$, $\lambda_j^{[2]} = \left[\begin{smallmatrix} \alpha_j & \beta_j \\ -\beta_j & \alpha_j \end{smallmatrix}\right]$, $\alpha_j, \beta_j \in \mathbb{R}$, $\beta_j > 0$ and that both square matrices $[X_1, X_2]$ and X_3 are nonsingular.

Assumption A2 is for practical purpose since typically n is large and k is small. It should be noted, however, that assumptions A3 and A4 impose some mild limitation on the original model (M_0, C_0, K_0). For instance, the quadratic pencil

$$\lambda^2 I_3 + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

does not have such a 3-block partition with either $k = 1$ or 2.

It is easy to see that $Q_0(\lambda)\mathbf{x} = 0$ if and only if

$$L(\lambda)\begin{bmatrix} \mathbf{x} \\ \mathbf{z} \end{bmatrix} = 0, \tag{2.5}$$

where

$$L(\lambda) := \lambda \begin{bmatrix} C_0 & M_0 \\ M_0 & 0 \end{bmatrix} - \begin{bmatrix} -K_0 & 0 \\ 0 & M_0 \end{bmatrix},$$

and $\mathbf{z} = \lambda\mathbf{x}$ if M_0 is nonsingular. By A1, it is well known that we can normalize the eigenvectors X in such a way that

$$\begin{bmatrix} X \\ X\Lambda \end{bmatrix}^{\top} \begin{bmatrix} C_0 & M_0 \\ M_0 & 0 \end{bmatrix} \begin{bmatrix} X \\ X\Lambda \end{bmatrix} = S = \operatorname{diag}\{S_1, S_2, S_3\}, \tag{2.6}$$

$$\begin{bmatrix} X \\ X\Lambda \end{bmatrix}^{\top} \begin{bmatrix} -K_0 & 0 \\ 0 & M_0 \end{bmatrix} \begin{bmatrix} X \\ X\Lambda \end{bmatrix} = S\Lambda, \tag{2.7}$$

where

$$\begin{aligned}
S_1 &= \operatorname{diag}\left\{ \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \ldots, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}_{\ell_1 \text{ copies}}, \epsilon_{2\ell_1+1}, \ldots, \epsilon_k \right\}, \\
S_2 &= \operatorname{diag}\left\{ \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \ldots, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}_{\ell_2 \text{ copies}}, \epsilon_{k+2\ell_2+1}, \ldots, \epsilon_n \right\}, \\
S_3 &= \operatorname{diag}\left\{ \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \ldots, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}_{\ell_3 \text{ copies}}, \epsilon_{n+2\ell_3+1}, \ldots, \epsilon_{2n} \right\},
\end{aligned} \tag{2.8}$$

with $\epsilon_j = \pm 1$. Note that there are equal numbers of positive and negative signs among the ϵ_j's. We shall exploit this standard form to establish the solvability conditions for the MUP. Specifically, with the definition

$$\widetilde{\Lambda} := \operatorname{diag}\{\Lambda_1, \Lambda_2\}, \qquad \widetilde{X} := [X_1, X_2],$$

and by comparing the corresponding blocks on both sides of (2.6) and (2.7), respectively, we obtain the relationships

$$\widetilde{X}^{\top} C_0 X_3 + \widetilde{\Lambda}^{\top} \widetilde{X}^{\top} M_0 X_3 + \widetilde{X}^{\top} M_0 X_3 \Lambda_3 = 0, \tag{2.9}$$

$$X_3^{\top} C_0 X_3 + \Lambda_3^{\top} X_3^{\top} M_0 X_3 + X_3 M_0 X_3 \Lambda_3 = S_3, \tag{2.10}$$

$$-\widetilde{X}^{\top} K_0 X_3 + \widetilde{\Lambda}^{\top} \widetilde{X}^{\top} M_0 X_3 \Lambda_3 = 0, \tag{2.11}$$

$$-\widetilde{X}^{\top} K_0 X_2 + \widetilde{\Lambda}^{\top} \widetilde{X}^{\top} M_0 X_2 \Lambda_2 = \begin{bmatrix} 0 \\ S_2 \Lambda_2 \end{bmatrix}. \tag{2.12}$$

We also have the equalities

$$-X_1^{\top} K_0 X_2 + \Lambda_1^{\top} X_1^{\top} M_0 X_2 \Lambda_2 = 0, \tag{2.13}$$

$$-X_1^{\top} K_0 X_3 + \Lambda_1^{\top} X_1^{\top} M_0 X_3 \Lambda_3 = 0. \tag{2.14}$$

3. General solution to the incremental pencil

The QIEP associated with the eigenpair (Υ, Z) for the incremental pencil $\Delta Q(\lambda)$ defined in (2.4) is equivalent to solving the following algebraic system,

$$\begin{cases} \Delta M X_2 \Lambda_2^2 + \Delta C X_2 \Lambda_2 + \Delta K X_2 = 0, \\ \Delta M X_3 \Lambda_3^2 + \Delta C X_3 \Lambda_3 + \Delta K X_3 = 0, \\ \Delta M^\top = \Delta M, \\ \Delta C^\top = \Delta C, \\ \Delta K^\top = \Delta K, \end{cases} \tag{3.1}$$

for matrices ΔM, ΔC and ΔK. Obviously, the original pencil (M_0, C_0, K_0) is already a particular solution. We want to characterize the solution in general. Denote

$$\Phi_{ij} := \mathbf{e}_i \mathbf{e}_j^\top + \mathbf{e}_j \mathbf{e}_i^\top, \qquad 1 \leqslant i, j \leqslant k, \tag{3.2}$$

where $\mathbf{e}_i$ is the standard ith unit vector. We first provide the following sufficient condition for solving (3.1).

Theorem 3.1. *Define*

$$\Delta M_{ij} := -M_0 X_1 \Phi_{ij} X_1^\top M_0, \tag{3.3}$$

$$\Delta C_{ij} := M_0 X_1 \Phi_{ij} \Lambda_1^{-\top} X_1^\top K_0 + K_0 X_1 \Lambda_1^{-1} \Phi_{ij} X_1^\top M_0, \tag{3.4}$$

$$\Delta K_{ij} := -K_0 X_1 \Lambda_1^{-1} \Phi_{ij} \Lambda_1^{-\top} X_1^\top K_0. \tag{3.5}$$

Then each triplet $(\Delta M_{ij}, \Delta C_{ij}, \Delta K_{ij})$, $1 \leqslant i \leqslant j \leqslant k$, is a solution to the system (3.1).

Proof. It is clear that ΔM_{ij}, ΔC_{ij} and ΔK_{ij} are all symmetric. By direct substitution and using (2.13), we see that

$$\begin{aligned} \Delta M_{ij} X_2 \Lambda_2^2 + \Delta K_{ij} X_2 &= \left(-M_0 X_1 \Phi_{ij} X_1^\top M_0\right) X_2 \Lambda_2^2 + \left(-K_0 X_1 \Lambda_1^{-1} \Phi_{ij} \Lambda_1^{-\top} X_1^\top K_0\right) X_2 \\ &= \left(-M_0 X_1 \Phi_{ij} \Lambda_1^{-\top} X_1^\top K_0\right) X_2 \Lambda_2 + \left(-K_0 X_1 \Lambda_1^{-1} \Phi_{ij} X_1^\top M_0\right) X_2 \Lambda_2 \\ &= -\Delta C_{ij} X_2 \Lambda_2. \end{aligned}$$

Similarly, using (2.14), we see that every equation in (3.1) is satisfied. □

By the homogeneity of (3.1), any linear combination of $(\Delta M_{ij}, \Delta C_{ij}, \Delta K_{ij})$, $1 \leqslant i \leqslant j \leqslant k$, is also a solution to (3.1). Note that if M_0 is nonsingular, then it cannot be expressed as a linear combination of ΔM_{ij} which is rank deficient. It follows that the triplet

$$(\Delta M, \Delta C, \Delta K) := \sum_{1 \leqslant i \leqslant j \leqslant k} \alpha_{ij} (\Delta M_{ij}, \Delta C_{ij}, \Delta K_{ij}) + \beta (M_0, C_0, K_0), \tag{3.6}$$

where $\alpha_{ij}, \beta \in \mathbb{R}$ are arbitrary constants, is also a solution to (3.1). We claim that for almost all given original models (M_0, C_0, K_0) any other solution to (3.1) is always of the form (3.6). In other words, the set

$$\{(M_0, C_0, K_0)\} \bigcup \{(\Delta M_{ij}, \Delta C_{ij}, \Delta K_{ij})\}_{1 \leqslant i \leqslant j \leqslant k} \tag{3.7}$$

forms a basis for the solution space of (3.1). Note that by construction the coefficients of these incremental pencils are symmetric low-rank matrices.

To see the necessity of (3.6), we break down the argument into several steps. We first single out the second equation in (3.1) as a stand-alone QIEP associated with eigenpairs $(\Lambda_3, X_3) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}$. Introducing a free 'parameter'

$$U := X_3^\top \Delta M X_3 \tag{3.8}$$

in terms of ΔM, we reformulate the QIEP as

$$U\Lambda_3^2 + \left(X_3^\top \Delta C X_3\right)\Lambda_3 + \left(X_3^\top \Delta K X_3\right) = 0 \tag{3.9}$$

for the coefficient matrices $(U, X_3^\top \Delta C X_3, X_3^\top \Delta K X_3)$. The following result which gives rise to a parametric representation of the solution to (3.9) has been proved in [17, theorem 2.1].

Theorem 3.2. *With U given in (3.8) as a parameter, the general solution to (3.9) is given by*

$$X_3^\top \Delta C X_3 = -\left(U\Lambda_3 + \Lambda_3^\top U + D\right), \tag{3.10}$$

$$X_3^\top \Delta K X_3 = \Lambda_3^\top U \Lambda_3 + \Lambda_3^\top D, \tag{3.11}$$

where D is another parameter of the form

$$D = diag\left\{\begin{bmatrix}\xi_1 & \eta_1\\ \eta_1 & -\xi_i\end{bmatrix}, \ldots, \begin{bmatrix}\xi_{\ell_3} & \eta_{\ell_3}\\ \eta_{\ell_3} & -\xi_{\ell_3}\end{bmatrix}, \xi_{2\ell_3+1}, \ldots, \xi_n\right\}, \tag{3.12}$$

with arbitrary constants $\xi_i, \eta_j \in \mathbb{R}$.

The free parameters U and D must be further restricted in order to satisfy the first equation in (3.1). Towards that end, denote

$$W := X_3^{-1}X_2,$$

and rewrite the first equation after substitution as

$$UW\Lambda_2^2 - \left(U\Lambda_3 + \Lambda_3^\top U + D\right)W\Lambda_2 + \left(\Lambda_3^\top U\Lambda_3 + \Lambda_3^\top D\right)W = 0. \tag{3.13}$$

It will prove to be convenient to rewrite the parameter U as

$$U = X_3^\top M_0 \widetilde{X}\Phi\widetilde{X}^\top M_0 X_3, \tag{3.14}$$

where the new parameter $\Phi \in \mathbb{R}^{n\times n}$ is symmetric. Such a change of variables is permissible because all three matrices X_3, M_0 and $\widetilde{X}$ are nonsingular. Observe that

$$\begin{aligned}
UW\Lambda_2 &= X_3^\top M_0\widetilde{X}\Phi\widetilde{X}^\top M_0 X_2\Lambda_2\\
&= X_3^\top M_0\widetilde{X}\Phi\widetilde{\Lambda}^{-\top}\left(\widetilde{X}^\top K_0 X_2 + \begin{bmatrix}0\\ S_2\Lambda_2\end{bmatrix}\right)\\
&= X_3^\top M_0\widetilde{X}\Phi(\widetilde{\Lambda}^{-\top}\widetilde{X}^\top K_0)X_2 + X_3^\top M_0\widetilde{X}\Phi\begin{bmatrix}0\\ \Lambda_2^{-\top}S_2\Lambda_2\end{bmatrix}\\
&= U\Lambda_3 W + X_3^\top M_0\widetilde{X}\Phi\begin{bmatrix}0\\ \Lambda_2^{-\top}S_2\Lambda_2\end{bmatrix}.
\end{aligned} \tag{3.15}$$

In the above, the second equality follows from (2.12) whereas the fourth equality follows from (2.11). Equation (3.13) therefore can be simplified to

$$\left(X_3^\top M_0\widetilde{X}\Phi\begin{bmatrix}0\\ \Lambda_2^{-\top}S_2\Lambda_2\end{bmatrix} - DW\right)\Lambda_2 = \Lambda_3^\top\left(X_3^\top M_0\widetilde{X}\Phi\begin{bmatrix}0\\ \Lambda_2^{-\top}S_2\Lambda_2\end{bmatrix} - DW\right). \tag{3.16}$$

Because Λ_2 and Λ_3 have distinct eigenvalues, it must be that

$$X_3^\top M_0\widetilde{X}\Phi\begin{bmatrix}0\\ \Lambda_2^{-\top}S_2\Lambda_2\end{bmatrix} - DW = 0. \tag{3.17}$$

Partition the parameter matrix Φ into blocks,

$$\Phi = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{12}^\top & \Phi_{22} \end{bmatrix},$$

with $\Phi_{11} \in \mathbb{R}^{k\times k}$. We now gain some insight into the structure of the parameter matrix Φ.

Theorem 3.3. *In order to satisfy the first equation in (3.1), the parameter U defined in (3.14) cannot be totally free. While Φ_{11} can be any symmetric matrix in $\mathbb{R}^{k\times k}$, the other part of Φ is completely determined by the parameter D through the relationship*

$$\begin{bmatrix} \Phi_{12} \\ \Phi_{22} \end{bmatrix} = \widetilde{X}^{-1} M_0^{-1} X_3^{-\top} D W \Lambda_2^{-1} S_2^{-1} \Lambda_2^\top. \tag{3.18}$$

We need to further restrict D so that the resulting $\Phi_{22} \in \mathbb{R}^{(n-k)\times(n-k)}$ is symmetric. For simplicity, let

$$P := \left(X_3^\top M_0 \widetilde{X}\right)^{-\top} \begin{bmatrix} 0 \\ I_{n-k} \end{bmatrix} = [\mathbf{p}_1, \ldots, \mathbf{p}_{n-k}] \in \mathbb{R}^{n\times(n-k)}, \tag{3.19}$$

$$Q := W \Lambda_2^{-1} S_2^{-1} \Lambda_2^\top = [\mathbf{q}_1, \ldots, \mathbf{q}_{n-k}] \in \mathbb{R}^{n\times(n-k)}. \tag{3.20}$$

Then for Φ_{22} to be symmetric, the parameter matrix D must satisfy the linear equation

$$P^\top D Q - Q^\top D P = 0_{n-k}. \tag{3.21}$$

Recall that D is of the diagonal form defined in (3.12). Introducing the operator $\delta : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ by

$$\delta(\mathbf{t}) := \operatorname{diag}\left\{ \begin{bmatrix} t_1 & t_2 \\ -t_2 & t_1 \end{bmatrix}, \ldots, \begin{bmatrix} t_{2\ell_3-1} & t_{2\ell_3} \\ -t_{2\ell_3} & t_{2\ell_3-1} \end{bmatrix}, t_{2\ell_3+1}, \ldots, t_n \right\},$$

if $\mathbf{t} = [t_1, \ldots, t_n]^\top \in \mathbb{R}^n$ and the sequence of truncated matrices

$$A_j := [\mathbf{a}_{j+1}, \ldots, \mathbf{a}_{n-k}], \qquad j = 1, \ldots n-k-1,$$

if $A = [\mathbf{a}_1, \ldots, \mathbf{a}_{n-k}] \in \mathbb{R}^{n\times(n-k)}$, we can rewrite the off-diagonal entries of system (3.21) in the equivalent form

$$B\mathbf{d} = 0, \tag{3.22}$$

where

$$B := \begin{bmatrix} P_1^\top \delta(\mathbf{q}_1) - Q_1^\top \delta(\mathbf{p}_1) \\ P_2^\top \delta(\mathbf{q}_2) - Q_2^\top \delta(\mathbf{p}_2) \\ \vdots \\ P_{n-k-1}^\top \delta(\mathbf{q}_{n-k-1}) - Q_{n-k-1}^\top \delta(\mathbf{p}_{n-k-1}) \end{bmatrix} \in \mathbb{R}^{\frac{(n-k)(n-k-1)}{2}\times n}, \tag{3.23}$$

$$\mathbf{d} := [\xi_1, \eta_1, \xi_2, \eta_2, \ldots, \xi_{\ell_3}, \eta_{\ell_3}, \xi_{2\ell_3+1}, \ldots, \xi_n]^\top. \tag{3.24}$$

Any solution to (3.22) will guarantee a symmetric parameter matrix Φ which, in turn, will lead to a solution to system (3.1). It remains to characterize the solution space to (3.22).

We make several observations.

Theorem 3.4. *The system (3.22) has a nontrivial solution. That is, its solution space has dimensionality at least one.*

Proof. If $\mathbf{d} = 0$ is the only solution, then by (3.18) and (3.14) we find that every solution $(\Delta M, \Delta C, \Delta K)$ to (3.1) must satisfy

$$\Delta M = M_0 X_1 \Phi_{11} X_1^\top M_0,$$

for some symmetric matrix Φ_{11} in $\mathbb{R}^{k\times k}$. On the other hand, we already know that (M_0, C_0, K_0) is also a solution to (3.1). This is an obvious contradiction because M_0 is of full rank where ΔM in the above form is of rank at most k. □

Indeed, we can be more specific about the dimension of the solution space. Let r denote the rank of the matrix B. If $k > n - \frac{1+\sqrt{8n-7}}{2}$, that is, if k is sufficiently large (and note that this is not of practical interest in MUP applications), then $n - r \geqslant 2$; otherwise, we will have $n - r \geqslant 1$. In the latter case, recall the fact that rank deficient matrices of any fixed size, say, $m \times n$, form a measure zero subset in its ambient space $\mathbb{R}^{m\times n}$. The coefficient matrix B is an algebraic function of the eigenvalues and eigenvectors of the original pencil (M_0, C_0, K_0) and is already rank deficient. The set of pencils that make the corresponding matrices B further rank deficient to $r < n - 1$ should have measure zero. For almost all original pencils (M_0, C_0, K_0), the matrix B is of rank $n - 1$. We believe our first main result concluded below, which precisely characterizes the solution to the QIEP for the incremental pencil, is new in the field.

Theorem 3.5. *If the coefficient matrix B defined in (3.23) is of rank $r = n - 1$ (this automatically implies that k must be sufficiently small), then the general solution to (3.1) is given by (3.6). That is, the solution space of (3.1) is spanned by the $1 + \frac{k(k+1)}{2}$ matrices in (3.7).*

As far as the MUP is concerned, we have just provided a basis for the possible incremental pencils which maintain no spillover to the original pencil (M_0, C_0, K_0). It is critically important to note that the general solution (3.6) does *not* require any knowledge of the remaining $2n - k$ eigeninformation (Λ_2, X_2) and (Λ_3, X_3) at all.

4. Solvability of the MUP

Suppose now that new eigenpairs $(\Sigma, Y) \in \mathbb{R}^{k\times k} \times \mathbb{R}^{n\times k}$ are measured and we desire to update the original model $Q_0(\lambda) = \lambda^2 M_0 + \lambda C_0 + K_0$ by replacing (Λ_1, X_1) by (Σ, Y) while maintaining the remaining eigenpairs (Υ, Z) invariant. We assume that $X_1 \in \mathbb{R}^{k\times k}$ is of full rank. The goal of this section is to characterize the condition on (Σ, Y) under which the MUP is solvable.

For convenience, let $\mathfrak{S}(A)$ denote the spectrum of the matrix A. We first observe that if $Q_0(\lambda)\mathbf{x} = \lambda^2 M_0\mathbf{x} + \lambda C_0\mathbf{x} + K_0\mathbf{x} = 0$ for some $\lambda \neq 0$, then for any scalar τ we can write

$$\begin{aligned} Q_0(\tau)\mathbf{x} = \tau^2 M_0\mathbf{x} + \tau C_0\mathbf{x} + K_0\mathbf{x} &= \tau^2 M_0\mathbf{x} - \tau\left(\lambda M_0\mathbf{x} + \frac{1}{\lambda}K_0\mathbf{x}\right) + K_0\mathbf{x} \\ &= (\tau - \lambda)\left(\tau M_0\mathbf{x} - \frac{1}{\lambda}K_0\mathbf{x}\right). \end{aligned}$$

It follows that if $\tau \notin \mathfrak{S}(\Lambda)$, then

$$Q_0(\tau)^{-1}\left(\tau M_0\mathbf{x} - \frac{1}{\lambda}K_0\mathbf{x}\right) = \frac{1}{\tau - \lambda}\mathbf{x}. \tag{4.1}$$

Recall that in order to solve the MUP, both equations (2.2) and (2.3) must be satisfied simultaneously. In the preceding section, we have already seen that generically the general solution to (2.2) is given by (3.6), provided $k < n - \frac{1+\sqrt{8n-7}}{2}$. In particular, the triplet $(\Delta\hat{M}, \Delta\hat{C}, \Delta\hat{k})$ given by

$$\begin{cases} \Delta\hat{M} := -M_0 X_1 \Phi_{11} X_1^\top M_0, \\ \Delta\hat{C} := M_0 X_1 \Phi_{11} \Lambda_1^{-\top} X_1^\top K_0 + K_0 X_1 \Lambda_1^{-1} \Phi_{11} X_1^\top M_0, \\ \Delta\hat{K} := -K_0 X_1 \Lambda_1^{-1} \Phi_{11} \Lambda_1^{-\top} X_1^\top K_0, \end{cases} \tag{4.2}$$

with an arbitrary $\Phi_{11} \in \mathbb{R}^{k\times k}$ solves (2.2). With this in mind, we now derive the necessary condition for solving (2.3).

Theorem 4.1. *Assume that in the newly measured eigenpairs $(\Sigma, Y) \in \mathbb{R}^{k\times k} \times \mathbb{R}^{n\times k}$ the matrix Σ has exactly the same block diagonal structure as that of Λ_1. If*

$$(M_0 + \Delta\hat{M}) Y \Sigma^2 + (C_0 + \Delta\hat{C}) Y \Sigma + (K_0 + \Delta\hat{K}) Y = 0, \tag{4.3}$$

then there exists a matrix $T \in \mathbb{R}^{k\times k}$ such that

$$Y = X_1 T. \tag{4.4}$$

If Y is of full rank, then T is invertible. In this case, for the MUP to be solvable, it is necessary that $\mathrm{Range}(Y) = \mathrm{Range}(X_1)$.

Proof. Let

$$\Omega := \mathrm{diag}\left\{ \underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ \mathrm{i} & -\mathrm{i} \end{bmatrix}, \ldots, \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ \mathrm{i} & -\mathrm{i} \end{bmatrix}}_{\ell_1 \text{ copies}}, \underbrace{1, \ldots, 1}_{k-\ell_1 \text{ copies}} \right\}.$$

It is easy to verify that Ω is a unitary matrix and has the effect of transforming real block diagonal form to complex diagonal form. Write

$$\begin{aligned} \hat{\Lambda}_1 &:= \Omega^H \Lambda_1 \Omega = \mathrm{diag}\{\lambda_1, \ldots, \lambda_k\}, \\ \hat{\Sigma} &:= \Omega^H \Sigma \Omega = \mathrm{diag}\{\sigma_1, \ldots, \sigma_k\} \\ \hat{Y} &:= Y\Omega = [\hat{\mathbf{y}}_1, \ldots, \hat{\mathbf{y}}_k] \equiv [\boldsymbol{\eta}_1, \bar{\boldsymbol{\eta}}_1, \ldots, \boldsymbol{\eta}_\nu, \bar{\boldsymbol{\eta}}_\nu, \boldsymbol{\eta}_{\nu+1}, \ldots, \boldsymbol{\eta}_k] \\ \hat{X}_1 &:= X_1 \Omega = [\hat{\mathbf{x}}_1, \ldots, \hat{\mathbf{x}}_k], \\ \hat{\Phi}_{11} &:= \Omega^H \Phi_{11} \Omega. \end{aligned}$$

It follows from (4.3) that

$$\begin{aligned} Q_0(\sigma_j)\hat{\mathbf{y}}_j &= -\left(\sigma_j^2 \Delta\hat{M}_0 + \sigma_j \Delta\hat{C}_0 + \Delta\hat{K}_0\right)\hat{\mathbf{y}}_j \\ &= \left(\sigma_j^2 M_0 \hat{X}_1 \hat{\Phi}_{11} \hat{X}_1^H M_0 - \sigma_j M_0 \hat{X}_1 \hat{\Phi}_{11} \hat{\Lambda}_1^{-H} \hat{X}_1^H K_0 \right. \\ &\quad \left. - \sigma_j K_0 \hat{X}_1 \hat{\Lambda}_1^{-1} \hat{\Phi}_{11} \hat{X}_1^H M_0 + K_0 \hat{X}_1 \hat{\Lambda}_1^{-1} \hat{\Phi}_{11} \hat{\Lambda}_1^{-H} \hat{X}_1^H K_0\right)\hat{\mathbf{y}}_j \\ &= \left(\sigma_j M_0 \hat{X}_1 - K_0 \hat{X}_1 \hat{\Lambda}_1^{-1}\right) \hat{\Phi}_{11} \left(\sigma_j \hat{X}_1^H M_0 - \hat{\Lambda}_1^{-H} \hat{X}_1^H K_0\right) \hat{\mathbf{y}}_j \\ &= \left[\left(\sigma_j M_0 - \frac{1}{\lambda_1} K_0\right)\hat{\mathbf{x}}_1, \ldots, \left(\sigma_j M_0 - \frac{1}{\lambda_k} K_0\right)\hat{\mathbf{x}}_k\right] \hat{\Phi}_{11}\left(\sigma_j \hat{X}_1^H M_0 - \hat{\Lambda}_1^{-H} \hat{X}_1^H K_0\right)\hat{\mathbf{y}}_j, \end{aligned}$$

for $j = 1, \ldots, k$. Applying (4.1), we obtain

$$\hat{\mathbf{y}}_j = \hat{X}_1 \,\mathrm{diag}\left\{\frac{1}{\sigma_j - \lambda_1}, \ldots, \frac{1}{\sigma_j - \lambda_k}\right\} \hat{\Phi}_{11}\left(\sigma_j \hat{X}_1^H M - \hat{\Lambda}_1^H \hat{X}_1^H K\right)\hat{\mathbf{y}}_j$$

for $j = 1, \ldots, k$. Upon substitution, it follows that

$$\begin{aligned} Y &= \hat{Y}\Omega^H \\ &= X_1 \underbrace{\Omega \operatorname{diag}\left\{\frac{1}{\sigma_j - \lambda_1}, \ldots, \frac{1}{\sigma_j - \lambda_k}\right\} \Omega^H \Phi_{11} \left(\Sigma X_1^\top M_0 - \Lambda_1^\top X_1^\top K_0\right) Y}_{T} . \end{aligned}$$

Note that $\Omega \operatorname{diag}\{\frac{1}{\sigma_j - \lambda_1}, \ldots, \frac{1}{\sigma_j - \lambda_k}\}\Omega^H$ is in $\mathbb{R}^{k\times k}$, so T is real valued. □

Theorem 4.1 is important because it points out that, in order to perform the updating with no spillover, the newly observed eigenvectors Y cannot be too arbitrary. The vectors of Y must reside in the range space of the original eigenvectors X_1. If this constraint is not satisfied, then the model cannot be updated.

Suppose now that $Y = X_1 T$ for some nonsingular matrix $T \in \mathbb{R}^{k\times k}$. It is interesting to ask under what conditions a symmetric matrix $\Phi_{11} \in \mathbb{R}^{k\times k}$ can be determined so that equality (4.3) holds. Towards this end, we denote

$$\Theta := T\Sigma T^{-1} \tag{4.5}$$

and make two additional assumptions which generally are true:

A5. Assume that $\mathfrak{S}(\Sigma)$ and $\mathfrak{S}\big((X_1^\top M_0 X_1, -X_1^\top K_0 X_1 \Lambda_1^{-1})\big)$ are disjoint;
A6. Assume that $0 \notin \mathfrak{S}\big(X_1^\top M_0 X_1 \Theta - \Lambda_1^{-\top} X_1^\top K_0 X_1\big)$.

From the fact that

$$C_0 X_1 = -M_0 X_1 \Lambda_1 - K_0 X_1 \Lambda_1^{-1},$$

and the assumption $Y = X_1 T$, we can rewrite (4.3) as

$$\begin{aligned} &M_0 X_1 \left(\underbrace{\left[\Theta - \Lambda_1 - \Phi_{11}\big(X_1^\top M_0 X_1 \Theta - \Lambda_1^{-\top} X_1^\top K_0 X_1\big)\right]}_{V} T\Omega \right) \hat{\Sigma} - K_0 X_1 \Lambda_1^{-1} \\ &\qquad \times \left(\underbrace{\left[\Theta - \Lambda_1 - \Phi_{11}\big(X_1^\top M_0 X_1 \Theta - \Lambda_1^{-\top} X_1^\top K_0 X_1\big)\right]}_{V} T\Omega \right) = 0. \end{aligned} \tag{4.6}$$

Assumption A5 implies that

$$V := \Theta - \Lambda_1 - \Phi_{11}\big(X_1^\top M_0 X_1 \Theta - \Lambda_1^{-\top} X_1^\top K_0 X_1\big) = 0, \tag{4.7}$$

because, otherwise, $VT\Omega \neq 0$ and there would exist a nonzero column vector, say, $\mathbf{v}_j$, of the matrix $VT\Omega$ and a scalar $\sigma_j \in \mathfrak{S}(\Sigma)$ such that

$$\sigma_j M_0 X_1 \mathbf{v}_j - K_0 X_1 \Lambda_1^{-1} \mathbf{v}_j = 0,$$

which would imply that σ_j is an eigenvalue of the linear pencil $\big(X_1^\top M_0 X_1, -X_1^\top K_0 X_1 \Lambda_1^{-1}\big)$ and would contradict assumption A5. It follows from assumption A6 that Φ_{11} is given by

$$\Phi_{11} = (\Theta - \Lambda_1)\big(X_1^\top M_0 X_1 \Theta - \Lambda_1^{-\top} X_1^\top K_0 X_1\big)^{-1}. \tag{4.8}$$

Obviously, not all nonsingular matrices $T \in \mathbb{R}^{k\times k}$ are feasible. The resulting matrix Φ_{11} defined in (4.8) must be symmetric. With this in mind, we have finally consummated our second main result which completely characterizes when the MUP is solvable.

Theorem 4.2. *Given newly measured eigenpairs* (Σ, Y)*, assume that* $Y = X_1 T$ *for some nonsingular* $T \in \mathbb{R}^{k\times k}$ *and that the two assumptions A5 and A6 hold. Define* Θ *as in (4.5). Then the MUP is solvable if and only if the matrix* T *(which ties* Σ *to* Θ*) is such that*

$$\left(X_1^\top M_0 X_1 - X_1^\top K_0 X_1 \Lambda_1^{-1}\right)(\Theta - \Lambda_1) = \left(\Theta^\top - \Lambda_1^\top\right)\left(X_1^\top M_0 X_1 - \Lambda_1^{-\top} X_1^\top K_0 X_1\right). \tag{4.9}$$

In this case, the matrix Φ_{11} *is given by (4.8) which defines the incremental pencil (4.2) for the update.*

Observe from (2.7) that

$$X_1^\top K_0 X_1 = \Lambda_1^\top X_1^\top M_0 X_1 \Lambda_1 - S_1 \Lambda_1.$$

Upon substitution into (4.8), we see that

$$\Phi_{11} = (\Theta - \Lambda_1)\left(X_1^\top M_0 X_1 (\Theta - \Lambda_1) + \Lambda_1^{-\top} S_1 \Lambda_1\right)^{-1} = \left(X_1^\top M_0 X_1 + S_1 (\Theta - \Lambda_1)^{-1}\right)^{-1}.$$

It is worth mentioning that if the matrix T is S_1-symplectic, that is, if T satisfies the relationship $T^\top S_1 T = S_1$ where S_1 is defined in (2.8), then Φ_{11} is automatically symmetric for arbitrary Σ, so long as the newly measured eigenvalue information Σ has exactly the same block structure as Λ_1 and the difference $\Sigma - \Lambda_1$ is invertible. To see this point, note that

$$\Sigma^\top S_1 = S_1 \Sigma.$$

Together with the S_1-symplecticity of T, it can easily be established that

$$T^\top \left(\Theta^\top - \Lambda_1^\top\right) S_1 T = T^\top S_1 (\Theta - \Lambda_1) T,$$

showing that Φ_{11} is symmetric. The special case when $T = I$ is of particular interest, that is, when the eigenvectors in Y are kept the same as those in X_1, the quadratic model can be updated with arbitrary eigenvalues so long as values in Σ are kept in the same block structure as that in the original Λ_1.

5. Conclusion

Model updating with no spillover has been a longstanding open problem. Many efforts have been made, both theoretically and computationally, in response to the demand of its many critical applications. Thus far, the results are limited and hardly satisfactory. One of the most fundamental challenges is to characterize when this model updating problem with no spillover is solvable.

This paper provides a complete theory on when such an updating with no spillover is possible. In particular, we think two contributions made in this paper are worthy of attention. First, we describe a formula for the basis of the solution space of the quadratic inverse eigenvalue problem associated with the incremental pencil. An important characteristic in our construction for this general solution is that it does not involve knowledge of the remaining $2n - k$ eigenstructure at all, nicely fitting in the situation where no such knowledge is available in practice. Second, we develop a necessary and sufficient condition on the newly measured eigenpair (Σ, Y) that gives an account of whether the corresponding model updating problem is solvable. A distinguishing feature in our condition of solvability is its simplicity—roughly speaking, the newly measured eigenvectors Y need to be in the range space of the original eigenvectors X_1.

Because the model updating with no spillover has important applications in many areas of discipline, we think that our results in this paper fully addressing the issue of solvability should be of interest to the community. On the other hand, we want to point out that in practice the coefficient matrices M, C and K are often structured or parametrized. Can such a structured model be updated with no spillover? In addition to matching or maintaining the spectral data, can the adjustments be made with minimal norm or maximal robustness? Can the physical feasibility of the updated M, C and K be maintained? These are some open questions that might be worthy of further study.

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