On the Differential Equation $\frac{dX}{dt} = [X, k(X)]$ where k Is a Toeplitz Annihilator

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Abstract

The differential equation $\frac{dX}{dt} = [X, k(X)]$ where k is a Toeplitz annihilator has been suggested as a means to solve the inverse Toeplitz eigenvalue problem. Starting with the diagonal matrix whose entries are the same as the given eigenvalues, the solution flow has been observed numerically to always converge a symmetric Toeplitz matrix as $t \longrightarrow \infty$. This paper is an attempt to understand the dynamics involved in the case $n = 3$.

1. Introduction.

Consider the differential system

$$\frac{dX}{dt} = [X, k(X)] \tag{1}$$

where $X = X(t) \in R^{n\times n}$, $t \in R$, k stands for a certain matrix-valued operator such that the action $k(X)$ on X is skew-symmetric, and

$$[A, B] := AB - BA \tag{2}$$

denotes the Lie bracket. It is easy to see that the subspace $\mathcal{S}(n)$ of all symmetric matrices in $R^{n\times n}$ is invariant under (1). Associated with (1) is the initial value problem:

$$\frac{dQ}{dt} = Qk(X) \tag{3}$$

$$Q(0) = I. \tag{4}$$

Since k is skew-symmetric and $Q(0) = I$, the solution $Q(t)$ stays in the group $\mathcal{O}(n)$ of orthogonal matrices. It is important to observe that $X(t)$ and $Q(t)$ are related by

$$X(t) = Q(t)^T X(0) Q(t). \tag{5}$$

For this reason, we may rewrite (3) so that the flow $Q(t)$ is determined independently of $X(t)$, except $X(0)$. Another important consequence of (5) is that the solution $X(t)$ is orthogonally similar to the initial value $X(0)$ and, hence, $X(t)$ is defined and (Frobenius) norm-preserved for all $t \in R$.

With appropriately formulated k, the differential system (1) has found a variety of applications in the field of linear algebra. Tactics on how to construct k for solving eigenvalue value problems, singular value problems, spectrally constrained least squares approximation problems, inverse eigenvalue problems, nearest normal matrix problem, quadratic programming problems, and simultaneous reduction problems can be found in [4]. In certain cases there exist discrete counterparts which happen to be well-known numerical algorithms and, hence, the differential equation approach enable us to better understand the dynamical behavior of these algorithms. But what is more significant is when the underlying linear algebra problem seems impossible to be tackled by any conventional discrete method, the differential equation approach can generate a path that leads to a solution. Inverse Toeplitz eigenvalue problem (ITEP), where a symmetric Toeplitz matrix is to be constructed from a prescribed set of eigenvalues, is such an example of applications.

There are two questions involved in the (ITEP). The first is the existence question of whether symmetric Toeplitz matrices can have arbitrary real eigenvalues. To this date, the problem remains open when $n \geq 5$ [6]. The second is the computation question of numerically constructing the Toeplitz matrix. To this purpose, very few algorithms are available and mostly converge only locally. See, for example, [11].

Recently, we have proposed to solve the (ITEP) by using (1) where k is an *annihilator* of the subspace $\mathcal{T}$ of all symmetric Toeplitz matrices. That is, we want k to be such that

$$k(X) = 0 \text{ if and only if } X \in \mathcal{T}. \tag{6}$$

In view of the dimensions of the three subspaces involved, the construction of a linear map $k : \mathcal{S}(n) \longrightarrow \mathcal{S}(n)^\perp$ with property (6) is possible. In fact, there are many ways to define such a linear map k with property (6). See [4]. Perhaps the simpliest way is by defining

$$k_{ij} := \begin{cases} x_{i+1,j} - x_{i,j-1}, & \text{if } 1 \leq i < j \leq n \\ 0, & \text{if } 1 \leq i = j \leq n \\ x_{i-1,j} - x_{i,j+1}, & \text{if } 1 \leq j < i \leq n \end{cases} \tag{7}$$

where k_{ij} denotes the (i,j)-component of $k(X)$.

Our idea is motivated by the following intuitive thinking: The flow of (1) starting with $X(0) = \Lambda := \text{diag}\{\lambda_1, \ldots, \lambda_n\}$ stays on the isospectral manifold

$$\mathcal{M}(\Lambda) := \{Q^T \Lambda Q | Q \in \mathcal{O}(n)\}. \tag{8}$$

Generically, the prescribed eigenvalues $\lambda_1, \ldots, \lambda_n$ should be distinct. In this case, $[X, k(X)] = 0$ if and only if $k(X)$ is a polynomial of X [9] and, therefore, $k(X) \in \mathcal{S}(n) \cap \mathcal{S}(n)^\perp = \{0\}$. If condition (6) holds, then all equilibria of the flow $X(t)$ are necessarily in $\mathcal{T}$. Since $\|X(t)\| = \|\Lambda\|$ for all $t \in R$, $X(t)$ must have a non-empty invariant ω-limit set [3]. From this, we raise the hope that $X(t)$ might converge to a single point $\hat{X}$ as $t \longrightarrow \infty$ [1, Theorem2.3,page23] and, in this way, the (ITEP) is solved.

Needless to say, the argument outlined above is incomplete in many aspects. For example, nothing seems to be wrong when the subspace $\mathcal{T}$ in (6) is replaced by an arbitrary n-dimensional subspace $\mathcal{U}$ of $\mathcal{S}(n)$ and then the equation (1) is applied to solve the inverse eigenvalue problem (IEP) on $\mathcal{U}$. However, we can easily construct a subspace $\mathcal{U}$ on which the (IEP) is not solvable. Thus the solvability of the (ITEP) by using (1) must somehow be reflected in the special structure of $\mathcal{T}$.

On the other hand, we have experimented the differential system (1), with $X(0) = \Lambda$ and with k defined by (7), extensively on various sets of spectral data for the (ITEP). We have observed that the orbit *always* converges to an equilibrium point. Thus, we conjecture that the (ITEP) is always solvable. What remains to be proved, however, is that the solution of (1) does has a simple asymptotic behavior as $t \longrightarrow \infty$. In this way, the existence question of the (ITEP) could have been settled. Unfortunately, the analysis of dynamical behavior is just as difficult as the (ITEP) itself. At the present time we still do not have the answer. Despite of the theoretical difficulty, we suggest that following the solution flow of (1) is a feasible numerical method for solving the (ITEP).

The dynamical system involved in (1) with k defined by (7) is of theoretical interest in its own right even for small n — The vector field is described by a system of homogeneous polynomials of degree 2 and its solution is norm preserving. We recall that our situation is quite similar to a much smaller system discussed in [2, Example 4, page193],

$$
\begin{aligned}
\frac{dy_1}{dt} &= y_2y_3 + y_5y_4 - 2\,y_2y_5, \\
\frac{dy_2}{dt} &= y_3y_4 + y_1y_5 - 2y_3y_1, \\
\frac{dy_3}{dt} &= y_5y_4 + y_2y_1 - 2y_2y_4, \\
\frac{dy_4}{dt} &= y_1y_5 + y_2y_3 - 2y_5y_3, \\
\frac{dy_5}{dt} &= y_2y_1 + y_3y_4 - 2y_1y_4,
\end{aligned}
\tag{9}
$$

only that the behavior of (9) was conjectured to be random. It is curious to ask what can happen to our system.

For $n = 2$, the differential system (1) with k defined by (7) can be written as

$$
\frac{dX}{dt} = \left[\begin{array}{cc} 2x_{12}(x_{11} - x_{22}) & -(x_{11} - x_{22})^2 \\ -(x_{11} - x_{22})^2 & -2x_{12}(x_{11} - x_{22}) \end{array} \right]. \tag{10}
$$

As the trace remains constant for all t, we may assume, without loss of generality, that $\lambda_1 + \lambda_2 \equiv 0$. We then see trivially from (10) that x_{12} decreases monotonically to $-|\lambda_1|$ and x_{11} converges to 0. For $n \geq 3$, the analysis becomes much more complicated. The primary goal of this paper is to discuss some of the properties we have observed for the case when $n = 3$.

This paper is organized as follows: We first give an interpretation of our differential system in terms of rigid body motion. Then we classify the stability of all critical points. We study several possible invariant sets from which we discover the existence of periodic solutions. We illustrate the orbital unstability of the period solutions. Finally, we summarize our conjectures. Many of our observations can be generalized to higher dimensional space. It is hoped that the exposition in this paper would stir up further discussion that would ultimately settle the existence question for the (ITEP).

2. Rigid Body Motion.

From now on, we shall restrict our attention to the case when $n = 3$.

The dual system (3) has an interesting physical interpretation in terms of rigid body motion. Let $k(t)$ be written as

$$k(t) = \begin{bmatrix} 0 & \omega_3(t) & -\omega_2(t) \\ -\omega_3(t) & 0 & \omega_1(t) \\ \omega_2(t) & -\omega_1(t) & 0 \end{bmatrix} \tag{11}$$

and define

$$\omega(t) := [\omega_1(t), \omega_2(t), \omega_3(t)]^T. \tag{12}$$

Also let the transpose of $Q(t)$ be written in columns

$$Q(t)^T = [p_1(t), p_2(t), p_3(t)]. \tag{13}$$

Then the dual system (3) implies, for $i = 1, 2, 3$,

$$\begin{aligned} \frac{dp_i}{dt} &= \omega \times p_i \\ p_i(0) &= e_i \end{aligned} \tag{14}$$

where e_i is the i-th standard unit basis vector in R^3 and $\times$ denotes the usual cross-product in R^3. The differential equation in (14) describes the linear velocity of the vector $p_i(t)$ and the vector $\omega(t)$ may be interpreted as the angular velocity of the motion. As a whole, the dual system then describes the rotation of an orthogonal coordinate system about the origin. From (7), it is trivial to see that the differential equation in (1) is invariant under the translation $X + \sigma I$. Shifting $\Lambda + \sigma I$ by large enough $\sigma \in R$ if necessary, we may assume the initial matrix Λ is positive definite. In this case, $X(t)$ stays to be positive definite and may be interpreted as the *moment of inertia tensor* of a rigid body motion [10]. The total kinetic energy of motion is then given by [10]

$$T(t) := \frac{1}{2}\omega(t)^T X(t) \omega(t). \tag{15}$$

If the kinetic energy is dissipated to zero as $t \longrightarrow \infty$ the motion will come to a stop, which is the case that we have observed.

3. Critical Points.

We now take a closer look at the differential system (1) with k defined by (7).

Because of symmetry, only the upper triangular part of a matrix needs to be considered. For simplicity, we assume henceforth that

$$\lambda_1 + \lambda_2 + \lambda_3 = 0. \tag{16}$$

Since the trace is identically zero for all t, we eliminate one more dimension by setting

$$x_{22} = -x_{11} - x_{33}. \tag{17}$$

With entries arranged in the order $(x_{11}, x_{12}, x_{13}, x_{23}, x_{33})$, the differential system involved is equivalent to the following system in R^5:

$$\begin{aligned}
\frac{dx_{11}}{dt} &= 4x_{12}x_{11} + 2x_{12}x_{33} - 2x_{13}x_{23} + 2x_{13}x_{12}, \\
\frac{dx_{12}}{dt} &= -4x_{11}^2 - 4x_{11}x_{33} - 2x_{13}x_{33} - x_{13}x_{11} - x_{33}^2 - x_{23}^2 + x_{23}x_{12}, \\
\frac{dx_{13}}{dt} &= 3x_{11}x_{23} + 3x_{12}x_{33}, \\
\frac{dx_{23}}{dt} &= x_{23}x_{12} - x_{12}^2 - 4x_{11}x_{33} - x_{11}^2 - 4x_{33}^2 - 2x_{13}x_{11} - x_{13}x_{33}, \\
\frac{dx_{33}}{dt} &= 2x_{13}x_{23} - 2x_{13}x_{12} + 4x_{23}x_{33} + 2x_{11}x_{23}.
\end{aligned} \tag{18}$$

The right-hand side of (18) is still a system of homogeneous polynomials of degree 2. The scalar multiplication of any non-trivial solution of a homogeneous system is also a solution. We, therefore, do not expect any *isolated* equilibrium for the differential system (18). In fact,

LEMMA 3.1. *The differential system (18) has only two kinds of real equilibria:*

$$x_{11} = c_1,\ x_{12} = 0,\ x_{13} = -3c_1,\ x_{23} = 0,\ x_{33} = c_1 \tag{19}$$

or

$$x_{11} = 0,\ x_{12} = c_2,\ x_{13} = c_3,\ x_{23} = c_2,\ x_{33} = 0 \tag{20}$$

where c_i, $i = 1, 2, 3$ are arbitrary real constants.

Proof. The result follows from a tedious but straightforward computation by using the theory of Gröbner bases [12]. □

The local stability of these critical points is easy to comprehend. The coefficient matrix of a linearized system at $(c_1, 0, -3c_1, 0, c_1)$ has eigenvalues:

$$0,\ \pm 3\sqrt{6}c_1,\ \pm 6\sqrt{2}|c_1|i. \tag{21}$$

while that at $(0, c_2, c_3, c_2, 0)$ has eigenvalues:

$$0,\ 0,\ 6c_2,\ 2(c_2 + c_3),\ 2(c_2 - c_3). \tag{22}$$

Thus, except the trivial case where $c_1 = 0$, the first kind of equilibria can *never* be stable. Near such an equilibrium, (21) also indicates that there are periodic solutions. We shall investigate this situation later. The second kind of equilibria can be stable only if

$$c_2 < 0 \text{ and } |c_2| \geq c_3. \tag{23}$$

It is interesting to note that an equilibrium of the second kind corresponds to a Toeplitz matrix

$$\begin{bmatrix} 0 & c_2 & c_3 \\ c_2 & 0 & c_2 \\ c_3 & c_2 & 0 \end{bmatrix}$$

whose eigenvalues are:

$$\frac{c_3 - \sqrt{c_3^2 + 8c_2^2}}{2}, \ -c_3, \ \frac{c_3 + \sqrt{c_3^2 + 8c_2^2}}{2}$$

in ascending order whereas the equality in (23) corresponds to the case of multiple eigenvalues. The two zero eigenvalues in (22) simply indicate that the the entire Toeplitz matrices (with trace 0) form a 2-dimensional manifold of equilibria.

4. Invariant Sets.

We now consider invariant sets of the differential system (1) with k defined by (7). We first observe that

LEMMA 4.1. *If $\mathcal{I}$ is an invariant set for the differential equation (1), then so is the set $\alpha\mathcal{I} := \{\alpha X | X \in \mathcal{I}\}$ for arbitrary real constant α.*

Proof. If $X(t)$ satisfies the differential equation then, due to the homogeneity of the vector field, so does $Y(t) := \frac{1}{\alpha}X(\frac{t}{\alpha})$ for any real constant α. □

We have already found that

1. For each given Λ, we have already seen that the isospectral surface $\mathcal{M}(\Lambda)$ is an invariant set.
2. The manifolds of equilibria (19) and (20) are invariant.

A matrix is said to be (*skew-*) *persymmetric* if it is (skew-) symmetric about its northeast-southwest diagonal. We now claim that

LEMMA 4.2. *The subspace $\mathcal{W}$ of all symmetric and persymmetric matrices is invariant under the differential equation (1).*

Proof. It is easy to see that if X is symmetric and persymmetric, then $k(X)$ with k defined by (7) is skew-symmetric and skew-persymmetric. It follows that $[X, k(X)]$ stays symmetric and persymmetric. □

In our setting, because of the additional condition (16), we think of $\mathcal{W}$ as a 3-dimensional subspace:

$$\mathcal{W} = \{(x_{11}, x_{12}, x_{13}, x_{12}, x_{11}) | x_{11}, x_{12}, x_{13} \in R\}.$$

Note that the subspace $\mathcal{T}$ subject to condition (16) is embedded in $\mathcal{W}$ with $x_{11} = 0$.

A direct computation shows that, for any given $\{\lambda_1, \lambda_2, \lambda_3\}$ satisfying (16), the intersection of $\mathcal{W}$ and $\mathcal{M}(\Lambda)$ consists of three "ellipses":

$$\begin{aligned}
(x_{11} - \frac{\lambda_3}{4})^2 + \frac{1}{2}x_{12}^2 &= \tfrac{(2\lambda_1+\lambda_3)^2}{16}, \quad x_{13} = x_{11} - \lambda_3; \\
(x_{11} - \frac{\lambda_1}{4})^2 + \frac{1}{2}x_{12}^2 &= \tfrac{(\lambda_1+2\lambda_3)^2}{16}, \quad x_{13} = x_{11} - \lambda_1; \qquad (24) \\
(x_{11} + \frac{\lambda_1+\lambda_3}{4})^2 + \frac{1}{2}x_{12}^2 &= \tfrac{(\lambda_1-\lambda_3)^2}{16}, \quad x_{13} = x_{11} + \lambda_1 + \lambda_3.
\end{aligned}$$

It is interesting to note that the projections of these ellipses onto the (x_{11}, x_{12})-plane must be such that one circumscribes the other two. Typical cases are shown in Figure 1. By counting the x_{12}-intercepts of these ellipses, it is readily proved that

LEMMA 4.3. *For $n = 3$ the (ITEP) has exactly four real solutions if all given eigenvalues are distinct, and two real solutions if one eigenvalue has multiplicity 2.*

The dynamics of (18) restricted to $\mathcal{W}$, on the other hand, is governed by:

$$\begin{aligned}
\frac{dx_{11}}{dt} &= 6x_{11}x_{12}, \\
\frac{dx_{12}}{dt} &= -9x_{11}^2 - 3x_{11}x_{13}, \qquad (25) \\
\frac{dx_{13}}{dt} &= 6x_{11}x_{12}.
\end{aligned}$$

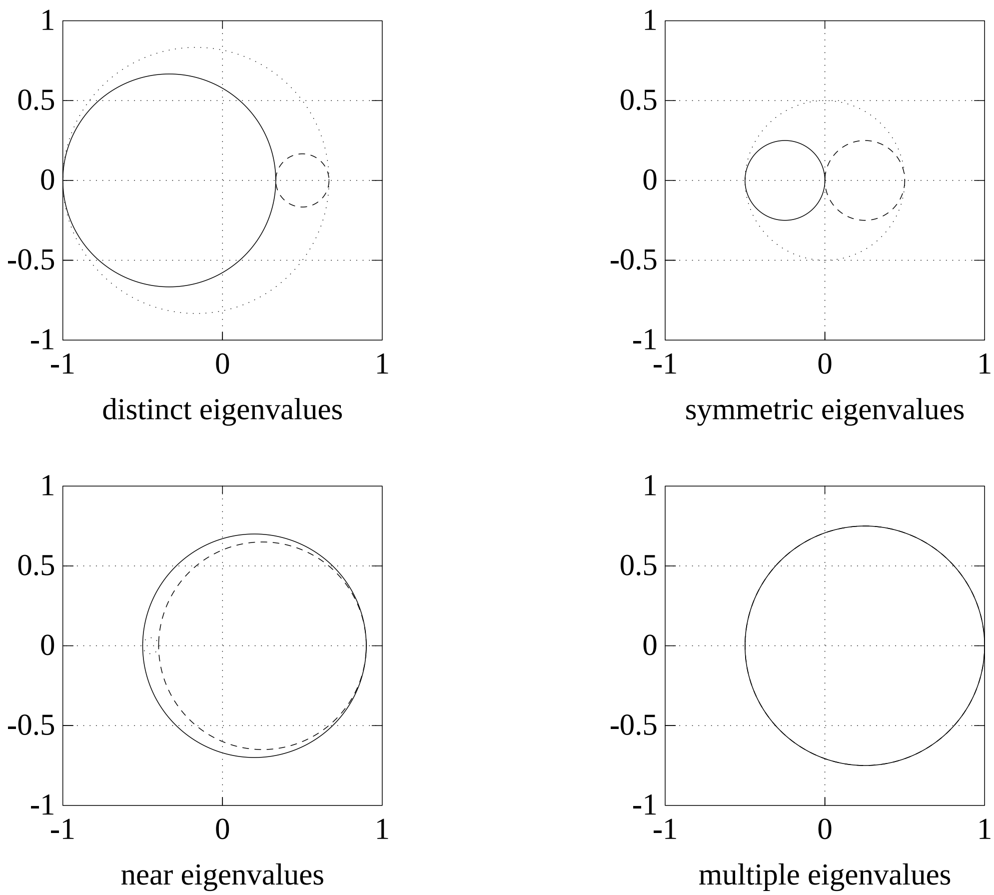

FIG. 1. *Plots of x_{11} versus x_{12} where $\mathcal{W}$ intersects $\mathcal{M}(\Lambda)$*

Thus it is clear that $\mathcal{W}$ itself consists of layers of 2-dimensional invariant affine subspaces. Each affine subspace is determined by

$$x_{13} = x_{11} + c_4 \tag{26}$$

for a certain real constant c_4. For any given c_4, the integral curves on the invariant affine subspace are determined by

$$(x_{11} + \frac{c_4}{4})^2 + \frac{1}{2}x_{12}^2 = c_5^2 \tag{27}$$

for real constants c_5. These elliptic orbits are concentric with the center $(-\frac{c_4}{4}, 0, \frac{3c_4}{4})$ which is an equilibrium of the first kind. It is also clear that there are periodic solutions near that equilibrium. On the other hand, for large enough c_5^2 (i.e., $|c_5| > |\frac{c_4}{4}|$), a non-periodic solution of (25) will converge as shown in Figure 2. The limit point corresponds

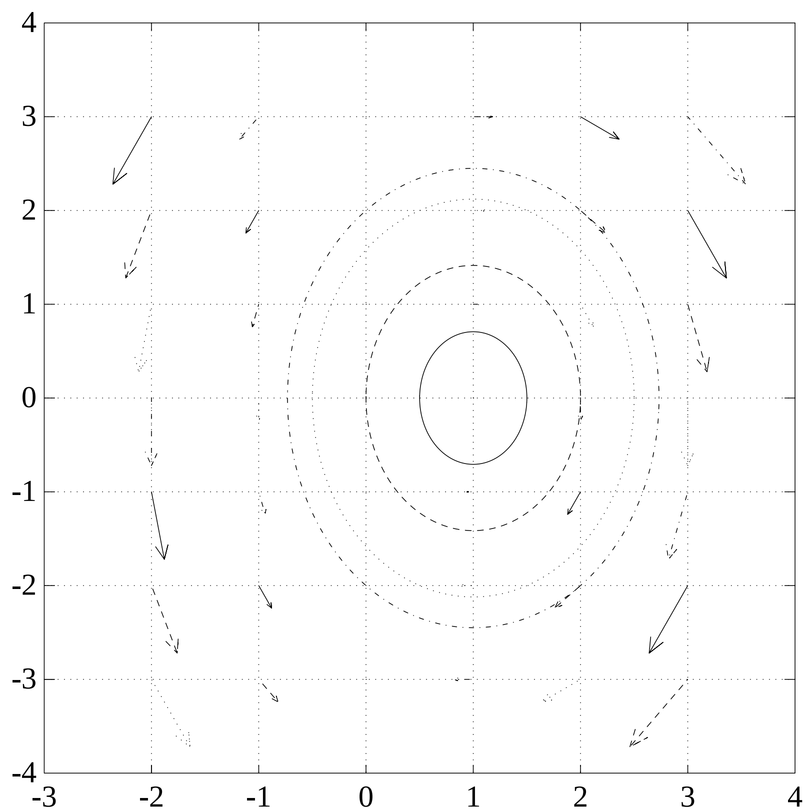

FIG. 2. *Flows in the persymmetric subspace (with x_{13} suppressed).*

to an equilibrium of the second kind $(0, -\sqrt{2c_5^2 - \frac{c_4^2}{8}}, c_4, -\sqrt{2c_5^2 - \frac{c_4^2}{8}}, 0)$ which, according to the condition (23), will be stable with respect to the system (18) only if

$$c_5^2 \geq \frac{9c_4^2}{16} \tag{28}$$

is satisfied.

Finally we conclude that

THEOREM 4.4. *For any given $\{\lambda_1, \lambda_2, \lambda_3\}$ satisfying (16), the surface $\mathcal{M}(\Lambda)$ can have one and only one equilibrium which is stable for the differential system (18).*

Proof. The proof is simply by checking the x_{12}−intercepts of the ellipses (24) against the condition (28). □

5. Orbital Stability.

The existence of periodic solutions discovered in the preceding sections is disappointing. If any of these closed orbits were proved to be the ω-limit set of other solution flows, then our plan of setting up (7) to solve the (ITEP) would have failed.

Figure 3 is a computer plot of $x_{12}(t)$ versus $x_{11}(t)$ for a single trajectory of (18) for $0 \leq t \leq 11.05$. The initial conditions, $x_{11}(0) = 1.0, x_{12}(0) = 1.0, x_{13} = -3.0, x_{14} = 1.0, x_{15} = 1.0$, indicate that the true trajectory should be an ellipse.

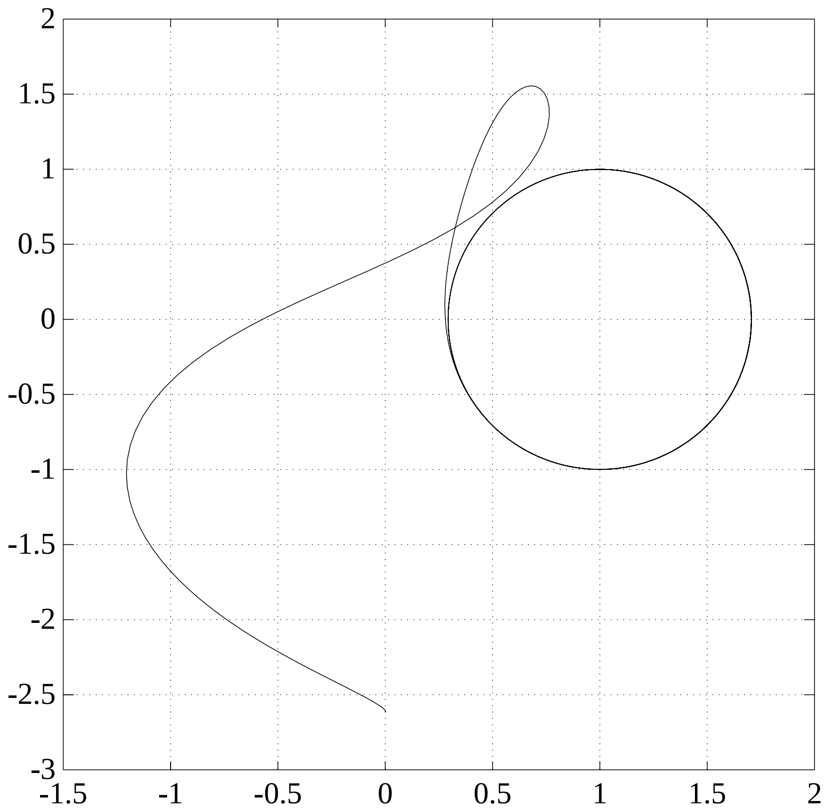

FIG. 3. *Plot of $x_{11}(t)$ versus $x_{12}(t)$ for a single trajectory*

Despite the high accuracy (10^{-14}) we have demanded from the package ODE [13], Figure 3 clearly shows that the numerical solution even fails to stay close to the ellipse. Once the solution is off the track even by a small amount, the errors quickly accumulate to the extent that the solution leaves the ellipse entirely. This is a strong indication that the periodic solution is not orbitally stable [5, Page323].

It is worth noting that although the numerical solution fails to track down the elliptic orbit, it stays close to the surface $\mathcal{M}(\Lambda)$. Figure 4 plots the difference of eigenvalues (measured in the 2-norm) between $X(0)$ and $X(t)$. The error is certainly acceptable within machine roundoff. It is amazing to see that although the numerical solution is meaningless to the original initial value problem, the final *false* limit point does solve the (ITEP).

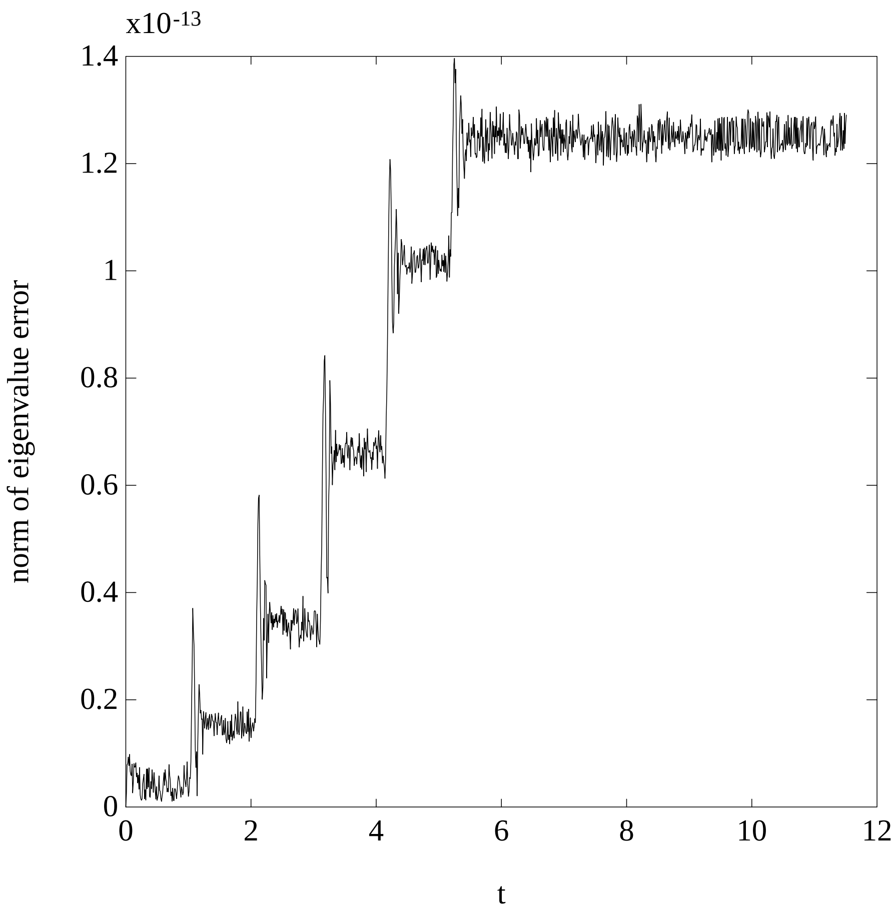

FIG. 4. *Errors of eigenvalues between $X(0)$ and $X(t)$*

The orbital unstability can be confirmed by calculating the *characteristic exponents* of the linearized system of (18) at a periodic solution. The theory can be found, for example, in [5, Theorem 2.2, Chapter 13]. For the above example, we first estimate by numerical experiments that the period is about $\omega = 1.04719755120$ (accurate up to the 11-th digit). Then we calculate a fundamental matrix $\Phi(t)$ satisfying $\Phi(0) = I$ for the linearized system of (18) at the periodic solution. Since $\Phi(\omega) = e^{\omega R}$ for a certain constant matrix R [5, Theorem5.1, Chapter 3], the corresponding characteristic exponents are estimated to be:

$$\pm 7.8998, \ \pm 2.0222 \times 10^{-5}, \ 4.5297 \times 10^{-14}$$

The last small number should be regarded as machine zero. It is the first positive characteristic exponent 7.8998064636 that clearly confirms the orbital unstability.

6. Other Observations.

There are some other interesting facts about the differential system (1) with k defined by (7.

We have observed a special property of symmetry as stated below:

LEMMA 6.1. *The diagonals of the solution $X(t)$ with initial value $X(0) = \Lambda$ alternate with symmetry of evenness and oddness. More precisely, all the entries on the main diagonal of $X(t)$ are even functions, and those on the first super-diagonal are odd functions, and so on.*

Proof. Replace entries of a matrix by symbols e or o, respectively, depending upon if they are even or odd functions. If

$$X = \begin{bmatrix} e & o & e & \dots & & & \\ o & e & o & \dots & & & \\ e & o & e & \dots & & & \\ \vdots & & & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & & \vdots \\ & & & & & e & o \\ & & & & & o & e \end{bmatrix}, \tag{29}$$

then by the definition of k it is easy to check that

$$[X, k(X)] = \begin{bmatrix} o & e & o & \dots & & & \\ e & o & e & \dots & & & \\ o & e & o & \dots & & & \\ \vdots & & & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & & \vdots \\ & & & & & o & e \\ & & & & & e & o \end{bmatrix}.$$

Note that Λ obviously has the symmetric structure (29). It follows that the Picard's iteration [5],

$$X_{m+1}(t) := \Lambda + \int_0^t [X_m(s), k(X_m(s))]\, ds \tag{30}$$

with $X_0(t) \equiv \Lambda$ will have the same symmetric structure. The well-known Picard-Lindelöf theorem guarantees that $\{X_m(t)\}$ converges uniformly to the unique solution $X(t)$. The assertion follows. $\square$

Let $\mathcal{W}^\perp$ denote the orthogonal complement of $\mathcal{W}$ in the space $\mathcal{S}(n)$ with respect to the Frobenius norm. That is, $\mathcal{W}^\perp$ is the subspace of all symmetric and skew-persymmetric matrices. Identifying $X \in \mathcal{S}(n)$ as the direct sum

$$\begin{aligned} X(t) &= \begin{bmatrix} p_1 & p_2 & p_3 \\ p_2 & -2p_1 & p_2 \\ p_3 & p_2 & p_1 \end{bmatrix} + \begin{bmatrix} s_1 & s_2 & 0 \\ s_2 & 0 & -s_2 \\ 0 & -s_2 & -s_1 \end{bmatrix} \\ &:= P(t) \oplus S(t) \end{aligned} \tag{31}$$

with $P \in \mathcal{W}$ and $S \in \mathcal{W}^{\perp}$, we may then split the vector field of (7) as the direct sum of

$$\frac{dP}{dt} = [P, k(P)] + [S, k(S)] \tag{32}$$

$$\frac{dS}{dt} = [P, k(S)] + [S, k(P)]. \tag{33}$$

Equation (33) clearly reproves Lemma 4.2 when $S = 0$.

For $n = 3$ case, the solution flow must satisfy the equation

$$6p_1^2 + 4p_2^2 + 2p_3^2 + 2s_1^2 + 4s_2^2 = ||X(0)||^2 \tag{34}$$

because $||X(t)|| = ||X(0)||$. We have observed that generally $||S(t)||$ is not a monotone function because

$$< \frac{dS}{dt}, S >= -12s_1s_2p_1 + 12s_1p_3s_2 + 4p_2s_1^2 + 8p_2s_2^2. \tag{35}$$

changes signs. When $X(0) = \Lambda$, however, numerical experimentation seems to indicate that $< \frac{dS}{dt}, S > \leq 0$ for $t > 0$. If this is true, then we have the global convergence.

7. Conclusion.

The matrix differential equation (1) offers a new avenue of attacking the inverse Toeplitz eigenvalue problem. Numerical experimentation suggests that this differential equation approach is globally convergent and that the (ITEP) is always solvable. We have studied the dynamics for the case $n = 3$ with the discovery of several interesting facts.

From a mathematical point of view, the differential system (1) is of theoretical interest in its own right. On the other hand, while very few methods are available for solving the (ITEP) numerically, we suggest integrating (1) with $X(0) = \Lambda$ by any available ODE solver as a feasible numerical approach.

REFERENCES

[1] B. Aulbach, Continuous and Discrete Dynamics near Manifolds of Equilibria, Lecture Notes in Math., vol. 1058, Springer=Verlag, Berlin, 1980.

[2] C. M. Bender and S. A. Orszag, Advanced Mathematical Methods for Scientists and Engineers, McGraw-Hill, New York, 1978.

[3] F. Brauer and J. A. Nohel, Qualitative Theory of Ordinary Differential Equations, Benjamin, New York, 1969.

[4] M. T. Chu, Matrix differential equations: A continuous realization process for linear algebra problems, preprint, 1990.

[5] E. A. Coddington and N. Levinson, Theory of Ordinary Differential Equations, McGraw-Hill, New York, 1955.

[6] P. Delsarte and Y. Genin, Spectral properties of finite Toeplitz matrices, Proceedings of the 1983 International Symposium of Mathematical Theory of Networks and Systems, Beer-Sheva, Israel, 194-213.

[7] K. R. Driessel and M. T. Chu, Can real symmetric Toeplitz matrices have arbitrary real spectra?, SIAM J. Matrix Anal. Appl., under revision.

[8] S. Friedland, J. Nocedal and M. L. Overton, The formulation and analysis of numerical methods for inverse eigenvalue problems, SIAM J. Numer. Anal., 24(1987), 634-667.

[9] F. R. Gantmacher, Matrix Theory, Vol. 1 and 2, Chelsea, New York, 1959.

[10] H. Goldstein, Classical Mechanics, Addison-Wesley, Massachusetts, 1965.

[11] D. P. Laurie, A numerical approach to the inverse Toeplitz eigenproblem SIAM J. Sci. Stat. Comput., 9(1988), 401-405.

[12] F. Pauer and M. Pfeifhofer, The theory of Gröbner bases, L'Enseignement Mathématique, t. 34(1988), 215-232.

[13] Shampine L. F. and Gordon, M. K., Computer Solution of Ordinary Differential Equations: The Initial Value Problems, Freeman, San Francisco, 1975.