Constructing a Hermitian Matrix from Its Diagonal Entries and Eigenvalues

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Abstract

Given two vectors $a, \lambda \in R^n$, the Schur-Horn theorem states that a majorizes λ if and only if there exists a Hermitian matrix H with eigenvalues λ and diagonal entries a. While the theory is regarded as classical by now, the known proof is not constructive. To construct a Hermitian matrix from its diagonal entries and eigenvalues therefore becomes an interesting and challenging inverse eigenvalue problem. Two algorithms for determining the matrix numerically are proposed in this paper. The lift and projection method is an iterative method which involves an interesting application of the Wielandt-Hoffman theorem. The projected gradient method is a continuous method which, besides its easy implementation, offers a new proof of existence because of its global convergence property.

Key words. Schur-Horn theorem, majorization, inverse eigenvalue problem, lift and projection, projected gradient.

AMS(MOS) subject classifications. 65F15, 65H15.

1. Introduction.

The well known Schur-Horn theorem [14] deals with the relationships between the main diagonal entries and eigenvalues of a Hermitian matrix. For the reference, we restate the theorem in the following form [15, Theorems 4.3.26, 4.3.32 and 4.3.33] :

THEOREM 1.1. *(Schur-Horn Theorem)*

1. Let H be a Hermitian matrix. Let $\lambda = [\lambda_i] \in R^n$ and $a = [a_i] \in R^n$ denote the vectors of eigenvalues and diagonal entries of H, respectively. If the entries are arranged in increasing order $a_{j_1} \leq \ldots \leq a_{j_n}$, $\lambda_{m_1} \leq \ldots \leq \lambda_{m_n}$, then

$$\sum_{i=1}^{k} a_{j_i} \geq \sum_{i=1}^{k} \lambda_{m_i}, \tag{1}$$

for all $k = 1, 2, \ldots, n$ with equality for $k = n$.

2. Given any $a, \lambda \in R^n$ satisfying (1), there exists a Hermitian matrix H with eigenvalues λ and diagonal entries a.

The notion of (1) is also known as *a majorizing* λ, which has arisen as the precise relationship between two sets of numbers in many areas of disciplines, including matrix theory and statistics. There are extensive research results on this subject. See, for example, [2, 16] and the references contained therein. The Schur-Horn theorem itself has many important applications. For instance, through the Schur-Horn theorem the total least squares problem can be seen to be equivalent to a linear programming problem [3]. Some other applications can be found, for example, in [6, 7].

The second part of the Schur-Horn Theorem gives rise to an interesting inverse eigenvalue problem, namely, to construct such a Hermitian matrix from the given eigenvalues and diagonal entries. For convenience, we shall refer to this problem as (SHIEP). The proof of existence is usually known as the harder part of the Schur-Horn Theorem. One point worthy of note is that there are far more variable in the (SHIEP) than constraints, which presumably implies that the solution is far from unique. It turns out that most of the proofs in the literature are not practicable for the (SHIEP) in that a construction by mathematical induction, if possible at all, would be overwhelmingly complicated. See, for example, [14, 15, 17]. In this paper we propose numerical algorithms that are different from the classical ways of authenticating the existence.

Henceforth, we shall denote the diagonal matrix whose main diagonal entries are the same as those of the matrix M as $\text{diag}(M)$, and the diagonal matrix whose diagonal entries are formed from the vector v as $\text{diag}(v)$. This notation will prove to be convenient in the discussion. Any ambiguity can be clarified from the context. Also, we shall define

$$\mathcal{T}(a) := \{T \in R^{n\times n} | \text{diag}(T) = \text{diag}(a)\} \tag{2}$$

and

$$\mathcal{M}(\Lambda) := \{Q^T \Lambda Q | Q \in \mathcal{O}(n)\} \tag{3}$$

where $\Lambda := \text{diag}(\lambda)$ and $\mathcal{O}(n)$ is the group of all orthogonal matrices in $R^{n\times n}$.

Our algorithms are based on the idea of finding the shortest distance between $\mathcal{T}(a)$ and $\mathcal{M}(\Lambda)$, i.e., we want to solve

$$\min_{T\in\mathcal{T}(a), Z\in\mathcal{M}(\Lambda)} \|T-Z\|_F \tag{4}$$

where $\|\cdot\|_F$ means the Frobenius norm. Clearly, the Schur-Horn Theorem attests that $\mathcal{T}(a)$ and $\mathcal{M}(\Lambda)$ intersect and hence the minimal value of (4) should be zero. Our goal is to, starting with an arbitrary point on either $\mathcal{T}(a)$ or $\mathcal{M}(\Lambda)$, find the intersection point.

The (SHIEP) is fundamentally different from a classical inverse eigenvalue problem (CIEP) that has been discussed, for example, in [12]. Given symmetric matrices $A_0, A_1, \ldots A_n$, the (CIEP) is to find a vector $c \in R^n$ such that the matrix $A(c)$ where

$$A(c) := A_0 + c_1 A_1 + \ldots + c_n A_n \tag{5}$$

has the prescribed spectrum λ. In relating the (CIEP) to the (SHIEP), one has to specify each A_i of the basis matrices. To characterize the matrices $A_1, \ldots, A_n$ a priori so that a solution to the (SHIEP) may be written in the form of (5), however, is by no means easy. One may select, for example, $A_0 = \operatorname{diag}(a)$ and all other A_i, $1 \le i \le n$, such that $\operatorname{diag}(A_i) = 0$. Even so, the off-diagonal entries of A_i are still completely free. Picking the remaining part of A_i arbitrarily would be an absurd thing to do since the resulting (CIEP) may very well not have a solution at all. In fact, one can easily construct a 2×2 example to demonstrate a *near miss* case where A_i's are such that a certain combination $A(c)$ from a special c is very near a solution of the (SHIEP), yet the entire affine subspace of $A(c)$ from every possible c does not intersect $\mathcal{M}(\Lambda)$. One may then wonder why not to exploit the freedom in the (SHIEP) by further restricting the structure of the matrix. For example, it seems to be a sensible requirement that the matrix being constructed should be a Jacobi matrix, since there are really $2n-1$ given data elements (both a and λ hav the same sum.) Again, one can easily check that there is no real numbers b_1, b_2 so that the 3×3 matrix

$$\begin{bmatrix} 1 & b_1 & 0 \\ b_1 & 2 & b_2 \\ 0 & b_2 & 3 \end{bmatrix}$$

will have eigenvalues $\{-5, -4, 15\}$. That is, the (SHIEP) for structured matrices is not necessarily solvable. It is an interesting question to study what additional conditions must be imposed in order that a more specified problem to have a solution, but in this paper attention is paid only to the (SHIEP) arisen from the Schur-Horn Theorem. Consequently, until $A_1, \ldots, A_n$ are properly selected, any numerical techniques proposed for (CIEP) are not directly applicable for the (SHIEP).

In contrast, a much easier iterative method that alternates points between $\mathcal{T}$ and $\mathcal{M}(\Lambda)$ is possible. This procedure, called *lift-and-projection* for the reason that will become clear from the geometry, will be discussed in section 2. The lift and projection method is essentially the same as the so called alternating projection method

[4, 8, 11, 13] except that the latter requires the underlying sets to be convex. The set $\mathcal{M}(\Lambda)$ is not convex. Nevertheless, we shall see for our problem that the so called *proximity map* can still be formulated. In particular, the projection is almost free and the Wielandt-Hoffman theorem makes the action of lifting possible at the cost of a spectral decomposition per step. We think this connection is worth mentioning even though the rate of convergence is expected to be linear only.

Our main contribution in this paper is in the construction of a gradient flow on the surface $\mathcal{M}(\Lambda)$ that moves toward the desired intersection point. No spectral decomposition is needed. Our approach is similar to that developed in [9] with slight modifications, but the application to the Schur-Horn theorem is apparently new. The gradient flow is derived in Section 3. We should emphasize that our goal in this paper is not to redo the proof of the Schur-Horn theorem, but rather to develop an algorithm that can compute the results promised by the theorem. On the other hand, if we can show that our algorithm always finds a solution, then in return we have indeed offered a different proof for the Schur-Horn theorem. Numerical examples are demonstrated in Section 4.

2. Lift and Projection.

In [10] we have introduced a notion that interprets numerical methods proposed in [12] for the (CIEP) as a *coordinate-free* Newton method. For the (SHIEP), however, one quickly discover that the same idea does not work. The search for a $\mathcal{T}$-intercept of a tangent array from $\mathcal{M}(\Lambda)$ amounts to a nonlinear system of $\frac{n(n+1)}{2}$ equations in $n(n-1)$ unknowns. When $n > 3$, this is an underdetermined system. Unlike those methods discussed in [12], there is no clear strategy on how the this system could be solved. In this section, we replace the concept of "tangent" by that of "projection" and propose an analogous but easier iteration method, called lift and projection.

The main idea is to alternate between $\mathcal{T}$ and $\mathcal{M}(\Lambda)$ in the following way: From a given $T^{(k)} \in \mathcal{T}$, first we find the point $Z^{(k)} \in \mathcal{M}(\Lambda)$ such that $\|T^{(k)} - Z^{(k)}\|_F = \mathrm{dist}(T^{(k)}, \mathcal{M}(\Lambda))$. Then we find $T^{(k+1)} \in \mathcal{T}$ such that $\|T^{(k+1)} - Z^{(k)}\|_F = \mathrm{dist}(\mathcal{T}, Z^{(k)})$. Here, as usual, the distance is measured in the Frobenius norm. A schematic diagram of the iteration is illustrated in Figure 1, even though the topology of $\mathcal{M}(\Lambda)$ is much more complicated. We call $Z^{(k)}$ a *lift* of $T^{(k)}$ onto $\mathcal{M}(\Lambda)$ and $T^{(k+1)} \in \mathcal{T}$ a *projection* of $Z^{(k)}$ onto $\mathcal{T}$.

The projection is easy to formulate. In fact, the projection $T = [t_{ij}]$ of any $Z = [z_{ij}] \in \mathcal{M}(\Lambda)$ onto $\mathcal{T}$ must be given by

$$t_{ij} := \begin{cases} z_{ij}, & \text{if } i \neq j \\ a_i, & \text{if } i = j. \end{cases} \tag{6}$$

The Wielandt-Hoffman theorem [15, Theorem 6.3.5], on the other hand, furnishes a mechanism for lifting. For demonstration purpose, we shall assume that both Λ and the given $T \in \mathcal{T}$ have simple spectrum. (For the case of multiple eigenvalues, the result only needs a slight modification.) Suppose $T = Q^T D Q$ is a spectral decomposition of T where D is a diagonal matrix of eigenvalues. Then the shortest distance between T

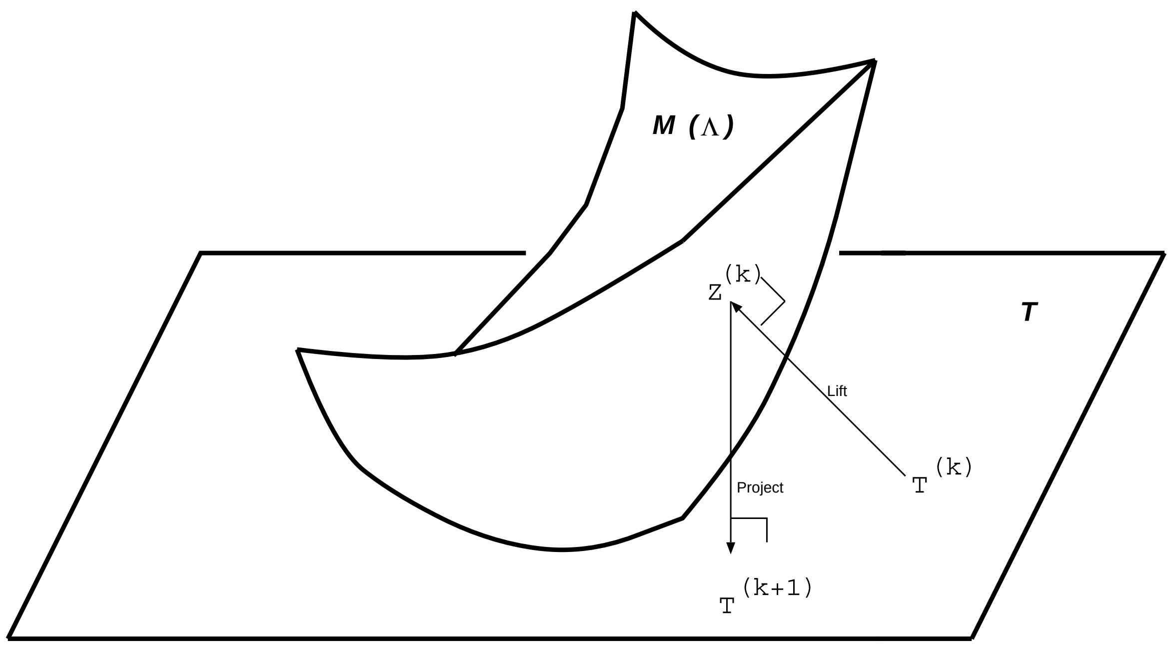

FIG. 1. *Geometric sketch of lifting and projection*

and $\mathcal{M}(\Lambda)$ is attained at the point (i.e., the lift of T onto $\mathcal{M}(\Lambda)$)

$$Z := Q^T \text{diag}(\lambda_{\pi_1}, \dots, \lambda_{\pi_n}) Q \tag{7}$$

where π is a permutation so that $\lambda_{\pi_1}, \dots, \lambda_{\pi_n}$ are in the same algebraic ordering as the diagonal entries in D. The justification on why this assertion is correct has already been proved in [7, 9].

Since in either step of lifting or projection we are minimizing the distance between a point and a set, we have

$$\|T^{(k+1)} - Z^{(k+1)}\|_F^2 \leq \|T^{(k+1)} - Z^{(k)}\|_F^2 \leq \|T^{(k)} - Z^{(k)}\|_F^2. \tag{8}$$

The lift and projection clearly is a descent method. The sequence $\{(T^{(k)}, Z^{(k)})\}$ will converge to a stationary point for the problem (4).

Because $\mathcal{M}(\Lambda)$ is not a convex set, a stationary point for (4) is not necessarily an intersection point of $\mathcal{T}$ and $\mathcal{M}(\Lambda)$. This is a major difference between our method and the alternating projection method [4, 8, 11, 13]. On the other hand, the application of the Wielandt-Hoffman theorem to formulate the proximity map despite of the non-convexity is remarkably simple and quite interesting. The rate of convergence being expected to be linear only, this method might not be very efficient.

3. Gradient Flow.

We now consider a continuous approach for the (SHIEP) that does not need any spectral decomposition. Similar to (4), our idea now is to solve the problem

$$\min_{Q \in \mathcal{O}(n)} F(Q) := \frac{1}{2}\|\mathrm{diag}(Q^T \Lambda Q) - \mathrm{diag}(a)\|_F^2. \tag{9}$$

The Schur-Horn theorem guarantees that there exists a Q at which F vanishes. We now explain how to improve the matrix Q when $F(Q)$ is not minimal.

In terms of the Frobenius inner product

$$< A, B >:= \sum_{i,j} a_{ij} b_{ij}, \tag{10}$$

the Fréchet derivative of F at Q acting on an arbitrary matrix $U \in R^{n\times n}$ can be calculated as follows:

$$\begin{aligned} F'(Q)U &= 2\langle \mathrm{diag}(Q^T\Lambda Q) - \mathrm{diag}(a), \mathrm{diag}(Q^T \Lambda U)\rangle \\ &= 2\langle \mathrm{diag}(Q^T\Lambda Q) - \mathrm{diag}(a), Q^T \Lambda U\rangle \\ &= 2\langle \Lambda Q(\mathrm{diag}(Q^T\Lambda Q) - \mathrm{diag}(a), U\rangle. \end{aligned} \tag{11}$$

The second equality in (11) follows from the observation that the first entry in the inner product is a diagonal matrix which results in the same inner product with either $\mathrm{diag}(Q^T\Lambda U)$ or $Q^T\Lambda U$. The third equality follows from the adjoint property of the inner product. The gradient ∇F at Q can now be represented as

$$\nabla F(Q) = 2\Lambda Q \beta(Q) \tag{12}$$

with $\beta(Q) := \mathrm{diag}(Q^T\Lambda Q) - \mathrm{diag}(a)$.

Once we have (12), the entire framework developed in [9] for projected gradient method can be applied. In particular, the projected gradient is readily available.

THEOREM 3.1. *The projection $g(Q)$ of $\nabla F(Q)$ onto $\mathcal{O}(n)$ is given by*

$$g(Q) = Q[Q^T\Lambda Q, \beta(Q)] \tag{13}$$

where $[A, B] := AB - BA$ is the Lie bracket.

Proof. See [9, formulas (20), (21) and (22)]. □

We may also calculated the projected Hessian as follows.

THEOREM 3.2. *Let $Q \in \mathcal{O}(n)$ be a stationary point of (9). Any tangent vector H of the manifold $\mathcal{O}(n)$ at Q is of the form $H = QK$ for some skew symmetric matrix K. The projected Hessian $g'(Q)$ acting on H is given by*

$$< g'(Q)QK, QK >=< \mathrm{diag[Q^T\Lambda Q, K] - [\beta(Q), K], [Q^T \Lambda Q, K]} > . \tag{14}$$

Proof. See [9, formulas (27), (28) and (29)]. □

The vector field

$$\dot{Q} = -g(Q) \tag{15}$$

defines a steepest descent flow on the manifold $\mathcal{O}(n)$ for the function $F(Q)$. Let $X := Q^T \Lambda Q$ and $\alpha(X) := \beta(Q) = \text{diag}(X) - \text{diag}(a)$, then correspondingly

$$\dot{X} = [X, [\alpha(X), X]] \tag{16}$$

defines an isospectral flow on $\mathcal{M}(\Lambda)$ that moves to decrease the distance between $\text{diag}(X)$ and $\text{diag}(a)$.

The algorithm for solving the (SHIEP) is then simply to integrate the differential equation (16) from a starting point $X_0 \in \mathcal{M}(\Lambda)$.

We now take a closer look at where the flow of (16) will converge to. By the way we construct (16), a natural Lyapunov's function

$$G(t) := \frac{1}{2}\|\text{diag}(X(t)) - \text{diag}(a)\|_F^2 \tag{17}$$

exists. Lyapunov's second method [5, Theorem5.5] implies that a limit point of (16) must satisfy $[\alpha(X), X] = 0$. For simplicity, we shall consider only the generic case where all $\lambda_1, \ldots \lambda_n$ are distinct. The case with some eigenvalues equal to each other is a little bit more complicate to analyze, although our numerical experiences seem to indicate that the convergence behavior should be similar. Under our assumption, a stationary point Q of (9) necessarily corresponds to an equilibrium point $X = Q^T \Lambda Q$ of (16), and vise versa.

Recall that $\beta(Q) = \text{diag}(Q^T \Lambda Q) - \text{diag}(a)$. Obviously if $\beta(Q) = 0$ at a stationary point Q, then the corresponding $X = Q^T \Lambda Q$ is a solution to the (SHIEP). Indeed, we have the following observation that shows the stationary point Q in this case satisfies the second order *necessary* condition for being a minimum of (9).

THEOREM 3.3. *If $\beta(Q) = 0$ for some $Q \in \mathcal{O}(n)$, then for all skew-symmetric matrices K it is true that $< g'(Q)QK, QK > = \|\text{diag}[\mathrm{Q}^{\mathrm{T}}\Lambda\mathrm{Q}, \mathrm{K}]\|_{\mathrm{F}}^2 \geq 0$. In other words, the projected Hessian of F at Q is positive semi-definite.*

Proof. The result follows directly from (14) and the definition of Frobenius inner product (10). □

The strict inequality is not true in the above theorem. In fact, if we denote $\Omega := \text{diag}[X, K] = \text{diag}\{\omega_1, \ldots, \omega_n\}$, then

$$\omega_i = \sum_{s=1}^{i-1} x_{si} k_{si} - \sum_{t=i+1}^{n} x_{it} k_{it}. \tag{18}$$

Let X be fixed. Since $\sum_{i=1}^n \omega_i = 0$, the system $\omega_i = 0$ for $i = 1, \ldots, n$ contains only $n-1$ independent equations in the $\frac{n(n-1)}{2}$ unknowns k_{ij}. We should be able to find non-trivial skew symmetric matrix K that makes $\Omega = 0$. However, we now show that the X's such that $\beta(Q) = 0$ are the only possible *asymptotically stable* equilibrium point for (16).

THEOREM 3.4. *If $\beta(Q) \neq 0$ at a stationary point Q, then there exists a skew-symmetric matrix K such that $< g'(Q)QK, QK >< 0$. Thus, Q cannot be a local minimum of (9).*

Proof. Transforming similarly by a permutation matrix if necessary, we may assume that $\beta(Q)$ is of the form

$$\beta(Q) = \mathrm{diag}\{\beta_1 I_{n_1}, \ldots, \beta_k I_{n_k}\} \tag{19}$$

where I_{n_i} is the $n_i \times n_i$ identity matrix and $\beta_1 > \ldots > \beta_k$. Since $[Q^T \Lambda Q, \beta(Q)] = 0$, it follows that $X = Q^T \Lambda Q$ must be block diagonal of the form

$$X = \mathrm{diag}\{X_{11}, \ldots, X_{kk}\} \tag{20}$$

where each X_{ii} is a real symmetric matrix of size $n_i \times n_i$. Define $E := Q\beta(Q)Q^T$. Since $[\Lambda, E] = 0$ and all entries of Λ are distinct, E is a diagonal matrix whose entries $E = \mathrm{diag}(e_1, \ldots, e_n)$ are simply a permutation of those of $\beta(Q)$. Note that Q^T is the orthogonal matrix of eigenvectors of X. So Q must also have the same structure as X. For any $n \times n$ skew-symmetric matrix $K = [K_{ij}]$, partitioned in the same way as in (20), that satisfies $K_{ii} = 0$ for all $i = 1, \ldots, k$, it is easy to check that $\mathrm{diag}[Q^T \Lambda Q, K] = 0$. The projected Hessian now becomes

$$\begin{aligned} < g'(Q)QK, QK > &= -< [\beta(Q), K], [Q^T \Lambda Q, K] > \\ &= -< E\tilde{K} - \tilde{K}E, \Lambda\tilde{K} - \tilde{K}\Lambda > \\ &= -2\sum_{i<j} (\lambda_i - \lambda_j)(e_i - e_j)\tilde{k}_{ij}^2 \end{aligned} \tag{21}$$

where $\tilde{K} = [\tilde{k}_{ij}] := QKQ^T$ is still a skew symmetric matrix with the same structure as K. From (21), it is now easy to pick up values of $\tilde{k}_{ij}$ so that $< g'(Q)QK, QK >< 0$. $\square$

Theorem 3.4 implies that at a stationary point Q where $\beta(Q) \neq 0$ there exists a certain direction along which the functional value F is increasing. The corresponding equilibrium point $X = Q^T \Lambda Q$, therefore, has at least one unstable (repelling) direction. To converge to such an unstable equilibrium point, the descent flow $X(t)$ must stay on very special directrices which form a measure zero, nowhere dense subset in R^n. From the numerical analysis standpoint, this kind of equilibrium point will never be realized because computation along the directrix can easily be derailed by round-off errors and hence pushed away from the unstable equilibrium point.

4. Numerical Examples.

Since $\alpha(X)$ is a diagonal matrix, all diagonal matrices on $\mathcal{M}(\Lambda)$ are equilibrium points for (16). Thus one should avoid using Λ as the initial value X_0. One way to generate a reasonable initial value is by defining $X_0 := Q^T \Lambda Q$ with Q a random orthogonal matrix. There are many techniques for generating such a Q [1, 19].

All the following computations are done on a DECstation 5000/200 with double precision. But we display all the numbers with only five digits so as to fit the data comfortably in the running text.

EXAMPLE 1. To simulate reasonable test data, we start with a randomly generated symmetric matrix

$$M_0 = \begin{bmatrix} 4.3792\times10^{-1} & 4.3055\times10^{-1} & 1.2086\times10^{+0} & 1.0968\times10^{+0} & 1.4616\times10^{+0} \\ 4.3055\times10^{-1} & 1.0388\times10^{+0} & 1.5021\times10^{+0} & 7.2134\times10^{-1} & 1.4543\times10^{-1} \\ 1.2086\times10^{+0} & 1.5021\times10^{+0} & 1.5396\times10^{-2} & 9.7239\times10^{-1} & 7.2076\times10^{-1} \\ 1.0968\times10^{+0} & 7.2134\times10^{-1} & 9.7239\times10^{-1} & 1.8609\times10^{+0} & 1.2622\times10^{+0} \\ 1.4616\times10^{+0} & 1.4543\times10^{-1} & 7.2076\times10^{-1} & 1.2622\times10^{+0} & 1.4024\times10^{+0} \end{bmatrix}.$$

The diagonal entries

$$a = [4.3792\times10^{-1}, 1.0388\times10^{+0}, 1.5396\times10^{-2}, 1.8609\times10^{+0}, 1.4024\times10^{+0}]$$

and eigenvalues

$$\lambda = [-1.4169\times10^{+0}, -5.6698\times10^{-1}, 4.3890\times10^{-1}, 1.4162\times10^{+0}, 4.8842\times10^{+0}]$$

of M_0 are used as the test data. Now we randomly generate an orthogonal matrix

$$Q_1 = \begin{bmatrix} -6.4009\times10^{-1} & -5.3594\times10^{-1} & -1.8454\times10^{-1} & -3.3375\times10^{-2} & -5.1757\times10^{-1} \\ 2.1804\times10^{-1} & -1.2359\times10^{-1} & -5.0336\times10^{-1} & -8.2193\times10^{-1} & 9.0802\times10^{-2} \\ -7.2099\times10^{-1} & 5.6072\times10^{-1} & 1.4302\times10^{-2} & -2.4876\times10^{-1} & 3.2199\times10^{-1} \\ 2.8417\times10^{-3} & -1.9828\times10^{-1} & 8.4401\times10^{-1} & -4.9375\times10^{-1} & -6.7297\times10^{-2} \\ -1.5134\times10^{-1} & -5.8632\times10^{-1} & 3.0406\times10^{-3} & 1.3284\times10^{-1} & 7.8464\times10^{-1} \end{bmatrix},$$

and define $X_0 = Q_1^T \Lambda Q_1$. We use the subroutine ODE in [18] as the integrator where local control parameters ABSERR and RELERR in ODE are set to be 10^{-12}. We examine the output values at time interval of 1, and assume the path has reached an equilibrium point when two consecutive output points are within a distance of 10^{-10}. At $t \approx 11$, the gradient flow converges to the matrix

$$M_1 = \begin{bmatrix} 4.3792\times10^{-1} & 2.6691\times10^{-1} & -1.9178\times10^{-1} & -6.1356\times10^{-1} & -1.5920\times10^{+0} \\ 2.6691\times10^{-1} & 1.0388\times10^{+0} & -7.2845\times10^{-1} & -8.6726\times10^{-1} & -1.9618\times10^{+0} \\ -1.9178\times10^{-1} & -7.2845\times10^{-1} & 1.5396\times10^{-2} & -6.3601\times10^{-1} & 1.6256\times10^{-1} \\ -6.1356\times10^{-1} & -8.6726\times10^{-1} & -6.3601\times10^{-1} & 1.8609\times10^{+0} & 1.5032\times10^{+0} \\ -1.5920\times10^{+0} & -1.9618\times10^{+0} & 1.6256\times10^{-1} & 1.5032\times10^{+0} & 1.4024\times10^{+0} \end{bmatrix}.$$

The flow in theory should stay in the surface $\mathcal{M}(\Lambda)$. The eigenvalues of M_1 are checked to agree with λ to 10 digits.

If we use another random orthogonal matrix

$$Q_2 = \begin{bmatrix} -4.7879\times10^{-1} & 8.7948\times10^{-2} & -4.1424\times10^{-3} & 3.0041\times10^{-1} & 8.2022\times10^{-1} \\ -4.1099\times10^{-1} & -5.7368\times10^{-1} & -6.8750\times10^{-1} & -7.0455\times10^{-2} & -1.5607\times10^{-1} \\ 1.7225\times10^{-1} & 6.1511\times10^{-1} & -6.1521\times10^{-1} & -4.2281\times10^{-1} & 1.8634\times10^{-1} \\ -7.1440\times10^{-1} & 2.5656\times10^{-1} & 3.2325\times10^{-1} & -5.0226\times10^{-1} & -2.5895\times10^{-1} \\ 2.4860\times10^{-1} & -4.6795\times10^{-1} & 2.1060\times10^{-1} & -6.8830\times10^{-1} & 4.4845\times10^{-1} \end{bmatrix},$$

then at $t \approx 13$, the gradient flow converges to the matrix

$$M_2 = \begin{bmatrix} 4.3792{\times}10^{-1} & -1.4087{\times}10^{+0} & 4.8811{\times}10^{-1} & -2.0882{\times}10^{+0} & 1.2285{\times}10^{+0} \\ -1.4087{\times}10^{+0} & 1.0388{\times}10^{+0} & 2.3067{\times}10^{-1} & 1.1160{\times}10^{+0} & -8.8543{\times}10^{-1} \\ 4.8811{\times}10^{-1} & 2.3067{\times}10^{-1} & 1.5396{\times}10^{-2} & -7.2958{\times}10^{-2} & 7.2054{\times}10^{-1} \\ -2.0882{\times}10^{+0} & 1.1160{\times}10^{+0} & -7.2958{\times}10^{-2} & 1.8609{\times}10^{+0} & -3.7601{\times}10^{-1} \\ 1.2285{\times}10^{+0} & -8.8543{\times}10^{-1} & 7.2054{\times}10^{-1} & -3.7601{\times}10^{-1} & 1.4024{\times}10^{+0} \end{bmatrix}$$

whose eigenvalues again agree reasonably well with λ.

EXAMPLE 2. Under the same stopping criterion, we repeat the experiment in Example 1 with $2,000$ test data. The diagonal entries a and eigenvalues λ are generated from symmetric matrices with normal distribution entries. The orthogonal matrices Q are generated from the QR decomposition of other stochastically independent (non-symmetric) random matrices [19]. We collect the length of integration required for reaching convergence in each case. This length should be inherent only to the individual problem data (and the stopping criterion), but should be independent of the machine used in computation. The histogram of the lengths is presented in Figure 2 where for better display the frequency distribution is plotted in its natural logarithm. When there is no distribution for a particular length (so the logarithm is negative infinity), the plot is left blank. As can be seen, about 77% of the cases converge with the length of integration less than 7 and about 93% converge with length less than 17. The maximal length of integration occurred in this test is 296. It perhaps is also interesting to note that all the $2,000$ cases converge to a desirable solution. Confirming our previous argument over Theorem 3.4, none of the cases gets trapped at a point where $F(Q) \neq 0$ although this kind of equilibrium points do exist.

EXAMPLE 3. In this example, we experiment with the case when multiple eigenvalues $\lambda = [1, 1, 1, 1, 4]$ are present. We take $a = [1.0749, 1.3309, 1.1197, 2.3035, 2.1709]$. Using the random orthogonal matrix

$$Q = \begin{bmatrix} -4.8713{\times}10^{-2} & -1.3354{\times}10^{-1} & 9.4639{\times}10^{-1} & -1.1419{\times}10^{-1} & 2.6666{\times}10^{-1} \\ 9.8790{\times}10^{-1} & -4.3307{\times}10^{-2} & 7.2187{\times}10^{-2} & -5.1681{\times}10^{-2} & -1.1955{\times}10^{-1} \\ -6.9873{\times}10^{-2} & -4.2957{\times}10^{-1} & 1.8176{\times}10^{-1} & 6.5185{\times}10^{-1} & -5.9384{\times}10^{-1} \\ 3.0930{\times}10^{-2} & -8.5347{\times}10^{-1} & -2.5445{\times}10^{-1} & -8.1527{\times}10^{-2} & 4.4637{\times}10^{-1} \\ 1.2584{\times}10^{-1} & 2.5953{\times}10^{-1} & -3.6892{\times}10^{-2} & 7.4347{\times}10^{-1} & 6.0225{\times}10^{-1} \end{bmatrix},$$

we find that a limit point exists at

$$M = \begin{bmatrix} 1.0749{\times}10^{+0} & -1.5748{\times}10^{-1} & -9.4707{\times}10^{-2} & 3.1254{\times}10^{-1} & 2.9622{\times}10^{-1} \\ -1.5748{\times}10^{-1} & 1.3309{\times}10^{+0} & 1.9903{\times}10^{-1} & -6.5679{\times}10^{-1} & -6.2250{\times}10^{-1} \\ -9.4707{\times}10^{-2} & 1.9903{\times}10^{-1} & 1.1197{\times}10^{+0} & -3.9499{\times}10^{-1} & -3.7437{\times}10^{-1} \\ 3.1254{\times}10^{-1} & -6.5679{\times}10^{-1} & -3.9499{\times}10^{-1} & 2.3035{\times}10^{+0} & 1.2354{\times}10^{+0} \\ 2.9622{\times}10^{-1} & -6.2250{\times}10^{-1} & -3.7437{\times}10^{-1} & 1.2354{\times}10^{+0} & 2.1709{\times}10^{+0} \end{bmatrix}$$

when $t \approx 41$.

EXAMPLE 4. In this example, we consider the case when multiple diagonal entries $a = [1, 1, 1, 1, 1]$ are present. Using $\lambda = [1.9747, 2.3050, 3.8938, -0.8128, -2.3608]$ and

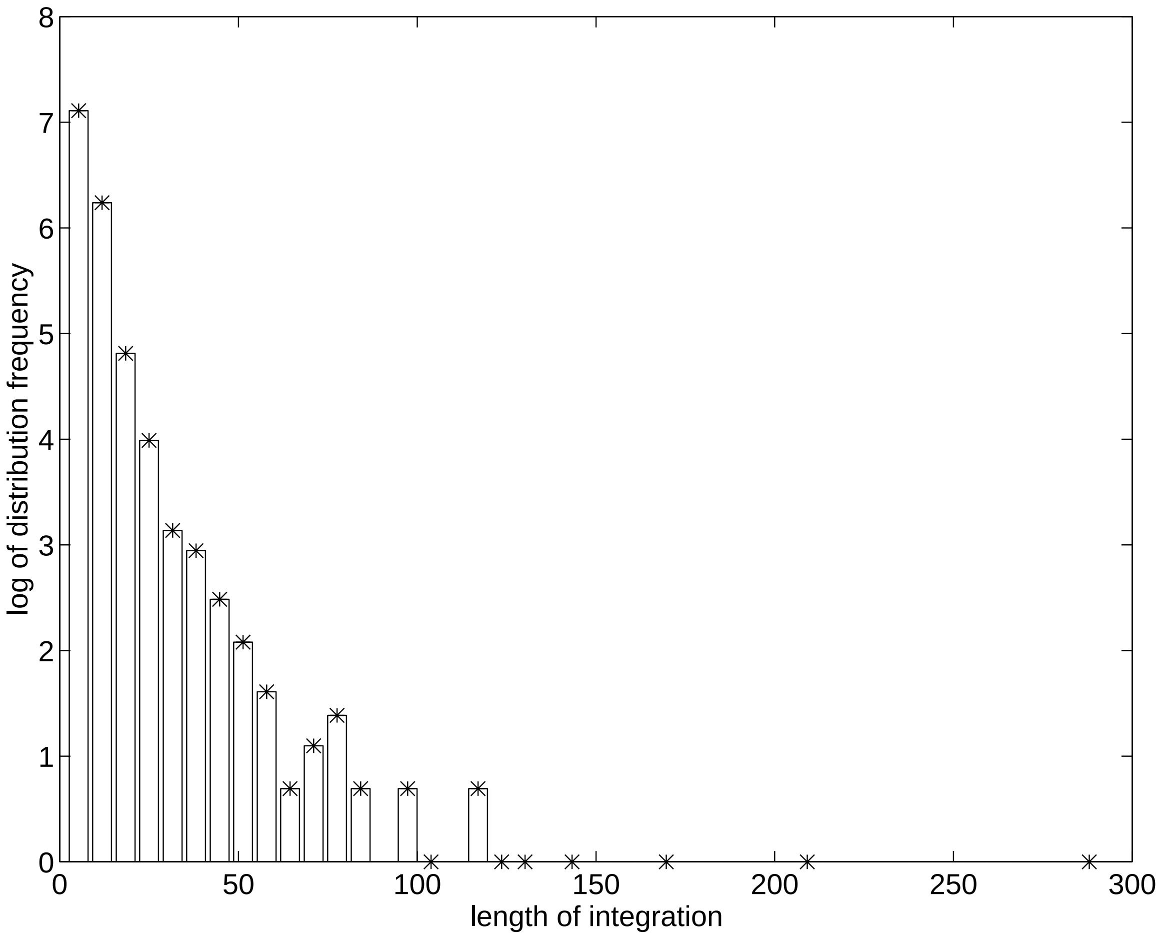

FIG. 2. *Histogram on the lengths of integration required for convergence*

the random orthogonal matrix

$$Q = \begin{bmatrix} -3.3399\times10^{-1} & 2.6628\times10^{-1} & -2.3522\times10^{-1} & -6.6904\times10^{-1} & 5.6089\times10^{-1} \\ -3.5191\times10^{-1} & -8.8924\times10^{-1} & 2.0821\times10^{-1} & -1.9279\times10^{-1} & 6.9964\times10^{-2} \\ 2.6488\times10^{-1} & -3.3460\times10^{-1} & -8.6998\times10^{-1} & 1.8120\times10^{-1} & 1.6787\times10^{-1} \\ 6.2901\times10^{-1} & -1.2402\times10^{-1} & 2.8591\times10^{-2} & -6.7641\times10^{-1} & -3.6141\times10^{-1} \\ 5.4662\times10^{-1} & -1.0493\times10^{-1} & 3.7899\times10^{-1} & 1.5765\times10^{-1} & 7.2230\times10^{-1} \end{bmatrix},$$

we find a limit point exists at

$$M = \begin{bmatrix} 1.0000\times10^{+0} & -1.4905\times10^{+0} & 1.1257\times10^{-1} & -1.4301\times10^{-1} & -1.6216\times10^{+0} \\ -1.4905\times10^{+0} & 1.0000\times10^{+0} & -8.1015\times10^{-2} & -4.5784\times10^{-1} & -7.5669\times10^{-1} \\ 1.1257\times10^{-1} & -8.1015\times10^{-2} & 1.0000\times10^{+0} & 1.4749\times10^{+0} & -2.1841\times10^{+0} \\ -1.4301\times10^{-1} & -4.5784\times10^{-1} & 1.4749\times10^{+0} & 1.0000\times10^{+0} & 4.3081\times10^{-1} \\ -1.6216\times10^{+0} & -7.5669\times10^{-1} & -2.1841\times10^{+0} & 4.3081\times10^{-1} & 1.0000\times10^{+0} \end{bmatrix},$$

when $t \approx 8$.

5. Conclusion.

The Schur-Horn theorem guarantees that the inverse eigenvalue problem of constructing a Hermitian matrix with prescribed diagonal entries and eigenvalues always has a solution. The numerical methods described in [12] will not work in general for finding such a solution. We propose two methods. The lift-and-project method makes a connection with the Wielandt-Hoffman theorem. The gradient flow method can be integrated by any available ordinary differential equation solver. We show the gradient flow method always converges. Numerical experiment seems to suggest that the method works reasonably well.

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