Constructing Symmetric Nonnegative Matrices with Prescribed Eigenvalues by Differential Equations

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Abstract

We propose solving the inverse eigenvalue problem for symmetric nonnegative matrices by means of a differential equation. If the given spectrum is feasible, then a symmetric nonnegative matrix can be constructed simply by following the solution curve of the differential system. The choice of the vector field is based on the idea of minimizing the distance between the cone of symmetric nonnegative matrices and the isospectral surface determined by the given spectrum. We explicitly describe the projected gradient of the objective function. Using center manifold theory, we also show that the ω-limit set of any solution curve is a single point. Some numerical examples are presented.

1. Introduction.

A matrix $A \in R^{n \times n}$ is said to be nonnegative if no entry of A is negative. Nonnegative matrices arise frequently in various applied areas [3]. The Perron-Frobenius theorem concerning the spectrum of nonnegative matrices may be regarded as the central result in the theory of nonnegative matrices. It is, therefore, of great interest to study the following inverse eigenvalue problem:

> **(Problem 1)** Given a set $\sigma := \{\lambda_1, \ldots, \lambda_n\} \subset C$, find necessary and sufficient conditions for σ to be the spectrum of some nonnegative matrix.

In practice, the prescribed eigenvalues $\lambda_1, \ldots, \lambda_n$ often are real numbers. So it is also interesting to ask:

> **(Problem 2)** Given a set $\sigma = \{\lambda_1, \ldots, \lambda_n\} \subset R$, find necessary and sufficient conditions for σ to be the spectrum of some symmetric nonnegative matrix.

For decades researchers have been trying to answer these problems. A few necessary and a few sufficient conditions can be found, for example, in [2, 10, 11, 12, 13, 14, 16, 18], or more recently in [4]. To our knowledge, however, neither Problem 1 nor Problem 2 has been completely solved.

In this paper we want to address the problem of constructing a nonnegative matrix with prescribed spectrum. The problem is stated as follows:

> **Problem 3** Given a set σ of n real values that is known a priori to be the spectrum of some nonnegative matrix, numerically construct a symmetric nonnegative matrix whose spectrum is exactly σ.

We have not found much discussion of Problem 3 in the literature. The most constructive result we have seen is the sufficient condition studied by Solues [18]. But Soules' condition is still limited because his construction depends on the specification of the Perron eigenvector — in particular, the components of the Perron eigenvector need to satisfy certain inequalities in order for his construction to work.

For our consideration, we shall need the following notation. Let $\mathcal{O}(n)$ denote the set of all orthogonal matrices in $R^{n \times n}$. Let Λ denote the diagonal matrix with diagonal entries $\lambda_1, \ldots, \lambda_n$; in symbols,

$$\Lambda := diag\{\lambda_1, \ldots, \lambda_n\}. \tag{1}$$

The set

$$\mathcal{M}(\Lambda) := \{Q^T \Lambda Q | Q \in \mathcal{O}(n)\} \tag{2}$$

will be called the isospectral surface corresponding to Λ. Although the assumption is not required in our discussion, it can be shown that $\mathcal{M}(\Lambda)$ is indeed a smooth manifold with dimension $\frac{n(n-1)}{2}$ if all λ_i are distinct. The set of all symmetric nonnegative matrices in $R^{n \times n}$ is denoted by $\pi_s(R^n_+)$.

We note that Problem 2 is equivalent to the following:

> **Problem 2'** Find necessary and sufficient condition conditions for the intersection of the isospectral surface $\mathcal{M}(\Lambda)$ and the cone $\pi_s(R^n_+)$ to be non-empty.

Thus we are motivated to explore the idea of developing a way to systematically reduce the distance between $\pi_s(R^n_+)$ and $\mathcal{M}(\Lambda)$. If $\mathcal{M}(\Lambda)$ does intersect $\pi_s(R^n_+)$, then of course the distance is zero. Otherwise, our approach still finds a matrix from $\mathcal{M}(\Lambda)$ and a matrix from $\pi_s(R^n_+)$ such that their distance is a local minimum. In the latter case, the matrix from $\pi_s(R^n_+)$ is expected to be on a face of the cone $\pi_s(R^n_+)$, i.e., some of the entries of the nonnegative matrix are zero. We shall see that this property indeed shows up naturally in the development of our theory. Another fact, obvious from the geometry, is also worth mentioning — if $\mathcal{M}(\Lambda)$ intersects $\pi_s(R^n_+)$ at an interior point, then $\mathcal{M}(\Lambda)$ intersects $\pi_s(R^n_+)$ in a relative neighborhood of that point. In this case there are infinitely many symmetric nonnegative matrices corresponding to the given spectrum.

We can precisely formulate our idea as a constrained optimization problem. We first note that the set $\pi(R^n_+)$ of all nonnegative matrices in $R^{n\times n}$ can be formed as

$$\pi(R^n_+) = \{B * B | B \in R^{n\times n}\} \tag{3}$$

where $X * Y$ denotes the Hadamard product of matrices X and Y. Let $\mathcal{S}(n)$ denote the set of all symmetric matrices in $R^{n\times n}$. Let

$$< A, B >:= trace(AB^T) = \sum_{i,j} a_{ij}b_{ij} \tag{4}$$

denote the Frobenius inner product of two matrices $A, B \in R^{n\times n}$. We shall consider the following minimization problem:

Problem 4 Minimize

$$F(Q, R) := \frac{1}{2}\|Q^T\Lambda Q - R * R\|^2, \tag{5}$$

subject to

$$(Q, R) \in \mathcal{O}(n) \times \mathcal{S}(n).$$

where $\|\cdot\|$ represents the Frobenius matrix norm. We shall show that the projection of the gradient vector of the objective function F onto the manifold $\mathcal{O}(n) \times \mathcal{S}(n)$ can be calculated explicitly. Consequently, we can introduce a steepest descent vector field on $\mathcal{O}(n) \times \mathcal{S}(n)$. This vector field can easily be transformed into a "flow" on the isospectral surface $\mathcal{M}(\Lambda)$ and a "flow" in the cone $\pi_s(R^n_+)$. Both flows are moving in the steepest descent direction to minimize their distance until an equilibrium is reached. Our approach to Problem 4, therefore, is a continuous realization process.

In our earlier works, we have applied similar ideas to tackle the inverse Toeplitz eigenvalue problems [9] and other least squares matrix approximation problems subject to spectral constraint [7]. Our approach there proves to be quite successful. In this paper, we shall use some of our previously developed ideas. In section 2 we develop the differential system; this is our main result. In section 3 we use center manifold theory to study the stability properties of the resulting differential system. We argue that generically the ω-limit set of a solution flow contains only a single point. This proves the global convergence of our method. In section 4, we study in detail the stability of equilibria for the case $n = 2$. Although this represents the simplest case, the stability analysis should shed some light on the behavior of our flow for higher dimension cases. We present some numerical examples in the last section.

2. Projected Gradient.

In the product space $R^{n\times n} \times R^{n\times n}$, we shall use the induced Frobenius inner product:

$$< (A_1, A_2), (B_1, B_2) > := < A_1, B_1 > + < A_2, B_2 > . \tag{6}$$

With this topology, the feasible set $\mathcal{O}(n) \times \mathcal{S}(n)$ of Problem 4 is clearly a smooth manifold. It is not difficult to show [7] that the space tangent to $\mathcal{O}(n) \times \mathcal{S}(n)$ at a point $(Q, R) \in \mathcal{O}(n) \times \mathcal{S}(n)$ is given by

$$\mathcal{T}_{(Q,R)}\mathcal{O}(n) \times \mathcal{S}(n) = \mathcal{T}_Q\mathcal{O}(n) \times \mathcal{T}_R\mathcal{S}(n) = Q\mathcal{S}(n)^\perp \times \mathcal{S}(n) \tag{7}$$

where $\mathcal{S}(n)^\perp$ denotes the orthogonal complement of $\mathcal{S}(n)$ and is composed of all skew-symmetric matrices in $R^{n\times n}$.

We first extend the definition of the function F in (5) in an obvious way to the entire space $R^{n\times n} \times R^{n\times n}$. A straightforward calculation shows that the Fréchet derivative of F at a general point $(A, B) \in R^{n\times n} \times R^{n\times n}$ acting on $(H, K) \in R^{n\times n} \times R^{n\times n}$ is:

$$\begin{aligned} F'(A, B)(H, K) &= < A^T\Lambda A - B * B, H^T\Lambda A + A^T\Lambda H - K * B - B * K > \\ &= < \Lambda A[(A^T\Lambda A - B * B)^T + (A^T\Lambda A - B * B)], H > \\ &\quad + < -2(A^T\Lambda A - B * B) * B, K > . \end{aligned} \tag{8}$$

The adjoint property $< A, BC > = < AC^T, B > = < B^TA, C >$ has been used to rearrange terms in (8). It follows that, with respect to the inner product (6), the gradient of F at (A, B) is a pair of matrices; in fact, we have

$$\begin{aligned} \nabla F(A, B) &= \Big(\Lambda A[(A^T\Lambda A - B * B)^T + (A^T\Lambda A - B * B)], \\ &\qquad -2(A^T\Lambda A - B * B) * B\Big) . \end{aligned} \tag{9}$$

We are interested only in the case when $(A, B) = (Q, R) \in \mathcal{O}(n) \times \mathcal{S}(n)$. In this case, (9) is simplified to:

$$\nabla F(Q, R) = (2\Lambda Q(Q^T\Lambda Q - R * R), -2(Q^T\Lambda Q - R * R) * R). \tag{10}$$

We now calculate the projection of $\nabla F(Q, R)$ on the manifold $\mathcal{O}(n) \times \mathcal{S}(n)$. Because we are using a product topology, the projection of $\nabla F(Q, R)$ on $\mathcal{O}(n) \times \mathcal{S}(n)$ is the direct product of the projections of the two components of $\nabla F(Q, R)$ on $\mathcal{O}(n)$ and $\mathcal{S}(n)$, respectively. Each of these projections can be calculated easily. In [7] we presented a simple way to do the projection on $\mathcal{O}(n)$: Since

$$R^{n\times n} = \mathcal{T}_Q\mathcal{O}(n) \oplus \mathcal{N}_Q\mathcal{O}(n) = Q\mathcal{S}(n)^\perp \oplus Q\mathcal{S}(n), \tag{11}$$

any matrix $A \in R^{n\times n}$ has a unique orthogonal splitting

$$A = Q\{\frac{1}{2}(Q^TA - A^TQ)\} + Q\{\frac{1}{2}(Q^TA + A^TQ)\} \tag{12}$$

as the sum of elements from $\mathcal{T}_Q\mathcal{O}(n)$ and $\mathcal{N}_Q\mathcal{O}(n)$. In particular, the projection of $2\Lambda Q(Q^T\Lambda Q - R * R)$ onto $\mathcal{T}_Q\mathcal{O}(n)$ is:

$$\begin{aligned} &\frac{1}{2}Q\left\{Q^T[2\Lambda Q(Q^T\Lambda Q - R * R)] - [2\Lambda Q(Q^T\Lambda Q - R * R)]^TQ\right\} \\ &= Q\left\{-Q^T\Lambda Q(R * R) + (R * R)Q^T\Lambda Q\right\} . \end{aligned} \tag{13}$$

On the other hand, $\mathcal{S}(n)$ is a vector space already, so the projection of $-2(Q^T\Lambda Q - R*R)*R$ onto $\mathcal{S}(n)$ is just itself. Thus we have found that the projection $g(Q,R)$ of $\nabla F(Q,R)$ onto the manifold $\mathcal{O}(n)\times\mathcal{S}(n)$ is given by the pair of matrices:

$$(14) g(Q,R) = \left(Q\left\{-Q^T\Lambda Q(R*R)+(R*R)Q^T\Lambda Q\right\}, -2(Q^T\Lambda Q - R*R)*R\right).$$

The differential equation

$$\frac{d(Q,R)}{dt} = -g(Q,R), \tag{15}$$

therefore, defines a "steepest" descent vector field on $\mathcal{O}(n)\times\mathcal{S}(n)$ for the objective function $F(Q,R)$.

We now transport the flow (15) to the surface $\mathcal{M}(\Lambda)$ and the cone $\pi_s(R^n_+)$. For $Q(t)\in\mathcal{O}(n)$ and $R(t)\in\mathcal{S}(n)$, let

$$X(t) := Q(t)^T\Lambda Q(t), \tag{16}$$
$$Y(t) := R(t)*R(t). \tag{17}$$

Upon differentiating $X(t)$ and $Y(t)$ with respect to the variable t and using (15), we find that $X(t)$ and $Y(t)$ are governed by the differential system

$$\frac{dX}{dt} = [X,[X,Y]], \tag{18}$$
$$\frac{dY}{dt} = 4Y*(X-Y). \tag{19}$$

In (18) we have used the Lie bracket notation $[A,B] := AB - BA$. Together with an initial value $(X(0),Y(0))\in\mathcal{M}(\Lambda)\times\pi_s(R^n_+)$, we have reformulated Problem 4 as an initial value problem for (X,Y). The initial value problem is readily solved by available software.

The vector field on the right-hand sides of (18) and (19) is well-defined for every $(X,Y)\in R^{n\times n}\times R^{n\times n}$. However, it is important to note that we intend to start the flow from an initial value $(X(0),Y(0))$ in $\mathcal{M}(\Lambda)\times\pi_s(R^n_+)$. Then $X(t)\in\mathcal{M}(\Lambda)$ and $Y(t)\in\pi_s(R^n_+)$ throughout the interval of existence. By the way these flows are constructed, we know both $X(t)$ and $Y(t)$ are bounded and, hence, exist for $t\in[0,\infty)$. In fact, if we define

$$G(t) := F(X(t),Y(t)) = \frac{1}{2}\|X(t)-Y(t)\|^2 \geq 0, \tag{20}$$

then it is easy to calculate that

$$\frac{dG}{dt} = -<[X,Y],[X,Y]> -4<(X-Y),Y*(X-Y)>\leq 0. \tag{21}$$

According to Lyapunov's second method [5, Theorem 5.5], the limit points of $(X(t),Y(t))\in\mathcal{M}(\Lambda)\times\pi_s(R^n_+)$ must satisfy the equation $\frac{dG}{dt}=0$. That is, $(\hat{X},\hat{Y})$ will be a limit point only if

$$[\hat{X},\hat{Y}] = 0, \tag{22}$$

and

$$\hat{Y} * (\hat{X} - \hat{Y}) = 0. \tag{23}$$

It is crucial to note from (18) and (19) that the conditions (22) and (23) are also sufficient for that $(\hat{X}, \hat{Y})$ be an equilibrium point for the system. (In fact, if all eigenvalues in σ are distinct, then it can be shown that conditions (22) and (23) are also necessary.) Let

$$\mathcal{L} := \{(X, Y) \in \mathcal{M}(\Lambda) \times \pi_s(R_+^n) | [X, Y] = 0, Y * (X - Y) = 0\}. \tag{24}$$

We conclude that if we start with any $(X(0), Y(0)) \in \mathcal{M}(\Lambda) \times \pi_s(R_+^n)$, then the solution flow $(X(t), Y(t))$ approaches the set $\mathcal{L}$ as $t \longrightarrow \infty$ [5, Lemma 5.4]. That is, for every $\epsilon > 0$, there exists a $T > 0$ such that for every $t > T$ there exist a point $(\hat{X}, \hat{Y}) \in \mathcal{L}$ (possibly depending on t) such that $\|(X(t), Y(t)) - (\hat{X}, \hat{Y})\| < \epsilon$.

The above convergence result is not entirely satisfactory. For example, the flow $(X(t), Y(t))$ might oscillate around a nontrivial limit set. It will be interesting and important if we can show that the ω-limit set of any orbit $(X(t), Y(t))$ contains only a singleton. In the next section we shall use center manifold theory to prove that if the ω-limit set of an orbit $(X(t), Y(t))$ contains a point of the type $(\hat{X}, \hat{X})$, then $(X(t), Y(t))$ indeed converges to $(\hat{X}, \hat{X})$.

For computation, we obviously may choose $X(0) = \Lambda$. We note (using (19)) that if one component of $Y(t)$ is zero, then that component remains zero. For a feasible set of "generic" values, therefore, we should begin the flow $Y(t)$ with an interior point (i.e., a positive matrix) of the cone $\pi_s(R_+^n)$. Other than this restriction, the choice of $Y(0)$ is arbitrary. Different initial values of $Y(0)$ may lead to different limit points. We shall see some numerical examples in the last section.

Finally, we remark that a limit point $\hat{Y}$ of a flow $Y(t)$ could lie in one of the faces of $\pi_s(R_+^n)$ even if the flow starts from the interior of $\pi_s(R_+^n)$. We expect this situation when $\mathcal{M}(\Lambda) \cap \pi_s(R_+^n) = \emptyset$, i.e., when the given spectrum σ is not associated with any element of $\pi_s(R_+^n)$. But the most interesting case occurs when no component of the limit point $\hat{Y}$ is zero. Then, by (23), the symmetric nonnegative matrix $\hat{Y}$ must be the same as the isospectral matrix $\hat{X}$. In this case, we have numerically constructed a symmetric nonnegative matrix that has a prescribed spectrum.

3. Convergence.

We have pointed out earlier that the ω-limit set of any orbit $(X(t), Y(t))$ is nonempty and invariant and that the orbit approaches its ω-limit set. In this section we shall take a closer look at the convergence behavior of the solution flow $(X(t), Y(t))$. We first use center manifold theory [6] to study the behavior of $(X(t), Y(t))$ near an equilibrium point. We then argue that the ω-limit set of any orbit $(X(t), Y(t))$ contains only a singleton.

Let $(\hat{X}, \hat{Y})$ be an equilibrium point of the system (18) and (19). If $\hat{X} \neq \hat{Y}$ or if $(\hat{X}, \hat{Y})$ does not belong to $\mathcal{M}(\Lambda) \times \pi_s(R^n_+)$, then we have not yet solved the inverse eigenvalue problem. We shall consider only the opposite case, namely $\hat{X} = \hat{Y} \in \pi_s(R^n_+)$.

Our first approach is similar to the work done in [7]. For convenience, we first briefly review center manifold theory: Consider the system

$$\frac{dx}{dt} = Ax + f(x, y) \tag{25}$$

$$\frac{dy}{dt} = By + g(x, y) \tag{26}$$

where $x \in R^n$, $y \in R^m$, and A, B are constant matrices such that all eigenvalues of A have zero real parts while all those of B have negative real parts, the functions f and g are C^2 with $f(0,0) = 0, f'(0,0) = 0, g(0,0) = 0$ and $g'(0,0) = 0$. Then there exists an invariant manifold, called the center manifold, for the system (25) and (26). The center manifold is characterized by a C^2 function h from R^n to R^m with the property

$$y = h(x), h(0) = 0, h'(0) = 0. \tag{27}$$

Furthermore, the stability of $(0,0) \in R^n \times R^m$ for the system (25) and (26) is equivalent to the stability of $0 \in R^n$ for the system

$$\frac{dz}{dt} = Az + f(z, h(z)). \tag{28}$$

In addition, if $0 \in R^n$ is stable for (28), then with $(x(0), y(0))$ sufficiently small, there exists a solution $z(t)$ of (28) such that as $t \longrightarrow \infty$,

$$x(t) = z(t) + O(e^{-\mu t}) \tag{29}$$

$$y(t) = h(z(t)) + O(e^{-\mu t}) \tag{30}$$

for some constant $\mu > 0$.

We now apply these results to the equations (18) and (19). Near an equilibrium point $(\hat{X}, \hat{X})$, we define

$$U(t) := X(t) - \hat{X}, \tag{31}$$

$$W(t) := X(t) - Y(t). \tag{32}$$

It is easy to see that (18) and (19) are equivalent to the following equations:

$$\begin{aligned} \frac{dU}{dt} &= [\hat{X}, [W, \hat{X}]] \\ &\quad + [U, [W, \hat{X}]] + [\hat{X}, [W, U]] + [U, [W, U]] \end{aligned} \tag{33}$$

$$\begin{aligned} \frac{dW}{dt} &= [\hat{X}, [W, \hat{X}]] - 4\hat{X} * W \\ &\quad + [U, [W, \hat{X}]] + [\hat{X}, [W, U]] + [U, [W, U]] - 4W * (U - W). \end{aligned} \tag{34}$$

Readers should distinguish between the linear and the nonlinear terms in each of the above expressions. We note that equation (33) is not quite in the same form as (25) since the linear term in (33) is in the variable W. But this discrepancy can easily be fixed through a simple linear transformation. Additionally, we are more interested in knowing whether $W(t)$ converges to zero than what $X(t)$ converges to. Thus we shall not be bothered to perform the transformation explicitly.

Since all underlying matrices are symmetric, it suffices to consider only the upper triangular parts of the matrices. Let Ω be the $\frac{n(n+1)}{2} \times \frac{n(n+1)}{2}$ matrix representing the upper triangular part of the linear operator $[\hat{X}, [W, \hat{X}]] - 4\hat{X} * W$. Applying center manifold theory, we first study the behavior of a solution flow near an equilibrium point.

LEMMA 3.1. *Suppose that all eigenvalues of Ω at an equilibrium point $(\hat{X}, \hat{X})$ have negative real part. Then, starting with any matrix $(X(0), Y(0))$ sufficiently close to $(\hat{X}, \hat{X})$, the solution flow $(X(t), Y(t))$ of (18) and (19) converges to a constant matrix of the form $(\hat{Z}, \hat{Z})$. (Note that $\hat{Z}$ may not be the same as $\hat{X}$).*

(pf): If all eigenvalues of Ω have negative real part, then obviously (See [6, Theorem 3, page5])

$$W \equiv h(U) \equiv 0 \tag{35}$$

is a center manifold for the system (33) and (34). It follows that the corresponding system (28) on the center manifold has constant solution. From (29) and (30), we conclude that $U(t)$ converges to a constant matrix while $W(t)$ converges to the zero matrix as $t \longrightarrow \infty$. We note that center manifold theory does not provide any information regarding which limit point $U(t)$ (and hence $X(t)$) is converging to, although it does guarantee that $X(t)$ and $Y(t)$ are converging to the same point.

The critical supposition that all eigenvalues of Ω have negative real part is difficult to justify in general. Even for the special case $n = 2$ to be discussed in the next section, the explicit expressions for eigenvalues of Ω are very complicated. Nonetheless, we have observed the following fact concerning this supposition:

LEMMA 3.2. *At any equilibrium point $(\hat{X}, \hat{X}) \in \mathcal{M}(\Lambda) \times \pi_s(R^n_+)$, no eigenvalue of the corresponding Ω can have positive real part.*

(pf): We recall the definition $W(t) = X(t) - Y(t)$ and the fact that the differential equations (18) and (19) are designed to fulfill the specific purpose of reducing $\|X(t) - Y(t)\|$. Thus the Frobenius norm of the upper triangular part of $W(t)$ cannot grow as a function of t. Since $W(t)$ is related to its derivative by equation (34), the assertion follows.

In order that some eigenvalues of Ω have zero real parts, the components of $\hat{X}$ must satisfy certain algebraic equations. (Some examples are demonstrated in the next section). The algebraic constraint, therefore, limits these special matrices, denoted by $(\tilde{X}, \tilde{X})$, to a lower dimensional manifold in $\mathcal{M}(\Lambda) \times \pi_s(R^n_+)$. Forming a set of measure zero in the relative topology of $\mathcal{M}(\Lambda) \times \pi_s(R^n_+)$, points like $(\tilde{X}, \tilde{X})$ should be regarded as non-generic. Thus, for *almost all* equilibrium points of the kind $(\hat{X}, \hat{X})$, all eigenvalues of the corresponding Ω have negative real part. Lemma 3.1, therefore, serves to explain the generic behavior of the dynamics of (18) and (19).

Near an equilibrium point of the kind $(\tilde{X}, \tilde{X})$, the corresponding center manifold becomes much more complicated than (35). However, it can be proved that any flow, starting sufficiently close to $(\tilde{X}, \tilde{X})$, still converges to a single point $(\tilde{Z}, \tilde{Z})$. The proof is tedious but straightforward. We shall not give the full account of details here. But examples in the next section should illustrate our point.

It should be noted that Lemma 3.1 proves only a *local* convergence result. But we also know in the earlier discussion that the semiorbit of $(X(t), Y(t))$ approaches arbitrarily close to its ω-limit set which is a subset of all equilibrium points. These observations together imply that a solution flow $(X(t), Y(t))$ converges *globally* to a single point [1, Theorem 2.3]. Indeed, we have the following result:

LEMMA 3.3. *Let $(X(t), Y(t))$ be a solution flow of the differential system (18) and (19). Suppose $(\hat{X}, \hat{X})$ is an ω-limit point of the orbit $(X(t), Y(t))$ where all eigenvalues of the corresponding Ω have negative real parts. Then $(X(t), Y(t)) \longrightarrow (\hat{X}, \hat{X})$ as $t \longrightarrow \infty$.*

(pf): Since $(\hat{X}, \hat{X})$ is an ω-limit point of $(X(t), Y(t))$, there exists $T > 0$ such that $(X(T), Y(T))$ is sufficiently close to $(\hat{X}, \hat{X})$. By Lemma 3.1, the solution flow that begins at $(X(T), Y(T))$ converges to a single point $(\hat{Z}, \hat{Z})$. It follows that $(X(t), Y(t))$ converges to $(\hat{Z}, \hat{Z})$. Since $(\hat{X}, \hat{X})$ is an ω-limit point, it must be that $\hat{X} = \hat{Z}$.

In the above lemma, the assumption that all eigenvalues of Ω have negative real parts can be weakened. In fact, all we need in the proof of global convergence is the fact of local convergence to a single point. Thus, we restate the lemma as follows:

LEMMA 3.4. *Let $(X(t), Y(t)) \in \mathcal{M}(\Lambda) \times \pi_s(R^n_+)$ be a solution of the differential system (18) and (19). If $(\hat{X}, \hat{X})$ is an ω-limit point of this solution, then $\lim_{t\to\infty}(X(t), Y(t)) = (\hat{X}, \hat{X})$.*

We conclude this section with one final remark on Lemma 3.4. It is obvious that not every given set σ of n real values can be the spectrum of some nonnegative matrix. If a non-feasible spectrum is given, we cannot expect the ω-limit set of any solution $(X(t), Y(t)) \in \mathcal{M}(\Lambda) \times \pi_s(R^n_+)$ to contain a point of the form $(\hat{X}, \hat{X})$. But even if σ is feasible, it is possible that an orbit $(X(t), Y(t))$ contains *no* limit point of the form $(\hat{X}, \hat{X})$. We have not analyzed this type of equilibrium points yet. In either case, however, our numerical experiment seems to suggest that the ω-limit set of $(X(t), Y(t))$ still contains a single point.

4. Stability Analysis for $n = 2$.

We shall now analyze the differential system (18) and (19) for the case $n = 2$ in detail. The answers to Problem 2 and Problem 3 are obviously known for this simple case. But we hope the following study will provide some interesting insight into the understanding of the higher dimensional case.

First we explain the geometry of $\mathcal{M}(\Lambda)$ and $\pi_s(R_+^n)$. Due to symmetry, it suffices to study the behavior of the six variables $(x_{11}, x_{12}, x_{22}; y_{11}, y_{12}, y_{22})$ only. We note that the set $\mathcal{O}(2)$ consists of two kinds of orthogonal matrices

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

and,

$$\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$$

with $\theta \in [0, 2\pi)$. From (16), it follows that

$$\begin{aligned} x_{11} &= \lambda_1 \cos^2\theta + \lambda_2 \sin^2\theta, \\ x_{12} &= (\lambda_1 - \lambda_2)\cos\theta\sin\theta, \\ x_{22} &= \lambda_1 \sin^2\theta + \lambda_2 \cos^2\theta. \end{aligned} \tag{36}$$

These equations provide a parametric representation of the ellipse in R^3 formed by the intersection of the plane Π with equation $x_{11} + x_{22} = \lambda_1 + \lambda_2$ and the ellipsoid with equation $x_{11}^2 + 2x_{12}^2 + x_{22}^2 = \lambda_1^2 + \lambda_2^2$. If $\lambda_1 = \lambda_2$, then this ellipse is degenerate. Thus, the isospectral surface $\mathcal{M}(\Lambda)$ for $n = 2$ is represented by an ellipse. The distance from the plane Π to the origin is $\frac{1}{2}|\lambda_1 + \lambda_2|$. The cone $\pi_s(R_+^2)$ is the set of points in the first octant of R^3. The inverse eigenvalue problem will have a solution if and only if the ellipse intersects the first octant (See Figure 1). It is clear from the geometry that this condition is equivalent to $\lambda_1 + \lambda_2 \geq 0$.

For $n = 2$, the set $\mathcal{L}$ defined in (24) contains points of the following eight types:

(1)	$(y_{11}, y_{12}, y_{22}; y_{11}, y_{12}, y_{22})$;	y_{ij} arbitrary but > 0;
(2)	$(0, y_{12}, y_{22}; 0, y_{12}, y_{22})$;	y_{ij} arbitrary but > 0;
(3)	$(y_{11}, 0, y_{22}; y_{11}, 0, y_{22})$;	y_{ij} arbitrary but > 0, $y_{11} \neq y_{22}$;
(4)	$(y_{11}, x_{12}, y_{11}; y_{11}, 0, y_{11})$;	x_{12}, y_{ij} arbitrary, but $y_{11} > 0$;
(5)	$(y_{11}, y_{12}, 0; y_{11}, y_{12}, 0)$;	y_{ij} arbitrary but > 0;
(6)	$(x_{11}, 0, y_{22}; 0, 0, y_{22})$;	x_{11}, y_{22} arbitrary, but $y_{22} > 0$;
(7)	$(x_{11}, y_{12}, x_{11}; 0, y_{12}, 0)$;	x_{11}, y_{12} arbitrary, but $y_{12} > 0$;
(8)	$(y_{11}, 0, x_{22}; y_{11}, 0, 0)$;	x_{22}, y_{11} arbitrary, but $y_{11} > 0$;

We note that the set $\mathcal{L}$ is the union [15] of one 3-dimensional manifold (points of type (1)) and several 2-dimensional manifolds (points of types (2)-(8)). The set $\mathcal{L}$ is represented in Figure 2. For convenience, we have identify a representative for each type of point in Figure 2. For example, the first open octant represents type (1) points; the first open quadrant in the yz-plane represents type (2) points, and so on. The extra lines sticking out from the coordinate axes or planes represent the freedom of variables for the matrix X. These are types (4), (6), (7) and (8) points, respectively. For these types of limit points, note that the matrix Y is fixed to be the single point at the foot of these lines.

We consider the case when

$$\hat{X} = \hat{Y} = \begin{bmatrix} c & a \\ a & b \end{bmatrix}.$$

The corresponding matrix Ω in equation (19) is given by

$$\Omega = \begin{bmatrix} -4c - 2a^2 & 2ca - 2ba & 2a^2 \\ ca - ba & 2cb - c^2 - b^2 - 4a & -ca + ba \\ 2a^2 & -2ca + 2ba & -2a^2 - 4b \end{bmatrix}. \tag{37}$$

A general formula for eigenvalues of Ω is difficult to compute even with the help of a symbolic package. However, we already know from Lemma 3.2 that all eigenvalues of Ω have nonpositive real part. As an example, when $a = 2, b = 3$ and $c = 5$, we find the eigenvalues of Ω are approximately -35.53793480, -15.53700242 and -8.925062779. It can be seen easily from the characteristic polynomial that Ω will have two purely imaginary eigenvalues only if a, b and c come from a very special 2-dimensional hypersurface in R^3.

It turns out that some eigenvalues of Ω can be 0 when $\hat{X}$ is on the faces or edges of the cone $\pi_s(R^n_+)$. In the following, we consider the local convergence for some of these special cases:

Case 1 Type (8) limit point where

$$\tilde{X} = \tilde{Y} = \begin{bmatrix} c & 0 \\ 0 & 0 \end{bmatrix}.$$

In this case, $\Omega = diag\{-4c, -c^2, 0\}$. So Lemma 3.1 cannot be applied. We give below a somewhat extended argument to demonstrate how the convergence of a flow near $(\tilde{X}, \tilde{X})$ can be reached. The differential system (33) and (34) becomes

$$\begin{aligned}
u_{11}' &= 2u_{12}w_{12}c + 2u_{12}w_{12}u_{11} + 2w_{22}u_{12}^2 - 2w_{11}u_{12}^2 - 2u_{12}w_{12}u_{22}; \\
u_{12}' &= -w_{12}c^2 - 2w_{12}u_{11}c + 2u_{22}w_{12}c + cw_{11}u_{12} - cw_{22}u_{12} + u_{11}w_{11}u_{12} \\
&\quad +2u_{11}w_{12}u_{22} - w_{12}u_{11}^2 - u_{11}w_{22}u_{12} - w_{11}u_{12}u_{22} - w_{12}u_{22}^2 + w_{22}u_{12}u_{22}; \\
u_{22}' &= -2u_{12}w_{12}c + 2w_{11}u_{12}^2 + 2u_{12}w_{12}u_{22} - 2u_{12}w_{12}u_{11} - 2w_{22}u_{12}^2; \\
w_{11}' &= -4w_{11}c + 2u_{12}w_{12}c + 2u_{12}w_{12}u_{11} + 2w_{22}u_{12}^2 - 2w_{11}u_{12}^2 \\
&\quad -2u_{12}w_{12}u_{22} - 4w_{11}u_{11} + 4w_{11}^2; \\
w_{12}' &= -w_{12}c^2 - 2w_{12}u_{11}c + 2u_{22}w_{12}c + cw_{11}u_{12} - cw_{22}u_{12} + u_{11}w_{11}u_{12} \\
&\quad +2u_{11}w_{12}u_{22} - w_{12}u_{11}^2 - u_{11}w_{22}u_{12} - w_{11}u_{12}u_{22} - w_{12}u_{22}^2 \\
&\quad +w_{22}u_{12}u_{22} - 4w_{12}u_{12} + 4w_{12}^2; \\
w_{22}' &= -2u_{12}w_{12}c + 2w_{11}u_{12}^2 + 2u_{12}w_{12}u_{22} - 2u_{12}w_{12}u_{11} - 2w_{22}u_{12}^2 \\
&\quad -4w_{22}u_{22} + 4w_{22}^2.
\end{aligned} \tag{38}$$

We note there are two negative linear terms in w_{11}' and w_{12}'. So the convergence is contingent upon how the flow behaves on the center manifold. By center manifold theory, the center manifold of (38) is given by

$$(w_{11}, w_{12}) = h(u_{11}, u_{12}, u_{22}, w_{22}) \tag{39}$$

for some smooth function h. The geometry is illustrated in Figure 3 where we use the x-axis to represent the three variables $(u_{11}, u_{12}, u_{22}) \in R^3$, the y-axis to represent the variable $w_{22} \in R$ and the z-axis to represent the two variables $(w_{11}, w_{12}) \in R^2$. We note that the x-axis where $W \equiv 0$ also represents the 3-dimensional equilibrium points of type (1). Although h is difficult to compute explicitly, the following function can be shown to be a $O(\|(U,W)\|^3)$ approximation to h [6] near the origin:

$$\begin{aligned} w_{11} &:= \frac{u_{12}^2 w_{22}}{2c}; \\ w_{12} &:= \frac{-cu_{12}w_{22} - u_{11}u_{12}w_{22} + u_{12}u_{22}w_{22}}{c^2}. \end{aligned} \tag{40}$$

Upon substitution, we find the flow (28) on the center manifold is given by

$$\begin{aligned} u_{11}^{'} &= -w_{22}u_{12}^2(-4u_{22}c + 4u_{11}c - 4u_{22}u_{11} + 2u_{11}^2 + u_{12}^2c + 2u_{22}^2)/c^2; \\ u_{12}^{'} &= w_{22}u_{12}(-12u_{22}u_{11}c + 6u_{11}^2c + 4u_{11}c^2 + 6u_{22}^2c - 4u_{22}c^2 \\ &\quad +u_{12}^2c^2 + u_{12}^2u_{11}c + 6u_{22}^2u_{11} - 6u_{11}^2u_{22} + 2u_{11}^3 \\ &\quad -u_{12}^2u_{22}c - 2u_{22}^3)/(2c^2); \\ u_{22}^{'} &= w_{22}u_{12}^2(-4u_{22}c + 4u_{11}c - 4u_{22}u_{11} + 2u_{11}^2 + u_{12}^2c + 2u_{22}^2)/c^2; \\ w_{22}^{'} &= w_{22}(-4u_{12}^2u_{22}c + 4u_{12}^2u_{11}c + u_{12}^4c + 2u_{12}^2u_{22}^2 - 4u_{12}^2u_{22}u_{11} \\ &\quad +2u_{12}^2u_{11}^2 - 4u_{22}c^2 + 4w_{22}c^2)/c^2. \end{aligned} \tag{41}$$

The most dominant term in (41) is

$$w_{22}^{'} = 4w_{22}(w_{22} - u_{22}) + \text{ higher order terms.} \tag{42}$$

It is also obvious that

$$(w_{22} - u_{22})^{'} = 4w_{22}(w_{22} - u_{22}). \tag{43}$$

We are interested only in the case where $Y(t) \in \pi_s(R_+^n)$. Therefore, $w_{22} = x_{22} - y_{22} = u_{22} - y_{22} \leq u_{22}$ since $y_{22} \geq 0$. By checking the signs of the right-hand sides of (42) and (43), we find that the projection of the vector field (41) at any point $(u_{11}, u_{12}, u_{22}, w_{22}) \in R^4$ onto the (u_{22}, w_{22})-plane must be within the shaded region as shown in Figure 4. It is obvious from the geometry that $u_{22}(t)$ converges to a fixed point and $w_{22}(t)$ converges to 0 as t converges to infinity. In other words, we have shown that near a equilibrium point $(\tilde{X}, \tilde{X})$ of type (8), the solution flow $(X(t), Y(t))$ with $Y(0) \in \pi_s(R_+^n)$ converges to a fixed point of the form $(\tilde{Z}, \tilde{Z})$. This should manifest our point made in the preceding section concerning the convergence when Ω has zero eigenvalues. In Figure 4 we have also drawn the projection of vector field of (41) when $Y(0)$ is not in $\pi_s(R_+^n)$. This corresponds to the region below the diagonal $u_{22} = w_{22}$. It is interesting to note that $w_{22}(t)$ may diverge to infinity. This is because the differential system (18) and (19) has the descending property only if $Y(t) \in \pi_s(R_+^n)$.

Case 2 Type (3) limit points where

$$\hat{X} = \hat{Y} = \begin{bmatrix} c & 0 \\ 0 & b \end{bmatrix}$$

and $b \neq c$. We find $\Omega = diag\{-4c, -(b-c)^2, -4b\}$. Thus Lemma 3.1 can be applied.

Case 3 Type (7) limit points where

$$\tilde{X} = \tilde{Y} = \begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix}.$$

At such a limit point, the matrix Ω has eigenvalues $\{0, -4a, -4a^2\}$. An argument similar to the one given in Case 1 can be made. The center manifold should become more complicated because the eigenvectors, $[1,0,1]^T, [0,1,0]^T$ and $[1,0,-1]^T$, of Ω indicate that there are couplings between components. Instead of using center manifold theory, we now take a geometric viewpoint to study this limit point. Limit points of Type (7) have a unique feature that makes them special — That is, the ellipse $\mathcal{M}(\Lambda)$ containing $(\tilde{X}, \tilde{X})$ intersects the first octant only at $(\tilde{X}, \tilde{X})$. Therefore, the ellipse corresponding to a slightly perturbed spectrum, say $\sigma = \{a - \epsilon, -a\}$ with $\epsilon > 0$, will not intersect the first octant at all. This observation perhaps explains why we experience some numerical difficulty in constructing the second example in [18] by our method. We shall report this difficulty in the next section.

Case 4 Type (2) limit points where

$$\hat{X} = \hat{Y} = \begin{bmatrix} 0 & a \\ a & b \end{bmatrix}.$$

It can be checked that the characteristic polynomial of Ω is given by $p(\lambda) = \lambda^3 + (4a^2 + b^2 + 4a + 4b)\lambda^2 + (4b^3 + 16a^3 + 8a^2b + 16ab)\lambda + 32a^3b$. Clearly $p(\lambda)$ must have one negative real root. The other two roots can be purely imaginary numbers only if $(4a^2 + b^2 + 4a + 4b)(4b^3 + 16a^3 + 8a^2b + 16ab) = 32a^3b$. We, therefore, conclude that for almost all values of a and b, all three roots of $p(\lambda)$ have negative real part.

5. Numerical Results.

In this section we briefly report some of our numerical experiments with the differential equations (18) and (19).

We use the subroutine ODE in [17] as the integrator. Both local control parameters ABSERR and RELERR are set to be 10^{-12}. This criterion is used to control the accuracy in following the solution path. We examine the output values at time interval of 1. Normally, we should expect the loss of 1 or 2 digits in the global error. Thus, when the norm of the difference between two consecutive output points becomes less than 10^{-9}, we assume the path has converged to an equilibrium point. The execution is then terminated automatically. We always use $X(0) = \Lambda$ as the starting value for $X(t)$.

Example 1 We consider the spectrum $\sigma = \{5, 0, -2, -2\}$ which satisfies the so called condition (K) in [10]. Let E denote the matrix whose components are all 1's. We report below various choices of $Y(0)$ and the corresponding approximate equilibrium points $\hat{Y}$. We also report the approximate length of t for convergence. These lengths may depend on the initial values and the integrators, but they should be independent of the computing machine.

(a) $Y(0) = E$, $\hat{Y} \approx Y(120) \approx$

$$\begin{bmatrix} .3035817600D{+}0 & .5849737957D{+}0 & .2062366873D{+}1 & .2062366873D{+}1 \\ .5849737957D{+}0 & .5568227490D{+}0 & .1257189377D{+}1 & .1257189377D{+}1 \\ .2062366873D{+}1 & .1257189377D{+}1 & .6979774550D{-}1 & .2069797746D{+}1 \\ .2062366873D{+}1 & .1257189377D{+}1 & .2069797746D{+}1 & .6979774550D{-}1 \end{bmatrix}$$

(b) $Y(0) = 2E$, $\hat{Y} \approx Y(120) \approx$

$$\begin{bmatrix} .2384681793D{+}0 & .5682102400D{+}0 & .2008259297D{+}1 & .2008259297D{+}1 \\ .5682102400D{+}0 & .6025576213D{+}0 & .1336108181D{+}1 & .1336108181D{+}1 \\ .2008259297D{+}1 & .1336108181D{+}1 & .7948709968D{-}1 & .2079487100D{+}1 \\ .2008259297D{+}1 & .1336108181D{+}1 & .2079487100D{+}1 & .7948709968D{-}1 \end{bmatrix}$$

(c) $Y(0) = 12E$, $\hat{Y} \approx Y(120) \approx$

$$\begin{bmatrix} .2054687518D{+}0 & .5725349916D{+}0 & .1995497108D{+}1 & .1995497108D{+}1 \\ .5725349916D{+}0 & .6259762947D{+}0 & .1349043940D{+}1 & .1349043940D{+}1 \\ .1995497108D{+}1 & .1349043940D{+}1 & .8427747670D{-}1 & .2084277477D{+}1 \\ .1995497108D{+}1 & .1349043940D{+}1 & .2084277477D{+}1 & .8427747670D{-}1 \end{bmatrix}$$

(d) $Y(0) = 200E$, $\hat{Y} \approx Y(120) \approx$

$$\begin{bmatrix} .2002571961D{+}0 & .5800168336D{+}0 & .2000122546D{+}1 & .2000122546D{+}1 \\ .5800168336D{+}0 & .6302391750D{+}0 & .1339883371D{+}1 & .1339883371D{+}1 \\ .2000122546D{+}1 & .1339883371D{+}1 & .8475181443D{-}1 & .2084751814D{+}1 \\ .2000122546D{+}1 & .1339883371D{+}1 & .2084751814D{+}1 & .8475181442D{-}1 \end{bmatrix}$$

(e) $Y(0) =$ a randomly generated symmetric positive matrix, $\hat{Y} \approx Y(240) \approx$

$$\begin{bmatrix} .5806553062D{+}0 & .8075689805D{+}0 & .2354051323D{+}1 & .2276884031D{+}1 \\ .8075689805D{+}0 & .5285565377D{-}1 & .1981177836D{+}0 & .4150231898D{+}0 \\ .2354051323D{+}1 & .1981177836D{+}0 & .3084574129D{+}0 & .2165951279D{+}1 \\ .2276884031D{+}1 & .4150231898D{+}0 & .2165951279D{+}1 & .5803162721D{-}1 \end{bmatrix}$$

It is interesting to note that different choices of $Y(0)$ lead to different equilibrium points. It is also interesting to note that it takes about the same length of integration to reach convergence, although this observation is not conclusive.

Example 2 We consider the spectrum $\sigma = \{3-t, 1+t, -1, -1, -1, -1\}$ for $0 < t < 1$. It can be checked easily that the sufficient condition (K) in [10] is not satisfied. But in [18] it is proved that the set σ is indeed the spectrum of the nonnegative matrix

$$N := \begin{bmatrix} A & B \\ B & A \end{bmatrix}$$

where

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

and

$$B = \frac{(1-t)}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Considering $t = \frac{1}{2}$, we find the matrix $\Omega \in R^{21\times 21}$ at the point $(\tilde{X}, \tilde{X})$ with $\tilde{X} = N$ has one zero eigenvalue. Thus, this example is one of the exceptional cases we mentioned earlier. Furthermore, the sum of elements of σ is equal to 0 for every value of t. A slight perturbation of σ, therefore, may make the spectrum unfeasible. We think this is a situation similar to Case 3 discussed in the previous section. Indeed, the matrix N has zeros on its diagonal, which indicates that N is at the intersection of n faces of $\pi_s\,(R_+^6)$.

(a) Suppose $Y(0) = N$. It can be calculated that $\|X(0) - Y(0)\| = 5$. We find $\hat{Y} \approx Y(130) \approx$

$$\begin{bmatrix}
.0000D{+}0 & .9928D{+}0 & .9978D{+}0 & .2279D{+}0 & .2279D{+}0 & .2279D{+}0 \\
.9928D{+}0 & .0000D{+}0 & .9985D{+}0 & .1096D{+}0 & .1096D{+}0 & .1096D{+}0 \\
.9978D{+}0 & .9985D{+}0 & .0000D{+}0 & .1631D{+}0 & .1631D{+}0 & .1631D{+}0 \\
.2279D{+}0 & .1096D{+}0 & .1631D{+}0 & .0000D{+}0 & .1000D{+}1 & .1000D{+}1 \\
.2279D{+}0 & .1096D{+}0 & .1631D{+}0 & .1000D{+}1 & .0000D{+}0 & .1000D{+}1 \\
.2279D{+}0 & .1096D{+}0 & .1631D{+}0 & .1000D{+}1 & .1000D{+}1 & .0000D{+}0
\end{bmatrix}$$

where $\|X(130) - Y(130)\| \approx 7.9105 \times 10^{-11}$. We note this nonnegative matrix is different from the one constructed in [18] even though $Y(0)$ itself is already a solution to Problem 3.

(b) Suppose $Y(0) = E$. Then $\|X(0) - Y(0)\| \approx 6.9642$. We are surprised to find that our flow does not converge to an equilibrium point of the form $(\hat{X}, \hat{X})$, but rather $X(t)$ converges to $\hat{X} \approx$

$$\begin{bmatrix}
-.3000D{+}0 & -.2041D{-}3 & .7000D{+}0 & .7000D{+}0 & .7000D{+}0 & .7000D{+}0 \\
-.2041D{-}3 & .1500D{+}1 & .5867D{-}3 & .5867D{-}3 & .5867D{-}3 & .5867D{-}3 \\
.7000D{+}0 & .5867D{-}3 & -.3000D{+}0 & .7000D{+}0 & .7000D{+}0 & .7000D{+}0 \\
.7000D{+}0 & .5867D{-}3 & .7000D{+}0 & -.3000D{+}0 & .7000D{+}0 & .7000D{+}0 \\
.7000D{+}0 & .5867D{-}3 & .7000D{+}0 & .7000D{+}0 & -.3000D{+}0 & .7000D{+}0 \\
.7000D{+}0 & .5867D{-}3 & .7000D{+}0 & .7000D{+}0 & .7000D{+}0 & -.3000D{+}0
\end{bmatrix}$$

while $Y(t)$ converges to $\hat{Y} \approx$

$$\begin{bmatrix} .5909-307 & .3325D-6 & .7000D+0 & .7000D+0 & .7000D+0 & .7000D+0 \\ .3325D-6 & .1500D+1 & .6963D-3 & .6963D-3 & .6963D-3 & .6963D-3 \\ .7000D+0 & .6963D-3 & .7348-307 & .7000D+0 & .7000D+0 & .7000D+0 \\ .7000D+0 & .6963D-3 & .7000D+0 & .7348-307 & .7000D+0 & .7000D+0 \\ .7000D+0 & .6963D-3 & .7000D+0 & .7000D+0 & .7348-307 & .7000D+0 \\ .7000D+0 & .6963D-3 & .7000D+0 & .7000D+0 & .7000D+0 & .7348-307 \end{bmatrix}$$

with $\|X(9120) - Y(9120)\| \approx .6708$. It is interesting to note that the eigenvalues of $\hat{Y}$ are $\{2.8, 1.5, -.7, -.7 - .7, -.7\}$. The true equilibrium point $(\hat{X}, \hat{Y})$ is where all the small components in the second row and the second column except the $(2,2)$-position of the above two matrices are zero. We have observed that all the significant components of $(\hat{X}, \hat{Y})$ are reached as early as $t \approx 500$. The overall slow convergence is due to the slow rate of change of components in the second row and the second column. To see this, we rerun the code by choosing

$$Y(0) = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

so that the small components becomes zero. Then the corresponding orbit converges to the limit point within $t \approx 80$. This example also illustrates that an orbit $(X(t), Y(t))$ may not necessarily have a limit point of the form $(\hat{X}, \hat{X})$.

(c) Suppose $Y(0) = 12E$. Then $\|X(0) - Y(0)\| \approx 72.0868$. Again, we find that $X(t)$ converges to

$$\begin{bmatrix} -.3000D+0 & -.2487D-3 & .7000D+0 & .7000D+0 & .7000D+0 & .7000D+0 \\ -.2487D-3 & .1500D+1 & .7158D-3 & .7158D-3 & .7158D-3 & .7158D-3 \\ .7000D+0 & .7158D-3 & -.3000D+0 & .7000D+0 & .7000D+0 & .7000D+0 \\ .7000D+0 & .7158D-3 & .7000D+0 & -.3000D+0 & .7000D+0 & .7000D+0 \\ .7000D+0 & .7158D-3 & .7000D+0 & .7000D+0 & -.3000D+0 & .7000D+0 \\ .7000D+0 & .7158D-3 & .7000D+0 & .7000D+0 & .7000D+0 & -.3000D+0 \end{bmatrix}$$

while $Y(t)$ converges to

$$\begin{bmatrix} .7206-307 & .7133D-6 & .7000D+0 & .7000D+0 & .7000D+0 & .7000D+0 \\ .7133D-6 & .1500D+1 & .8494D-3 & .8494D-3 & .8494D-3 & .8494D-3 \\ .7000D+0 & .8494D-3 & .7205-307 & .7000D+0 & .7000D+0 & .7000D+0 \\ .7000D+0 & .8494D-3 & .7000D+0 & .7205-307 & .7000D+0 & .7000D+0 \\ .7000D+0 & .8494D-3 & .7000D+0 & .7000D+0 & .7205-307 & .7000D+0 \\ .7000D+0 & .8494D-3 & .7000D+0 & .7000D+0 & .7000D+0 & .7205-307 \end{bmatrix}$$

with $\|X(7140) - Y(7140)\| \approx .6708$. We believe the true limit point is the same as the one in (b).

Example 3 We consider the spectrum $\sigma = \{11, -3, -2, -2, -1, -1\}$ which satisfies a sufficient condition in [10, Theorem 2.4]. Then,

(a) With $Y(0) = 2E$, then $\|X(0) - Y(0)\| \approx 16.6132$ and $\hat{Y} \approx Y(70) \approx$

$$\begin{bmatrix} .5514D{+}0 & .1920D{+}1 & .2194D{+}1 & .1620D{+}1 & .1551D{+}1 & .1920D{+}1 \\ .1920D{+}1 & .2243D{+}0 & .2550D{+}1 & .2265D{+}1 & .1920D{+}1 & .2224D{+}1 \\ .2194D{+}1 & .2550D{+}1 & .1158D{+}0 & .3058D{+}1 & .2194D{+}1 & .2550D{+}1 \\ .1620D{+}1 & .2265D{+}1 & .3058D{+}1 & .3328D{+}0 & .1620D{+}1 & .2265D{+}1 \\ .1551D{+}1 & .1920D{+}1 & .2194D{+}1 & .1620D{+}1 & .5514D{+}0 & .1920D{+}1 \\ .1920D{+}1 & .2224D{+}1 & .2550D{+}1 & .2265D{+}1 & .1920D{+}1 & .2243D{+}0 \end{bmatrix}$$

with $\|X(70) - Y(70)\| \approx 4.3653 \times 10^{-10}$.

(b) With $Y(0) = 12E$, then $\|X(0) - Y(0)\| \approx 72.6360$ and $\hat{Y} \approx Y(70) \approx$

$$\begin{bmatrix} .5741D{+}0 & .1946D{+}1 & .2206D{+}1 & .1604D{+}1 & .1574D{+}1 & .1946D{+}1 \\ .1946D{+}1 & .2292D{+}0 & .2550D{+}1 & .2230D{+}1 & .1946D{+}1 & .2229D{+}1 \\ .2206D{+}1 & .2550D{+}1 & .1100D{+}0 & .3025D{+}1 & .2206D{+}1 & .2550D{+}1 \\ .1604D{+}1 & .2230D{+}1 & .3025D{+}1 & .2833D{+}0 & .1604D{+}1 & .2230D{+}1 \\ .1574D{+}1 & .1946D{+}1 & .2206D{+}1 & .1604D{+}1 & .5741D{+}0 & .1946D{+}1 \\ .1946D{+}1 & .2229D{+}1 & .2550D{+}1 & .2230D{+}1 & .1946D{+}1 & .2292D{+}0 \end{bmatrix}$$

with $\|X(70) - Y(70)\| \approx 6.0561 \times 10^{-10}$.

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