On Robust Matrix Completion with Prescribed Eigenvalues

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Abstract

Matrix completion with prescribed eigenvalues is a special kind of inverse eigenvalue problems. Thus far, only a handful of specific cases concerning its existence and construction have been studied in the literature. The general problem where the prescribed entries are at arbitrary locations with arbitrary cardinalities proves to be challenging both theoretically and computationally. This paper investigates some continuation techniques by recasting the completion problem as an optimization of the distance between the isospectral matrices with the prescribed eigenvalues and the affine matrices with the prescribed entries. The approach not only offers an avenue for solving the completion problem in its most general setting but also makes it possible to seek a robust solution that is least sensitive to perturbation.

Key words: matrix completion, inverse eigenvalue problem, robust solution, isospectral matrices, continuation method, steepest descent gradient flow

1   Introduction

Let $\sigma(X)$ denote the spectrum of a given matrix X. A general setting of an inverse eigenvalue problem with prescribed entries (**PEIEP**) can be delineated as follows: Given a certain subset $\mathcal{L} = \{(i_\nu, j_\nu)\}_{\nu=1}^{\ell}$ of pairs of subscripts, $1 \leq i_\nu, j_\nu \leq n$, a certain

set of values $\{a_1, \ldots, a_\ell\}$ over a field $\mathbb{F}$, and a set of n values $\{\lambda_1, \ldots, \lambda_n\}$ over the algebraically closed extension of $\mathbb{F}$, find a matrix $X = [x_{ij}] \in \mathbb{F}^{n \times n}$ such that

$$
\begin{cases}
\sigma(X) & = \{\lambda_1, \ldots, \lambda_n\}, \\
x_{i_\nu j_\nu} & = a_\nu, \text{ for } \nu = 1, \ldots, \ell.
\end{cases}
\tag{1}
$$

The prescribed entries and locations that X must comply with imposes a *structural constraint* on the construction. Likewise, the prescribed eigenvalues imposes a *spectral constraint* on X. Let $|\mathcal{L}|$ denote the cardinality ℓ of the index set $\mathcal{L}$ in general. The PEIEP is to determine (complete) the values of the remaining $n^2 - |\mathcal{L}|$ positions that do not belong to $\mathcal{L}$ so that X satisfies the spectral constraint.

Many classical results in matrix theory, including the well-known Schur-Horn theorem [8], the Mirsky theorem [12], and the Sing-Thompson theorem [14,15], can be cast as PEIEPs with specially selected index sets $\mathcal{L}$ [4]. The sufficient conditions comprehended in these results are the harder part of the proof. Indeed, they constitute interesting yet challenging inverse eigenvalue problems with prescribed diagonal elements or diagonal blocks. PEIEPs also arise in practical applications. The design of a mass-spring system or an IC circuit, for instance, gives rise to either an additive inverse eigenvalue problem or a Jacobi inverse eigenvalue problem [6] that is a special PEIEP. Another example is the derivation of higher order Gauss-Kronrod quadrature rules [2,6] based on given lower order Gaussian quadratures. Except a few special cases, little is known about the PEIEP either theoretically or computationally in its general setting.

By comparing the coefficients in the characteristic polynomial $\det(\lambda I - X)$ with the symmetric functions of the prescribed eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$, we may consider that solving a PEIEP is equivalent to solving a polynomial system. However, it is known, at least in many of the classical cases [4], that the prescribed data often has to satisfy a certain intrinsic inequalities without which the resulting polynomial system simply is not solvable. Even if we assume that the polynomial is consistent, such a system would be under-determined if $|\mathcal{L}| < n^2 - n$ and should always have multiple solutions. We are interested in numerical methods that not only can track down a solution but also is able to approximate a solution that is least sensitive to perturbations.

The notion of robustness was first introduced by Kautsky et al in the context of pole assignment problem [11], followed by many other studies including a recent article [13] on the quadratic eigenstructure assignment problem. However, unlike the robust eigenstructure assignment problems where entries in either the gain matrix or the output matrix are totally free, the robust PEIEP imposes additional challenges in that the solution matrix has a fixed structure specified by the prescribed entries. Finding a solution for the PEIEP is already a hard problem by itself [4], not to mention finding a robust solution.

We need to quantify more carefully the meaning of robustness. Because there are multiple solutions which could be in continuum form, it does not seem feasible to compare

how one solution is close to another solution. Rather, the robustness of an approximate solution usually is measured by the sensitivity of its eigenvalues to perturbations. Toward that end, we recall the well-known Bauer-Fike theorem that characterizes the sensitivity of eigenvalues to perturbations.

Theorem 1 *If μ is an eigenvalue of $X + E \in \mathbb{C}^{n \times n}$ and $V^{-1}XV = diag\{\lambda_1, \ldots, \lambda_n\}$, then*

$$\min_{\lambda \in \sigma X} |\lambda - \mu| \le \kappa_p(V) \|E\|_p, \tag{2}$$

where $\|\cdot\|_p$ denotes any of the p-norms and $\kappa_p(V)$ denotes the condition number of V with respect to the p-norm.

It is therefore intuitively true that a solution X_1 to the PEIEP is relatively more robust than another solution X_2 if the corresponding matrices V_1 and V_2 of eigenvectors satisfy the inequality $\kappa_p(V_1) < \kappa_p(V_2)$. The quantity $\kappa_p(V)$ sometime serves only as a bulk estimate that does not necessarily reflect the texture of the underlying eigenstructure. A more refined way to assess the robustness is the notion of condition number with respect to each individual eigenvalue introduced by Wilkinson [16]. Assuming that $\mathbf{u}^T$ and $\mathbf{v}$ are the left and right eigenvectors, respectively, corresponding to the eigenvalue λ of a matrix X, then the condition number $c(\lambda)$ of λ which measures the rate of change of λ relative to change of X is given by the formula [16]

$$c(\lambda) = \frac{\|\mathbf{u}^T\|_2 \|\mathbf{v}\|_2}{|\mathbf{u}^T \mathbf{v}|}. \tag{3}$$

Let $V = [\mathbf{v}_1, \ldots, \mathbf{v}_n]$ denote the matrix of eigenvectors. If we assume that $U = V^{-T}$ and $U^T X V = diag\{\lambda_1, \ldots, \lambda_n\}$, then the sum

$$\nu^2(V) = \sum_{i=1}^{n} \|\mathbf{u}_i^T\|_2^2 \|\mathbf{v}_i\|_2^2, \tag{4}$$

can be considered as a total measure of robustness of X (and hence of V). It is clear that $\nu(V) \le \kappa_F(V)$. As an illustration, consider the PEIEP with $n = 2$, $\mathcal{L} = \{(1, 2)\}$, $a_1 = 4$, and $\sigma(X) = \{1, 2\}$. It is easy to check that the solution must appear like

$$X^{[a]} = \begin{bmatrix} a & 4 \\ \frac{a(3-a)-2}{4} & 3-a \end{bmatrix},$$

forming a one-parameter family with arbitrary a in $\mathbb{R}$. The condition numbers of $X^{[a]}$ as well as the corresponding matrix $V^{[a]}$ of eigenvectors are plotted in Figure 1. Each $X^{[a]}$ is a solution to the PEIEP, but our ultimate goal should be to minimize this condition number among all possible solutions. Such a quest in general would be too expensive to be practical. Thus, strictly speaking, in what follows we seek only to lower this condition number while constructing a solution to the PEIEP.

The best constructive result concerning the PEIEP is due to Hershkowitz [7], asserting
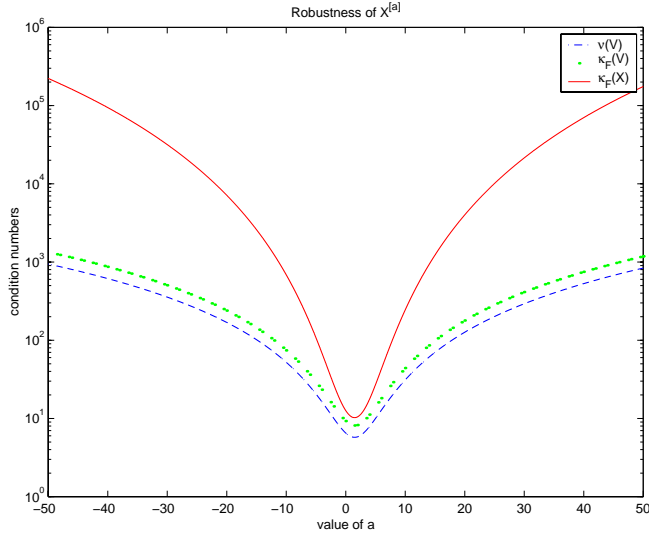
3

Fig. 1. Condition numbers of PEIEP solutions when $n = 2$, $\mathcal{L} = \{(1,2)\}$, $a_1 = 4$, and $\sigma(X^{[a]}) = \{1, 2\}$.

that, under some mild conditions, the PEIEP is solvable over the algebraically closed extension of $\mathbb{F}$ if $|\mathcal{L}| \leq 2n - 3$. Thus far, however, rational algorithms using only arithmetical operations have been developed only for $|\mathcal{L}| \leq n$ [9]. In [4], we have proposed a differential equation approach to tackle the PEIEP in its most general setting. None of these techniques addresses the issue of robustness. Our contribution in this paper is to modify the dynamical system to improve the robustness.

2 Solving the PEIEP

In this section we briefly review the differential equation approach proposed in [4]. We shall limit our discussion to solving the PEIEP over the field $\mathbb{R}$ of real numbers. Assuming that the prescribed eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$ are closed under complex conjugation, let $\Lambda \in \mathbb{R}^{n \times n}$ denote a matrix with eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$. Note that Λ is real-valued and is not limited to any specific structure. Let $\mathcal{G}l(n)$ denote the general group of $n \times n$ nonsingular matrices in $\mathbb{R}^{n \times n}$. The set

$$\mathcal{M}(\Lambda) = \{V \Lambda V^{-1} | V \in \mathcal{G}l(n)\} \tag{5}$$

consists of an *orbit* of all matrices that are isospectral to Λ under the group action of $\mathcal{G}l(n)$. We mention in passing that real-valued matrices isospectral to eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$ may consist of several disjoint orbits. It is entirely possible that a different choice of Λ might end up with a distinct orbit. Given an index subset of locations $\mathcal{L} = \{(i_\nu, j_\nu)\}_{\nu=1}^{\ell}$ and the prescribed values $\mathbf{a} = \{a_1, \ldots, a_\ell\}$, the set

$$\mathcal{S}(\mathcal{L}, \mathbf{a}) = \{X = [x_{ij}] \in \mathbb{R}^{n \times n} | x_{i_\nu j_\nu} = a_\nu, \nu = 1, \ldots, \ell\} \tag{6}$$

4

contains all matrices with the prescribed entries at the desired locations. One idea of solving the PEIEP is to find the intersection of the two geometric entities $\mathcal{M}(\Lambda)$ and $\mathcal{S}(\mathcal{L}, \mathbf{a})$.

There are several ways to find the intersection of geometric entities. Alternating projection or Newton-like iterative methods, for example, have been used successfully in other types of inverse eigenvalue problems [3]. We believe that a similar scheme can be devised for PEIEPs. However, we are interested in more than just finding a solution. For the sake of finding a robust solution, we find the differential equation approach proposed in [4] is particularly appealing.

The original idea in [4] was to minimize, for each given $X \in \mathcal{M}(\Lambda)$, the distance between X and $\mathcal{S}(\mathcal{L}, \mathbf{a})$. That is, we want to minimize the function defined by

$$f(X) = \frac{1}{2}\langle X - P(X), X - P(X)\rangle, \tag{7}$$

where $P(X)$ denotes the orthogonal projection of matrix X onto the affine subspace $\mathcal{S}(\mathcal{L}, \mathbf{a})$ with respect to the Frobenius inner product

$$\langle A, B \rangle = \text{trace}(AB^T) = \sum_{i,j} a_{ij}b_{ij}$$

for $A = [a_{ij}], B = [b_{ij}] \in \mathbb{R}^{n \times n}$. It is more convenient to rewrite this minimization as an unconstrained optimization problem in terms of the parameter V in the open set $\mathcal{G}l(n)$. Suppose that $X = V\Lambda V^{-1}$. The objective function $f(X)$ is equivalent to

$$g(V) = \frac{1}{2}\langle V\Lambda V^{-1} - P(V\Lambda V^{-1}), V\Lambda V^{-1} - P(V\Lambda V^{-1})\rangle. \tag{8}$$

It can be shown, upon computing the Fréchet derivative of g and applying the Rietz representation theorem, that the gradient ∇g of the objective function g is given by the expression

$$\nabla g(V) = \left(V\Lambda V^{-1} - P(V\Lambda V^{-1})\right)V^{-T}\Lambda - (V\Lambda V^{-1})^T \left(V\Lambda V^{-1} - P(V\Lambda V^{-1})\right)V^{-T}. \tag{9}$$

More concisely, the equation

$$\nabla g(V)V^T = \left[X - P(X), X^T\right], \tag{10}$$

where $[M, N] = MN - NM$ denotes the Lie bracket commutator, holds. It follows that the vector field

$$\frac{dV}{dt} = k(V\Lambda V^{-1})V^{-T}, \tag{11}$$

where

$$k(X) = [X^T, X - P(X)], \tag{12}$$

5

defines a flow in the open set $\mathcal{G}l(n)$ that moves in the steepest descent direction to reduce the value of $g(V)$. In the meantime, the vector field

$$\frac{dX}{dt} = [h_V(X), X] \tag{13}$$

with

$$h_V(X) = k(X)V^{-T}V^{-1} \tag{14}$$

defines the steepest descent flow on the manifold $\mathcal{M}(\Lambda)$ for $f(X)$. The system (13) is not particularly important in practice. All it matters is that the relationship $X(t) = V(t)\Lambda V(t)^{-1}$ translates (11) and (13) back and forth.

The gradient flow framework outlined above is general enough that it can be used to explore all types of PEIEPs regardless how the locations or cardinalities of the prescribed entries are given. For different $\mathcal{S}(\mathcal{L}, \mathbf{a})$, only the projection $P(X)$ needs modification. In the event that the PEIEP is not solvable, the approach will find a matrix on $\mathcal{M}(\Lambda)$ that is nearest to $\mathcal{S}(\mathcal{L}, \mathbf{a})$ in the sense of least squares.

A special case of the PEIEP is worth additional attention. If in the delineation of a PEIEP the matrix X is also required to be symmetric, then it suffices to consider the orbit defined by the group $\mathcal{O}(n)$ of orthogonal matrices instead of by $\mathcal{G}l(n)$. In such case, the differential equations can be much simplified. For example, (11) becomes

$$\frac{dQ}{dt} = k(Q\Lambda Q^T)Q \tag{15}$$

with

$$k(X) = [P(X), X],$$

whereas (13) is reduced to

$$\frac{dX}{dt} = [X, [X, P(X)]]. \tag{16}$$

The system (16) is known as the Brockett double bracket equation which, because of its many interesting properties, has attracted considerable attention in recent years [1,5]. All matrices from $\mathcal{O}(n)$ are equally (and perfectly) conditioned. Thus, for symmetric PEIEP, there is no issue of robustness involved.

Recall that Λ is a constant real matrix with prescribed eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$. Thus both dynamical systems (11) and (15) are autonomous. To maintain that the flow $Q(t)$ defined by (15) stays on the manifold $\mathcal{O}(n)$, special numerical schemes known as geometric integrators or Lie group methods [10] should be used for integration. In contrast, since the parameter $V(t)$ defined by (11) generally has no additional structure, the system (11) can be integrated by *any* available ODE solver starting with initial value $V(0) = I$. This naturally constitutes a reasonable numerical method for solving the general PEIEP.

3 Controlling the Robustness

We would like to not only integrate (11) to its equilibrium, but also to somehow control the condition number of $V(t)$. Obviously, as it stands now, $\kappa_p(V(t))$ is inherited once $V(t)$ is determined by (11) and there is not much we can do. To control the robustness, we must provide a mechanism to redirect the course of integration.

One possible approach is to build in the condition number of V in the objective function to be minimized. That is, instead of (8), we can consider one of the new objective functions,

$$\xi(V) = g(V) + \frac{\alpha}{2}\langle V, V\rangle\langle V^{-1}, V^{-1}\rangle, \tag{17}$$

or

$$\zeta(V) = g(V) + \frac{\beta}{2}\nu^2(V). \tag{18}$$

The new terms added to $g(V)$ stand for a *penalty* on either $\kappa_F^2(V)$ or $\nu^2(V)$. The multipliers α and β are positive numbers selected to reflect how much weight we want to emphasize the penalty. It remains to calculate the gradient of $\xi(V)$ or $\zeta(V)$, and everything described above about gradient flows should follow through. Toward that, we make claim of the following two dynamical systems after some manipulation of calculus.

Theorem 2 *If $\xi(V)$ is used as the objective function, then the steepest descent flow in the same spirit of the system (11) is defined by the modified equation*

$$\frac{dV}{dt} = \left(k(V\Lambda V^{-1}) - \alpha\left(\langle V^{-1}, V^{-1}\rangle VV^T - \langle V, V\rangle V^{-T}V^{-1}\right)\right)V^{-T}. \tag{19}$$

Similarly, if $\zeta(V)$ is used, then the steepest descent flow becomes

$$\frac{dV}{dt} = \left(k(V\Lambda V^{-1}) - \beta\left(V\Theta(V^{-T})V^T - V^{-T}\Theta(V)V^{-1}\right)\right)V^{-T}, \tag{20}$$

where

$$\Theta(V) = diag\{\|\mathbf{v}_1\|_2^2, \ldots, \mathbf{v}_n\|_2^2\}, \tag{21}$$

if $V = [\mathbf{v}_1, \ldots, \mathbf{v}_n]$.

We find from numerical experiments that the weights α and β are fairly difficult to manage. Too little penalty does not improve the condition number significantly, while too much penalty compromises the real purpose of finding an exact or nearly exact solution to the PEIEP and a least squares solution is found instead. It seems from our numerical experience that only a small fraction of the condition number in the dynamical system will suffice to tame the behavior. In some cases, we see that α or β should be in the range of 10^{-10} to avoid an inadvertent least squares solution. We think that a more sophisticated scheme in selecting α and β adaptively, such as those used in the interior point methods, might help to improve the situation, but that investigation will have to be reported in a separate paper.

7

On the other hand, observe that the initial value $V(0) = I$ is perfectly conditioned. By continuity, the conditioning of $V(t)$ will remains reasonable well for at least small values of t. As the integration continues, it is possible that $\kappa_F(V(t))$ (or $\nu(V(t))$) will grow as $V(t)$ converges to singularity or becomes ill-conditioned. Is it possible, when this occurs, to restart the integration? We exploit this idea more carefully in the sequel.

First of all, it is important to note that all three vector fields (11), (19) and (20) are in the same format that

$$\frac{dV}{dt} = \varphi(V; \Lambda) V^{-T}, \tag{22}$$

but with different definitions of $\varphi(V; \Lambda)$. Let $t_0, t_1, \ldots$ be a sequence of positive numbers whose values will be defined later. Let $V(t)$ denote the solution to any one of the three differential equations with initial value $V(0) = I$. For each $i = 0, 1, \ldots$, let $V_i(\tau)$ denote the solution to the initial value problem

$$\frac{dV_i}{d\tau} = \varphi(V_i; X_i) V_i^{-T}, \quad V_i(0) = I, \quad \tau \in [0, t_i], \tag{23}$$

where, starting with $X_0 = \Lambda$, we recursively define

$$X_{i+1} = V_i(t_i) X_i V_i^{-1}(t_i). \tag{24}$$

The following theorem concerning the factorization of $V(t)$ is a consequence of the fact that (22) is an autonomous differential system. A proof for the case that $\varphi(V; \Lambda) = k((V\Lambda V^{-1})$ can be found in [4], but the generalization to other types of $\varphi(V; \Lambda)$ is not difficult.

Theorem 3 *At each point $t = s + \sum_{j=0}^{i-1} t_j$ in the interval of existence for the initial value problem (22) with $V(0) = I$, the solution $V(t)$ can be factorized as*

$$V(t) = V_i(s) V_{i-1}(t_{i-1}) \cdots V_0(t_0), \tag{25}$$

where each $V_j(\tau)$ is the solution of (23).

Other than the obvious fact that the value of t_i for each $i = 0, 1 \ldots$ must be within the maximal interval of existence for each system (23), the choice of t_i can be quite flexible. One strategy is to continue integrating (23) within that interval until $\kappa_F(V_i(\tau))$ reaches some predesignated threshold. The value of τ in reaching that threshold is the *maximal* t_i we can define. We then update X_{i+1} according to (24) and switch to solve a new initial value problem (23). We call this process a *restart*. Restart is expensive because we have to update X_{i+1}. In the extreme (and not recommended) case, we can restart at every single integration step. By restarting, however, we effectively improve the computational stability because the condition number of $V_i(\tau)$ is well kept under some upper bound.

Thus far, we have not seen any hope of improving $\kappa_p(V(t))$. The theory implies that the solution $V(t)$ from the continuation scheme (22) and the product (25) from the

restart scheme (23) are identical. However, we now make an interesting and somewhat surprising remark. We have observed time and time again in our numerical experiments that the computed results by these two schemes behave very differently. More candidly, they converge to two entirely different limit points. The reason for this happening is not completely understood, perhaps partially due to the "curse" that the PEIEP is ill-posed in two regards: that the problem has multiple (and often continuum) solutions and that a small perturbation of the vector field in (23) by the restart changes the course of integration which moves the flow to a dramatically different limit point. Computations for ill-posed problems such as this generally would have been disastrous, but notice that in our flow approach we have built in the descent property by which the objective functions are allowed only to go downhill. For instance, if (11) is used, then the distance between $\mathcal{M}(X_i)$ and $\mathcal{S}(\mathcal{L}, \mathbf{a})$ can only be reduced. That is, it is always true that

$$\|X_{i+1} - P(X_{i+1})\|_F \leq \|V_i(t)X_iV_i(t)^{-1} - P(V_i(t)X_iV_i(t)^{-1})\|_F \leq \|X_i - P(X_i)\|_F$$

for all $t \in [0, t_i)$. Changing course of integration does not affect this descent property. Rather, it offers an opportunity to turn the curse of ill-posedness into a possibly blessing by helping to improve the stability and conditioning. We shall use numerical examples to demonstrate our points.

By now we have several ways to tackle the robust PEIEPs numerically. We categorize them into two classes of methods.

Algorithm 1 *Simply integrate any of the three differential equations (11), (19), or (20) by any available ODE integrator, starting with $V(0) = I$ and $X_0 = \Lambda$, until convergence.*

We make two comments on the application of Method 1. First, assuming that flow defined by the system (11) does converge to a true solution of the PEIEP, we find that such a solution usually enjoys some degree of robustness already. Nonetheless, *modifying the initial value X_0 by choosing different Λ can make a big difference.* Not only that different equilibrium points may emerge, but also that they may vary greatly in robustness. For instance, if the given spectrum $\{\lambda_1, \ldots, \lambda_n\}$ is all real, then we find that starting with $\Lambda = \Lambda_0 = \text{diag}\{\lambda_1, \ldots, \lambda_n\}$ generally is less desirable than starting with $\Lambda = \Lambda_0 + T_0$ where T_0 is an arbitrary strictly upper triangular matrix. The triangular starting value usually converges faster to more robust equilibrium point for the system (11) than the diagonal starting value. Again, we do not have a complete comprehension of the mathematical reason that leads to this observation. One possible explanation is that the Schur triangular form (or the real-valued Schur block triangular form with 2×2 diagonal blocks for complex conjugate eigenvalues) is reachable via orthogonal similarity transformations. Recall that the isospectral orbits by $\mathcal{O}(n)$ group action are closed and bounded. It seems intuitively true that such a point should be easier and more "stable" to locate than, say, the Jordan canonical form (which under the above assumption is precisely the diagonal matrix.) Since our integration process is in a sense a reversal of the reduction procedure used to find canonical forms, that is, we want to start with some points in the "reduced form" and travel to some other

points in the "full form" while in the meantime satisfying the structural constraints, starting with isospectral triangular matrices does seem to work better than starting with the diagonal matrix of eigenvalues.

Secondly, in the event that we want to further improve the robustness, we may employ the dynamical systems (19) or (20). In these cases, *we have to choose the weight α or β appropriately.* Too much emphasis on the penalty term translates into too much restriction on the condition number which, in turn, may jeopardize the original intention of finding an approximate solution to PEIEP and result in a least squares solution instead. Employing (19) or (20) will improve the condition number in general, but might slow down the convergence.

Algorithm 2 *Integrate the system (23) using any of the three aforementioned dynamical systems until the condition number of $V(t)$ violates a given threshold. By then, automatically applies a restart.*

This hybrid method has the ability to cap the condition number of the factors of $V(t)$ under a given threshold. The product of the condition numbers of these factors provides an upper bound on the condition number of $V(t)$, so this method does not control the robustness directly. However, we shall demonstrate by examples that this restart scheme has quite an impressive impact on the condition number after all. Additionally, the solution flow $V_i(t)$ in each segment $[0, t_i)$ being well conditioned (otherwise, a restart will be activated), it also improves the computational stability. Again, we realize that if the threshold for restart is set too low, then there will be frequent (expensive) restarts and more factors. The overall performance may be degraded due to this frequent restart.

4   Numerical Examples

In this section we use some numerical examples to demonstrate how our proposed methods perform. At the moment, our primary concern is not so much on the efficiency of these methods, though such a task could be greatly improved with careful programming. Rather, we focus on the behavior of the resulting flows from these differential systems. In what follows, we report the dynamics of the residual

$$R(t) = ||X(t) - P(X(t))||_F$$

that we would like to minimize as well as the corresponding condition numbers. For convenience, all numerical results are rounded to the fourth decimal number.

Example 1. We begin with the 2×2 example introduced earlier since we know all the solutions in closed form. Note that no solution can be symmetric in this case. By direct calculation, the best conditioned PEIEP solution is obtained when $a_m = 1.5$ at which $\kappa_F(X^{[a_m]}) = 10.2520$. It turns out that both $\kappa_F(V^{[a_m]}) = 8.1250$ and $\nu^2(V^{[a_m]}) =$

33.0078 attain their minimum value also at the same point a_m. Let $\Lambda_0 = \text{diag}\{1, 2\}$. Note that $X^{[a_m]} = V^{[a_m]}\Lambda_0 \left(V^{[a_m]}\right)^{-1}$.

Starting with $X_0 = \Lambda_0$, we integrate all three flows using ODE integrators available from MATLAB. Note that when $\alpha = \beta = 0$, both flows (19) and (20) are identical to (11). Numerical results are depicted in the top row of Figure 2. The flow associated
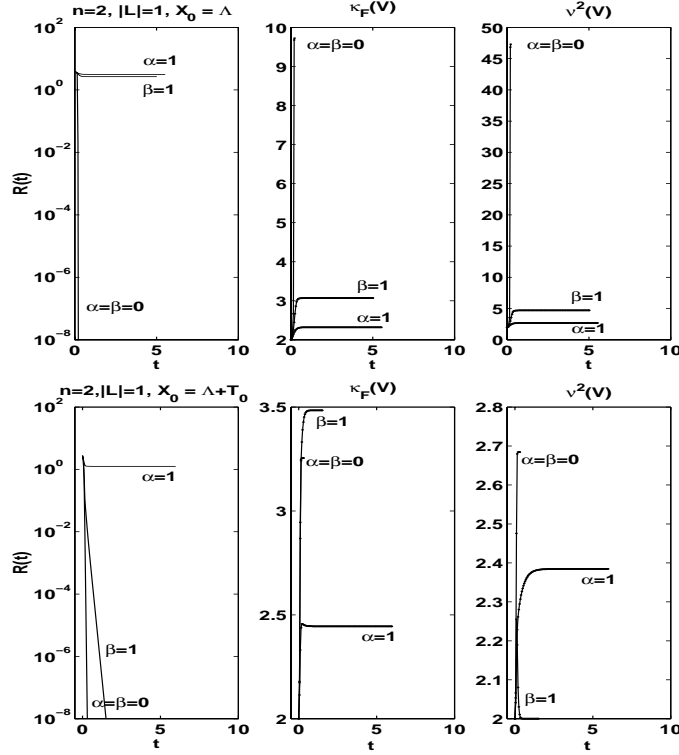

Fig. 2. Example 1: $n = 2$, $|\mathcal{L}| = 1$, $\alpha = \beta = 0, 1$ for $X_0 = \Lambda$ and $X_0 = \Lambda + T_0$

with (11) converges a true solution $X^{[-0.3125]}$ with reasonable condition numbers. The choices of either $\alpha = 1$ or $\beta = 1$ clearly improve the respective condition numbers as we have expected. However, the penalty apparently are too severe that the corresponding flows converge to least squares solutions.

With the same parameters α and β, but with $X_0 = \Lambda_0 + T_0$, say,

$$X_0 = \begin{bmatrix} 1 & 1.2533 \\ 0 & 2 \end{bmatrix},$$

the solution of (11) converges to a more robust solution $X^{[0.8994]}$. The condition number for the case $\alpha = \beta = 0$ in the bottom row of Figure 2 is smaller than that in the top row when a diagonal starting point is used. While the solution of (19) still converges to a least squares solution, it is nice to see that the solution of (20) now finds a PEIEP solution $X^{[1.1321]}$ whereas $|\kappa_F(X^{[1.1321]}) - \kappa_F(X^{[a_m]})| \approx 0.1338$.

Generally we do not have the luxury of knowing the set of completed matrices with

prescribed eigenvalues as in this example. Completing one single matrix with prescribed eigenvalues is indeed a hard problem by itself, let alone finding the most robust solution. Thus it should not be misunderstood that our methods can search among all values of a to locate $X^{[a_m]}$. Let $V^{[50]}$ be the matrix of eigenvectors of $X^{[50]}$ which is ill-conditioned. Suppose we start with a slightly perturbed isospectral matrix of $X^{[50]}$, say, $X_0 = (V^{[50]} + 10^{-3}R)\Lambda_0(V^{[50]} + 10^{-3}R)^{-1}$ where

$$R = \begin{bmatrix} -1.1465 & 1.1892 \\ 1.1909 & -0.0376 \end{bmatrix}$$

is a random matrix. The three flows (11), (19) and (20) converge to $X^{[48.1327]}$, $X^{[48.1284]}$ and $X^{[48.1202]}$, respectively. Though condition numbers are improved by about 15%, neither is anywhere near the optimal $X^{[a_m]}$.

Example 2. The problem where the prescribed entries are

$$A_{\mathcal{L}} = \begin{bmatrix} 0 & 0 & 0 & 0 & \mathbf{5.7980} \\ 0 & 0 & \mathbf{2.2594} & \mathbf{4.3290} & 0 \\ 0 & \mathbf{5.2982} & \mathbf{7.6036} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and the prescribed eigenvalues are $\{1, 2, 3, 4, 5\}$ has been considered in [4]. With the diagonal starting value $X_0 = \Lambda_0 = \text{diag}\{1, \ldots, 5\}$, it was demonstrated that the flow associated with (11) converges to an ill-conditioned limit point $X^*(\Lambda_0)$ at which $\kappa_F(X^*(\Lambda_0)) \approx 1.0165 \times 10^6$. It was also demonstrated that using the restart strategy (Method 2) where the threshold of allowable condition number was set at, say, 40, only one restart was needed to reach a better limit point $X_R^*(\Lambda_0)$ at which $\kappa_F(X_R^*(\Lambda_0)) \approx 528$, representing a significant improvement over $\kappa_F(X^*(\Lambda_0))$.

Suppose we now apply Method 1 by using (19) and (20), respectively, and "continually" vary the weights α and β in the range $[0, 1]$. We notice, as are depicted in Figure 3, that both condition numbers $\kappa_F(V)$ and $\nu^2(V)$ are depressed as the values of the weights are increased. The payoff, however, is that the residuals R(t) drops at a much slower rate. In fact, it seems that a true solution is attained only if puny penalties, that is α, $\beta \in [10^{-14}, 10^{-12}]$, are used, but then the resulting condition numbers remain high. Furthermore, it appears that if the penalty weights are greater than 10^{-4}, we can only retain a least squares solution.

Let X_0 be an arbitrary isospectral upper triangular matrix, say,

$$X_0 = \begin{bmatrix} 1 & 2.3406 & -11.8589 & -0.4123 & 1.2864 \\ 0 & 2 & -10.5590 & -11.2834 & 6.5647 \\ 0 & 0 & 3 & -13.4828 & -11.6782 \\ 0 & 0 & 0 & 4 & -4.6061 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}.$$
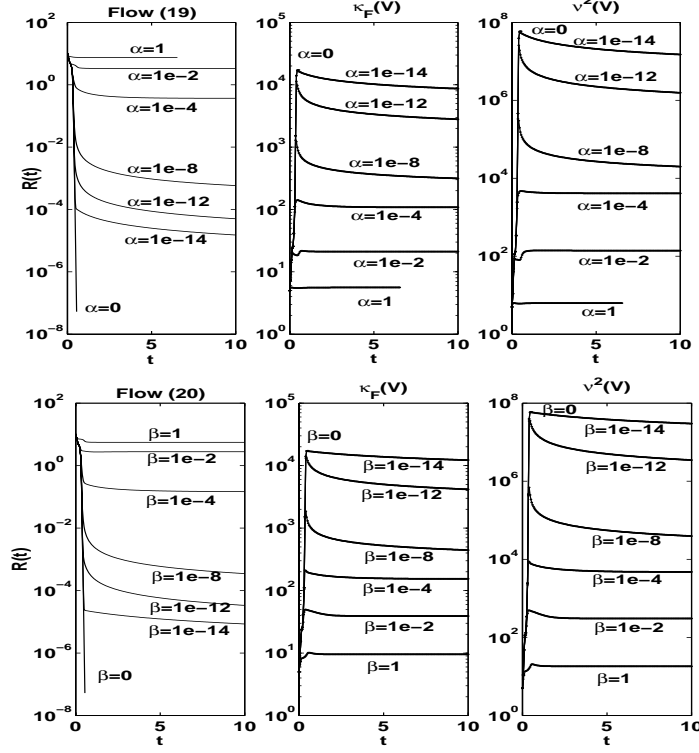
12

Fig. 3. $n = 5$, $|\mathcal{L}| = n$, $X_0 = \Lambda$, $\alpha = \beta = 0, 10^{-14}, 10^{-12}, 10^{-8}, 10^{-4}, 10^{-2}, 1$.

We observe that the flow (11) starting with $\Lambda_0 + T_0$ usually converges faster to a more robust equilibrium than starting with Λ_0. For diminutive weights, say, $\alpha, \beta \in [0, 10^{-8}]$, the flows associated with (19) and (20) behaves almost the same as that associated with(11). Again, the rule seems true that increasing penalty weights decreases both $\kappa_F(V)$ and $\nu^2(v)$, slows down the convergence, and eventually leads to a least squares solution, as is observed in Figure 4.

However, something remarkably different happens. We notice that with much larger penalty weights, say, $\alpha, \beta \geq 1$, an *optimal robust solution* is tracked down quickly by both flows (19) and (20). We also notice that the flow (20) usually converges slower than (19). A comparison of convergence's histories for $\alpha = \beta = 10$ is given in Figure 4.

The fact that the flows converge to an optimal robust solution deserves further attention. Recall that $\kappa_F(V) \geq \nu(V) \geq n$ in general and equality is obtained only when V is orthogonal. We check the limit point V^* of (19) with $\alpha = 1$ and find that

$$
V^* = \begin{bmatrix}
0.6182 & 0.7519 & -0.0585 & -0.2161 & 0.0484 \\
-0.5927 & 0.2765 & -0.4564 & -0.5585 & 0.2279 \\
-0.2980 & 0.2312 & 0.8816 & -0.2833 & 0.0155 \\
-0.3321 & 0.4599 & -0.0034 & 0.7347 & 0.3720 \\
0.2597 & -0.3052 & -0.1051 & -0.1469 & 0.8984
\end{bmatrix}
$$

at nearly $t^* \approx 6.8$ (as shown in Figure 4) is indeed nearly orthogonal. It is interesting to note from extensive numerical experiments that with a generic choice of T_0 and
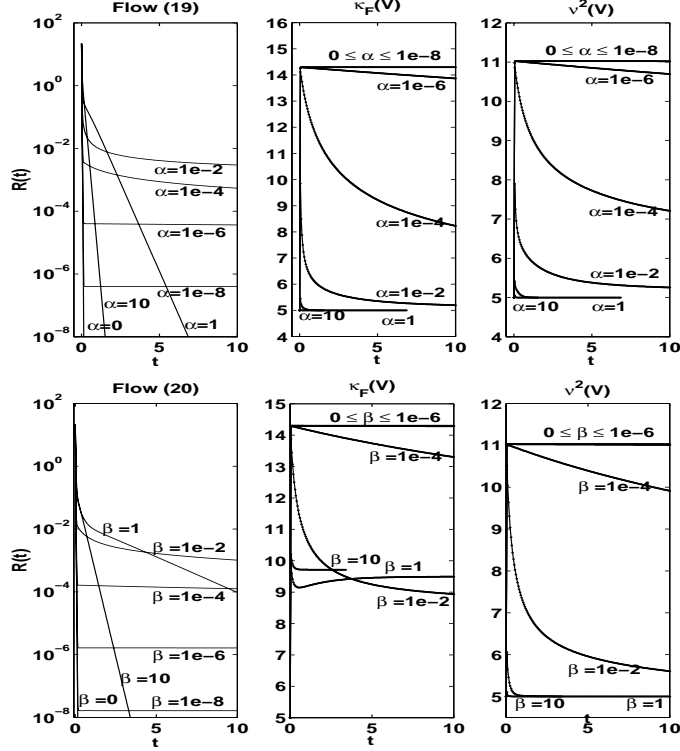
13

Fig. 4. $n = 5$, $|\mathcal{L}| = n$, $X_0 = \Lambda + T_0$, $\alpha = \beta = 0, 10^{-8}, 10^{-6}, 10^{-4}, 10^{-2}, 1, 10$.

large enough penalty weights, we almost always have $\kappa_F(V)$ converging to $n = 5$. The rate of convergence depends on T_0. If V^* is orthogonal, then $X^* = V^* X_0 V^{*T}$ is the Schur decomposition of X^*. This observation justifies numerically what we have claimed earlier that our continuation approach, starting from X_0 to X^*, is kind of the reversal of the usual QR algorithm. Starting from an upper triangular X_0 is a less stringent requirement than starting from the diagonal Λ_0 while we are looking for a transformation matrix V with nearly optimal condition numbers. The latter, if can be done, means that an additional structural constraint has been inadvertently imposed, that is, we are looking for a nearly "normal" PEIEP solution whose existence is not *a priori* guaranteed.

We summarize the observations above for this example in Table 1.

Table 1
Summary of Numerical Experiment with Example 2.

| Flow	$X_0 = \Lambda_0$, diagonal	Figure	$X_0 = \Lambda_0 + T_0$, upper triangular	Figure
(11)	ill-cond. w/o restart	3	$\kappa_F(V) \to c_0$, $\nu^2(V) \to m > 5$	4
(19), $\alpha \in [0, 10^{-8})$	conv., high cond.	3_{top}	similar to (11)	4_{top}
(20), $\beta \in [0, 10^{-8})$	conv., high cond.	3_{bottom}	similar to (11)	4_{bottom}
(19), $\alpha \in [10^{-8}, 1)$	LS solution	3_{top}	slow conv., $\kappa_F(V) \to c_1 \approx 5$, $\nu^2(V) \to 5$	4_{top}
(20), $\beta \in [10^{-8}, 1)$	LS solution	3_{bottom}	slow conv., $\kappa_F(V) \to c_2 < c_0$, $\nu^2(V) \to 5$	4_{bottom}
(19), $\alpha \geq 1$	LS solution	3_{top}	$\kappa_F(V) \to c_1 \approx 5$, $\nu^2(V) \to 5$	4_{top}
(20), $\beta \geq 1$	LS solution	3_{bottom}	$\kappa_F(V) \to c_2 < c_0$, $\nu^2(V) \to 5$	4_{bottom}
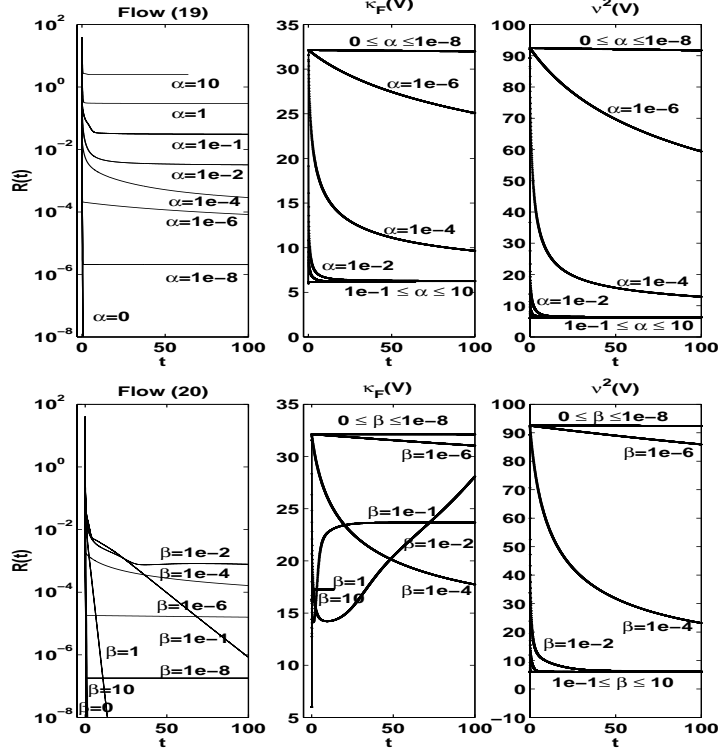
Fig. 5. Example 3: $n = 6$, $|\mathcal{L}| = 2n$, $X_0 = \Lambda + T_0$, $\alpha = \beta = 0, 10^{-8}, 10^{-6}, 10^{-4}, 10^{-2}, 10^{-1}, 1, 10$.

Example 3. In this example, we examine the convergence behavior when $|\mathcal{L}| = 2n$. This is a situation that goes beyond existing theory, so it is particularly interesting to see that our gradient flows can still find a solution. To fix the idea, we consider the case $n = 6$, with prescribed eigenvalues $\{1, \ldots, 6\}$, and want to complete the matrix:

$$
\mathcal{A}_{\mathcal{L}} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -2.3400 & 0 & -2.9589 \\
0.4031 & 7.3096 & 0 & 0 & 6.7709 & 0 \\
0 & 0 & 0 & -14.7513 & -3.7747 & 5.7786 \\
0 & 0 & 0 & 0 & 0 & 5.6890 \\
0 & 0 & -2.5565 & 8.9564 & 0 & 1.1844
\end{bmatrix}
$$

We experiment with both the diagonal starting value $\Lambda_0 = \text{diag}\{1, \ldots, 6\}$ and a (randomly generated) triangular starting matrix, say,

$$
X_0 = \begin{bmatrix}
1 & -2.3127 & -10.2326 & -6.8039 & -8.5953 & 2.1713 \\
0 & 2 & 9.3754 & -17.2577 & -7.5225 & -3.7346 \\
0 & 0 & 3 & 8.1320 & 12.2962 & -8.3203 \\
0 & 0 & 0 & 4 & 11.5075 & 2.8687 \\
0 & 0 & 0 & 0 & 5 & -18.1889 \\
0 & 0 & 0 & 0 & 0 & 6
\end{bmatrix}.
$$

The numerical results are summarized in Table 2. We note in particular that even if

15

Table 2
Summary of Numerical Experiment with Example 3.

Flow	$X_0 = \Lambda_0$, diagonal	$X_0 = \Lambda_0 + T_0$, upper triangular	Figure
(11)	mildly ill-cond. w/o restart	$\kappa_F(V) \to c_0$, $\nu^2(V) \to m > 6$	5
(19), $\alpha \in [0, 10^{-8})$	conv., high cond.	similar to (11)	5_{top}
(20), $\beta \in [0, 10^{-8})$	conv., high cond.	similar to (11)	5_{bottom}
(19), $\alpha \in [10^{-8}, 10^{-2})$	LS solution	slow conv., $\kappa_F(V) \to c_1 \approx 6$, $\nu^2(V) \to p \leq m$	5_{top}
(20), $\beta \in [10^{-8}, 10^{-2})$	LS solution	slow conv., $\kappa_F(V) \to c_2 < c_0$, $\nu^2(V) \to p \leq m$	5_{bottom}
(19), $\alpha \geq 10^{-2}$	LS solution	LS solution, $\kappa_F(V) \to c_1 \approx 6$, $\nu^2(V) \to 6$	5_{top}
(20), $\beta \geq 10^{-2}$	LS solution	$\kappa_F(V) \to c_2 < c_0$, $\nu^2(V) \to 6$	5_{bottom}

a larger weight is used, the flow (19) tends to a least squares solution while the flow (20) always attain a solution with minimum $\nu^2(V^*)$. We observe that the second flow (20) seems to converge more consistently with respect to different choices of triangular initial values. The convergence histories are shown in Figure 5.

5   Conclusion

Matrix completion with prescribed spectrum has presented a classical challenge in the literature. Most of the attention has been centering around cases where the prescribed entries at special locations. Beyond the Hershkowitz theorem that, with $|\mathcal{L}| = 2n - 3$, appears to be the most general result at present, little is known about the PEIEP either theoretically or numerically. In this paper, we propose using dynamical systems to track trajectories leading to a completed matrix numerically. The framework is applicable even under the situation when no existence theory is available at all. Extensive numerical experiments seem to suggest that our idea of gradient flow approach can serve as a reasonable means to tackle the most general PEIEPs where the prescribed entries are at arbitrary locations with arbitrary cardinalities. Work to be done includes more efficient integration schemes to track down the asymptotically stable equilibrium points of the dynamical systems and further mathematical analysis of these equilibrium points to see whether an existence theory of the underlying PEIEP can be established.

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