

# SPECTRAL DECOMPOSITION OF SELF-ADJOINT QUADRATIC PENCILS AND ITS APPLICATIONS

DRAFT AS OF March 28, 2007

MOODY T. CHU\* AND SHU-FANG XU†

**Abstract.** Spectral decomposition is of fundamental importance in many applications. Generally speaking, spectral decomposition provides a canonical representation of a linear operator over a vector space in terms of its eigenvalues and eigenfunctions. The canonical form often facilitates discussions which, otherwise, would be complicated and involved. This paper generalizes the classical results of eigendecomposition for self-adjoint linear pencils,  $\mathcal{L}(\lambda) = \lambda - A$  or  $B\lambda - A$ , to self-adjoint quadratic pencils  $\mathcal{Q}(\lambda) = M\lambda^2 + C\lambda + K$ . It is shown that the decomposition involves, in addition to the usual eigeninformation, certain free parameters. These parameters occur in such an intriguing way that properly selected parameters have a variety of interesting applications.

**AMS subject classifications.** 65F15, 15A22, 65F18, 93B55

**Key words.** quadratic pencil, spectral decomposition, inverse eigenvalue problem,

**1. Introduction.** Matrix decomposition has been a useful tool in many disciplines of sciences and engineering. The paradigmatic motive behind developing various kinds of decompositions is that canonical forms into which the original operator has been transformed usually are easier to manipulate. Some commonly used matrix factorizations for various purposes of applications include, for example, the LU decomposition, the Cholesky decomposition, the QR decomposition, the SVD decomposition, the polar decomposition, the Jordan decomposition, and so on [12]. The spectral decomposition involves the representation of an operator in terms of its eigenvalues and eigenvectors. Such a decomposition of linear pencils have important applications such as simplifying the representation of complicated systems, shedding light on the asymptotic behavior of differential equations, or facilitating the performance characterization of numerical algorithms, just to mention a few [6]. This paper concerns itself with the spectral decomposition of real-valued self-adjoint quadratic pencils. We believe that our parametric representation of the spectral decomposition is new. Its efficacy when applied to some challenging inverse problems should evidence the potential impact of our theory.

A classical theory about spectral decomposition is that a symmetric matrix  $A \in \mathbb{R}^{n \times n}$  can always be factorized as the product,

$$A = X\Lambda X^\top, \quad (1.1)$$

where

$$X = [\mathbf{x}_1, \dots, \mathbf{x}_n] \in \mathbb{R}^{n \times n}, \quad (1.2)$$

$$\Lambda = \text{diag}\{\lambda_1, \dots, \lambda_n\} \in \mathbb{R}^{n \times n}, \quad (1.3)$$

satisfy the relationships

$$A\mathbf{x}_i = \lambda_i\mathbf{x}_i, \quad i = 1, \dots, n, \quad (1.4)$$

$$X^\top X = I_n. \quad (1.5)$$

---

\*Department of Mathematics, North Carolina State University, Raleigh, NC 27695-8205. (chu@math.ncsu.edu) This research was supported in part by the National Science Foundation under grants CCR-0204157 and DMS-0505880.

†LMAM, School of Mathematical Sciences, Peking University, Beijing, 100871, China. (xsf@pku.edu.cn) This research was supported in part by NSFC under grant 10571007.

In other words, via solving the linear pencil equation  $\lambda \mathbf{x} - A\mathbf{x} = 0$  as shown in (1.4) and normalizing the eigenvector  $\mathbf{x}$  as indicated in (1.5), we are able to represent the data matrix  $A$  in terms of its spectral decomposition as declared in (1.1). Note that the orthogonality of  $X$  is an extra property due to the symmetry of  $A$ , which we shall explore later.

Consider now the general self-adjoint linear pencil,

$$\mathcal{L}(\lambda) = B\lambda - A, \quad (1.6)$$

where  $A$  and  $B$  are symmetric matrices in  $\mathbb{R}^{n \times n}$  and  $B$  is positive definite. Again, it is a well known fact the coefficients  $(B, A)$  of the pencil can be represented as

$$B = X^{-\top} X^{-1}, \quad (1.7)$$

$$A = X^{-\top} \Lambda X^{-1}, \quad (1.8)$$

where  $X$  and  $\Lambda$  are of the same format as in (1.2) and (1.3), except that they satisfy the relationship

$$A\mathbf{x}_i = \lambda_i B\mathbf{x}_i, \quad i = 1, \dots, n. \quad (1.9)$$

We shall call (1.7) and (1.8) the spectral decomposition of the linear pencil  $(B, A)$  because only the eigenvalues and eigenvectors of  $\mathcal{L}(\lambda)$  are involved in the representation. Note that ‘‘orthogonality condition’’ of  $X$  is built in the relationship (1.7) which, called *B-orthogonality*, can be written as  $X^\top BX = I_n$ .

It is natural to ask the question of generalizing the above notion to self-adjoint quadratic pencils,

$$\mathcal{Q}(\lambda) := M\lambda^2 + C\lambda + K, \quad (1.10)$$

where  $M$ ,  $C$  and  $K$  are real-valued symmetric matrices in  $\mathbb{R}^{n \times n}$  and  $M$  is nonsingular. That is, we want to represent the coefficient matrices  $(M, C, K)$  in terms of eigenvalues and eigenvectors of  $\mathcal{Q}(\lambda)$ . Such a task has been considered in the seminal book by Gohberg, Lancaster and Rodman via the notion of *Jordan chain* [11]. The challenge is that, even with symmetric matrix coefficients, a self-adjoint quadratic pencil  $\mathcal{Q}(\lambda)$  often has complex-valued eigenstructure. A real-valued spectral decomposition for the coefficients of a quadratic pencil in the same spirit of (1.7) and (1.8) for a linear pencil is not obvious. Additionally, we shall see that the spectral decomposition of  $\mathcal{Q}(\lambda)$  carries intrinsically a specially structured parameter matrix. Our main thrust in this paper is to investigate these properties and to demonstrate some interesting applications.

**2. Parameterized Spectral Decomposition.** From the given (1.10), define the linear pencil with the so called Lancaster structure,

$$\mathcal{L}(\lambda) := \mathcal{L}(\lambda; M, C, K) = \begin{bmatrix} C & M \\ M & 0 \end{bmatrix} \lambda - \begin{bmatrix} -K & 0 \\ 0 & M \end{bmatrix}. \quad (2.1)$$

Since  $\mathcal{Q}(\lambda)$  is self-adjoint, so is  $\mathcal{L}(\lambda)$ . It is easy to see that  $\mathcal{Q}(\lambda)$  and  $\mathcal{L}(\lambda)$  are equivalent in the sense that

$$\left( \begin{bmatrix} C & M \\ M & 0 \end{bmatrix} \lambda - \begin{bmatrix} -K & 0 \\ 0 & M \end{bmatrix} \right) \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = 0 \quad (2.2)$$

if and only if

$$\begin{cases} (\lambda C + K)\mathbf{x} + \lambda M\mathbf{y} & = 0, \\ \lambda M\mathbf{x} - M\mathbf{y} & = 0. \end{cases} \quad (2.3)$$

whereas, since  $M$  is nonsingular, we have  $\mathbf{y} = \lambda \mathbf{x}$ . Be cautious that the leading coefficient  $\begin{bmatrix} C & M \\ M & 0 \end{bmatrix}$  of  $\mathcal{L}(\lambda)$  is not positive definite, so the decomposition for (1.10) cannot be answered directly in the same way as in (1.7) and (1.8) for (1.6).

Using the same notation as before without causing ambiguity, assume that we can collect all the eigenvalues and eigenvectors into the matrices,

$$X = [\mathbf{x}_1, \dots, \mathbf{x}_{2n}] \in \mathbb{C}^{n \times 2n}, \quad (2.4)$$

$$\Lambda = \text{diag}\{\lambda_1, \dots, \lambda_{2n}\} \in \mathbb{C}^{2n \times 2n}. \quad (2.5)$$

Then obviously the  $2n \times 2n$  matrix  $\begin{bmatrix} X \\ X\Lambda \end{bmatrix}$  serves as the matrix of eigenvectors for the linear pencil  $\mathcal{L}(\lambda)$  and the algebraic equation

$$MX\Lambda^2 + CX\Lambda + KX = 0 \quad (2.6)$$

is satisfied. The above statement, however, does raise several concerns, including that  $(X, \Lambda)$  might be complex-valued and that the eigenvalues might have high geometric multiplicities and, hence,  $\mathcal{Q}(\lambda)$  does not have a complete set of  $2n$  eigenvectors. To remedy these issues, we make use of the following notion of *standard pair* first introduced in [11].

DEFINITION 2.1. *A pair of matrices  $(\mathfrak{X}, \mathfrak{T}) \in \mathbb{R}^{n \times 2n} \times \mathbb{R}^{2n \times 2n}$  is called a standard pair for the quadratic pencil  $\mathcal{Q}(\lambda)$  if and only if the matrix*

$$U = U(\mathfrak{X}, \mathfrak{T}) := \begin{bmatrix} \mathfrak{X} \\ \mathfrak{X}\mathfrak{T} \end{bmatrix} \quad (2.7)$$

*is nonsingular and the equation*

$$M\mathfrak{X}\mathfrak{T}^2 + C\mathfrak{X}\mathfrak{T} + K\mathfrak{X} = 0 \quad (2.8)$$

*holds.*

One advantage of the standard pair is that it induces eigeninformation of  $\mathcal{Q}(\lambda)$  by way of real-valued matrices. More specifically, we make the following claim.

LEMMA 2.2. *The eigeninformation  $(X, \Lambda)$  of  $\mathcal{Q}(\lambda)$  is completely attainable through the standard pair  $(\mathfrak{X}, \mathfrak{T})$  and vice versa.*

*Proof.* Observe that

$$\mathcal{Q}(\lambda)\mathbf{x} = 0 \Leftrightarrow (\mathcal{L}(\lambda)U)U^{-1} \begin{bmatrix} \mathbf{x} \\ \lambda\mathbf{x} \end{bmatrix} = 0.$$

From the equivalent expression of (2.8),

$$\begin{bmatrix} C & M \\ M & 0 \end{bmatrix} \begin{bmatrix} \mathfrak{X} \\ \mathfrak{X}\mathfrak{T} \end{bmatrix} \mathfrak{T} = \begin{bmatrix} -K & 0 \\ 0 & M \end{bmatrix} \begin{bmatrix} \mathfrak{X} \\ \mathfrak{X}\mathfrak{T} \end{bmatrix}. \quad (2.9)$$

we see that

$$\mathcal{L}(\lambda)U = \begin{bmatrix} C & M \\ M & 0 \end{bmatrix} U(\lambda I - \mathfrak{T}).$$

It follows that

$$\mathcal{Q}(\lambda)\mathbf{x} = 0 \Leftrightarrow (\lambda I - \mathfrak{T}) \left( U^{-1} \begin{bmatrix} \mathbf{x} \\ \lambda\mathbf{x} \end{bmatrix} \right) = 0.$$

The last expression not only shows that the matrix  $\mathfrak{T}$  has exactly the same spectrum as the pencil  $\mathcal{Q}(\lambda)$  but also indicates how the associated eigenvectors are related.  $\square$

To derive the spectral decomposition for  $\mathcal{Q}(\lambda)$ , it will be convenient to introduce the parameter matrix,

$$S = S(\mathfrak{x}, \mathfrak{T}) := \left( U^\top \begin{bmatrix} C & M \\ M & 0 \end{bmatrix} U \right)^{-1}. \quad (2.10)$$

Note that  $S$  is symmetric and is a function of the standard pair  $(\mathfrak{x}, \mathfrak{T})$ . The intertwinement of matrices and the inverse in the definition of  $S$  give evidence to precisely our point that the spectral decomposition of a quadratic pencil is not an easy subject. In view of (1.7), the matrix  $S^{-1}$  may be considered as a normalization factor for the “eigenvectors”  $U$  of the linear pencil  $\mathcal{L}(\lambda)$ . In (1.7),  $S^{-1}$  is taken to be the identity and, hence, the matter is simpler. The price we pay for having no parameter in (1.7), however, is less flexibility. Using  $S$  as a parameter, we now state our main result in terms of a standard pair.

**THEOREM 2.3.** *Given a self-adjoint quadratic pencil  $\mathcal{Q}(\lambda) := M\lambda^2 + C\lambda + K$ , let  $(\mathfrak{x}, \mathfrak{T})$  be a standard pair. Then the pencil enjoys a spectral decomposition in the sense that its matrix coefficients  $(M, C, K)$  can be factorized in terms of  $(\mathfrak{x}, \mathfrak{T})$  as follows:*

$$M = (\mathfrak{x}\mathfrak{T}S\mathfrak{x}^\top)^{-1}, \quad (2.11)$$

$$C = -M\mathfrak{x}\mathfrak{T}^2S\mathfrak{x}^\top M, \quad (2.12)$$

$$K = -M\mathfrak{x}\mathfrak{T}^3S\mathfrak{x}^\top M + CM^{-1}C. \quad (2.13)$$

*Proof.* By the definition of  $S$ , we trivially have

$$\begin{bmatrix} \mathfrak{x} \\ \mathfrak{x}\mathfrak{T} \end{bmatrix} S\mathfrak{x}^\top M = \begin{bmatrix} 0 \\ I_n \end{bmatrix}. \quad (2.14)$$

The expression (2.11) immediately follows. We also have a useful fact

$$\mathfrak{x}S\mathfrak{x}^\top = 0. \quad (2.15)$$

Post-multiplying (2.8) by  $S\mathfrak{x}^\top$  and applying (2.11) and (2.15), we obtain (2.12). Similarly, post-multiplying (2.8) by  $\mathfrak{T}S\mathfrak{x}^\top$ , we obtain

$$M\mathfrak{x}\mathfrak{T}^3S\mathfrak{x}^\top + C\mathfrak{x}\mathfrak{T}^2S\mathfrak{x}^\top + K\mathfrak{x}\mathfrak{T}S\mathfrak{x}^\top = 0.$$

Upon substitution by (2.11) and (2.12), the representation of  $K$  by (2.13) follows.  $\square$

The “orthogonality” of  $\mathfrak{x}$  with respect to  $S$  as indicated in (2.15) in the above proof is not a coincidence. It is worthy to point out another useful fact. By (2.9) we see that the product,

$$S^{-1}\mathfrak{T} = U^\top \begin{bmatrix} -K & 0 \\ 0 & M \end{bmatrix} U,$$

is symmetric. It follows that, given a standard pair  $(\mathfrak{x}, \mathfrak{T})$ , the product  $\mathfrak{T}S$  is symmetric. Together with the fact that  $S$  is symmetric, we can prove recursively the symmetry that

$$\mathfrak{T}^k S = (\mathfrak{T}^k S)^\top, \quad (2.16)$$

holds for all  $k = 1, 2, \dots$

The following result can be regarded as the converse of Theorem 2.3. It is important because it does not require beforehand the relationship (2.10) between  $S$  and  $(\mathfrak{X}, \mathfrak{T})$  before a quadratic pencil is defined. Rather, it qualifies whether a prescribed pair of matrices  $(\mathfrak{X}, \mathfrak{T})$  can ever serve as a standard pair at all and constructs, when feasible, the corresponding quadratic pencil.

**THEOREM 2.4.** *Let  $\mathfrak{X} \in \mathbb{R}^{n \times 2n}$ ,  $\mathfrak{T} \in \mathbb{R}^{2n \times 2n}$  be some given matrices. If there exists a symmetric and nonsingular matrix  $S \in \mathbb{R}^{n \times n}$  such that the product  $\mathfrak{X}\mathfrak{T}S\mathfrak{X}^\top$  is nonsingular and that the relationships (2.15) and (2.16) hold, then  $(\mathfrak{X}, \mathfrak{T})$  is a standard pair for the self-adjoint quadratic pencil  $\mathcal{Q}(\lambda)$  whose matrix coefficients  $M$ ,  $C$  and  $K$  are defined according to (2.11), (2.12) and (2.13), respectively.*

*Proof.* Since  $\mathfrak{X}\mathfrak{T}S\mathfrak{X}^\top$  is nonsingular,  $M$  can be defined. By the assumption of (2.16), we see that the three matrix coefficients  $M$ ,  $C$  and  $K$  are symmetric. By the assumption of (2.15) and the definition of  $C$ , we further observe that

$$\begin{bmatrix} \mathfrak{X} \\ \mathfrak{X}\mathfrak{T} \end{bmatrix} \begin{bmatrix} \mathfrak{T}S\mathfrak{X}^\top M + S\mathfrak{X}^\top C & S\mathfrak{X}^\top M \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ 0 & I_n \end{bmatrix},$$

implying that the matrix  $\begin{bmatrix} \mathfrak{X} \\ \mathfrak{X}\mathfrak{T} \end{bmatrix}$  is nonsingular. It follows that

$$M\mathfrak{X}\mathfrak{T}^2 \begin{bmatrix} \mathfrak{X} \\ \mathfrak{X}\mathfrak{T} \end{bmatrix}^{-1} = M\mathfrak{X}\mathfrak{T}^2 \begin{bmatrix} \mathfrak{T}S\mathfrak{X}^\top M + S\mathfrak{X}^\top C & S\mathfrak{X}^\top M \end{bmatrix} = \begin{bmatrix} -K & -C \end{bmatrix},$$

which is equivalent to  $M\mathfrak{X}\mathfrak{T}^2 + C\mathfrak{X}\mathfrak{T} + K\mathfrak{X} = 0$ .  $\square$

Thus far, we have not specified how the standard pair  $(\mathfrak{X}, \mathfrak{T})$  could be chosen from a given quadratic pencil. All we have is that Theorem 2.3 and Theorem 2.4 provide a necessary and sufficient condition on  $(\mathfrak{X}, \mathfrak{T})$  being a standard pair. Once a standard pair has been identified, we stress that the quadratic pencil can be decomposed in terms of  $(\mathfrak{X}, \mathfrak{T})$  via a parameter matrix  $S$ . In the subsequent discussion, we consider some specially selected standard pairs and study the consequential effect on the structure of the parameter matrix  $S$ .

**3. Structure of Parameter Matrix.** We shall limit our attention only to the case when  $\mathfrak{T}$  is block diagonal because it is perhaps the most commonly used structure in practice. It will be convenient to denote henceforth the spectrum of a given matrix  $W$  by the notation  $\sigma(W)$ .

The following result is true in general [9, Chapter VIII, Theorem 1], regardless whether  $S$  or  $\mathfrak{T}$  is derived from the context described in the preceding section or not.

**LEMMA 3.1.** *Suppose that  $\mathfrak{T} = \text{diag}\{\mathfrak{T}_1, \dots, \mathfrak{T}_k\}$  where  $\mathfrak{T}_j \in \mathbb{R}^{n_j \times n_j}$  for  $j = 1, \dots, k$  and  $\sigma(\mathfrak{T}_j) \cap \sigma(\mathfrak{T}_\ell) = \emptyset$  whenever  $j \neq \ell$ . Then a symmetric matrix  $S$  satisfies  $S\mathfrak{T}^\top = \mathfrak{T}S$  if and only if  $S = \text{diag}\{S_1, \dots, S_k\}$  where  $S_j \in \mathbb{R}^{n_j \times n_j}$  is symmetric and  $S_j\mathfrak{T}_j^\top = \mathfrak{T}_j S_j$  for  $j = 1, \dots, k$ .*

Now we move into more specific details pertaining to our investigation. In particular, we want to actually *see* a standard pair for a self-adjoint quadratic pencil. Let the distinct eigenvalues in the spectrum of the pencil  $\mathcal{Q}(\lambda)$  be denoted by

$$\lambda_1, \bar{\lambda}_1, \lambda_2, \bar{\lambda}_2, \dots, \lambda_\ell, \bar{\lambda}_\ell, \lambda_{\ell+1}, \dots, \lambda_k, \quad (3.1)$$

where  $\lambda_1, \dots, \lambda_\ell$  are distinct complex-valued eigenvalues and  $\lambda_{\ell+1}, \dots, \lambda_k$  are distinct real eigenvalues, each of which has algebraic multiplicity  $n_j$  (and, thus,  $2n_1 + \dots + 2n_\ell + n_{\ell+1} + \dots + n_k = 2n$ ). Associated with eigenvalue  $\lambda_j$ , let

$$J(\lambda_j) = \lambda_j I_{n_j} + N_j \quad (3.2)$$

denote its Jordan canonical form (which maybe made of several Jordan blocks) and  $X_j$  the  $n \times n_j$  submatrix of corresponding generalized eigenvectors. In the above,  $N_j$  is the  $n_j \times n_j$  nilpotent

matrix with at most 1's along its superdiagonal (see (3.13) for an example), depending on the geometric multiplicities of  $\lambda_j$ . It is known that the  $2n \times n_j$  matrix

$$\begin{bmatrix} X_j \\ X_j J(\lambda_j) \end{bmatrix} \quad (3.3)$$

is of full column rank and that the equation

$$MX_j J(\lambda_j)^2 + CX_j J(\lambda_j) + KX_j = 0 \quad (3.4)$$

is satisfied [11, Proposition 1.10]. In the event that  $\lambda_j = \alpha_j + i\beta_j$  is a complex-valued eigenvalue, write

$$X_j = X_{jR} + iX_{jI}, \quad (3.5)$$

with  $X_{jR}, X_{jI} \in \mathbb{R}^{n \times n_j}$ . We further combine the two corresponding  $n_j \times n_j$  complex-valued Jordan blocks  $J(\lambda_j)$  and  $J(\bar{\lambda}_j)$  into one  $2n_j \times 2n_j$  real-valued block  $J_r(\lambda_j)$  defined by

$$J_r(\lambda_j) := P_j^{-1} \begin{bmatrix} J(\alpha_j + i\beta_j) & 0 \\ 0 & J(\alpha_j - i\beta_j) \end{bmatrix} P_j = \begin{bmatrix} \alpha_j I_{n_j} + N_j & \beta_j I_{n_j} \\ -\beta_j I_{n_j} & \alpha_j I_{n_j} + N_j \end{bmatrix}, \quad (3.6)$$

where

$$P_j := \frac{1}{\sqrt{2}} \begin{bmatrix} I_{n_j} & -iI_{n_j} \\ I_{n_j} & iI_{n_j} \end{bmatrix}. \quad (3.7)$$

Finally, we arrive at the definition,

$$\mathfrak{X} := [X_{1R}, X_{1I}, \dots, X_{\ell R}, X_{\ell I}, X_{\ell+1}, \dots, X_k], \quad (3.8)$$

$$\mathfrak{T} := \text{diag}\{J_r(\lambda_1), \dots, J_r(\lambda_\ell), J(\lambda_{\ell+1}), \dots, J(\lambda_k)\}, \quad (3.9)$$

which by construction is a standard pair for  $\mathcal{Q}(\lambda)$ . The corresponding parameter matrix  $S$  therefore possesses the structure as is described in Lemma 3.1. That is,

$$S = \text{diag}\{S_1, \dots, S_k\}, \quad (3.10)$$

where all diagonal blocks  $S_j$  are symmetric. Additionally, for  $j = 1, \dots, \ell$ , the matrix  $S_j \in \mathbb{R}^{2n_j \times 2n_j}$  satisfies

$$S_j J_r(\lambda_j)^\top = J_r(\lambda_j) S_j, \quad (3.11)$$

and for  $j = \ell + 1, \dots, k$ , the matrix  $S_j \in \mathbb{R}^{n_j \times n_j}$  satisfies

$$S_j J(\lambda_j)^\top = J(\lambda_j) S_j. \quad (3.12)$$

What is interesting is that the structure of  $S$  contains far more subtle texture than its first glance.

**3.1. Upper Triangular Hankel Structure.** Recall that an  $m \times n$  matrix  $H = [h_{ij}]$  is said to have a Hankel structure if  $h_{ij} = \eta_{i+j-1}$ , where  $\{\eta_1, \dots, \eta_{m+n-1}\}$  are some fixed scalars. The matrix  $H$  is said to be *upper triangular Hankel* if  $\eta_k = 0$  for all  $k > \min\{m, n\}$ . Note that the zero portion of an upper triangular Hankel matrix occurs at the lower right corner of the matrix.

Assume that the geometric multiplicity of  $\lambda_j$  is  $m_j$ , that is, assume that there are  $m_j$  Jordan blocks corresponding to the eigenvalue  $\lambda_j$ . Write

$$N_j = \text{diag}\{N_1^{(j)}, N_2^{(j)}, \dots, N_{m_j}^{(j)}\}, \quad (3.13)$$

where  $N_i^{(j)}$  is the nilpotent block of size  $n_i^{(j)}$  for  $i = 1, \dots, m_j$ . A straightforward calculation shows that any symmetric solution  $Z$  to the equation

$$ZN_j^\top = N_j Z, \quad (3.14)$$

is necessarily of the form

$$Z = \begin{bmatrix} Z_{11} & Z_{12} & \dots & Z_{1m_j} \\ Z_{21} & Z_{22} & \dots & Z_{2m_j} \\ \vdots & \vdots & & \vdots \\ Z_{m_j 1} & Z_{m_j 2} & \dots & Z_{m_j m_j} \end{bmatrix}, \quad Z_{ik} \in \mathbb{R}^{n_i^{(j)} \times n_k^{(j)}}, \quad (3.15)$$

where  $Z_{ik}^\top = Z_{ki}$  and  $Z_{ik}$  is upper triangular Hankel. Equipped with this fact, we conclude from (3.12) that the matrices  $S_j$  corresponding to the real eigenvalues  $\lambda_j$ ,  $j = \ell + 1, \dots, k$ , are made of  $m_j \times m_j$  upper triangular Hankel blocks as described in (3.15). This structure persists in the following sense for the complex conjugate eigenvalues  $\lambda_j$  when  $j = 1, \dots, \ell$ .

**THEOREM 3.2.** *With  $J_r(\lambda_j)$  given by (3.6), a real-valued symmetric matrix  $S_j$  satisfies (3.11) if and only if  $S_j$  is of the form*

$$S_j = \begin{bmatrix} U_j & W_j \\ W_j & -U_j \end{bmatrix}, \quad (3.16)$$

where  $U_j$  and  $W_j$  are real-valued  $n_j \times n_j$  matrices whose entries can be partitioned into upper triangular Hankel blocks of the form described in (3.15).

*Proof.* For simplicity, denote the blocks of  $S_j$  in the form

$$S_j = \begin{bmatrix} U & W \\ W^\top & V \end{bmatrix},$$

where  $U$  and  $V$  are symmetric. Comparing the corresponding blocks in  $S_j J_r(\lambda_j)^\top = J_r(\lambda_j) S_j$ , we obtain

$$N_j U - U N_j^\top = \beta(W - W^\top), \quad (3.17)$$

$$N_j W - W N_j^\top = -\beta(U + V), \quad (3.18)$$

$$N_j W^\top - W^\top N_j^\top = \beta(U + V),$$

$$N_j V - V N_j^\top = \beta(W - W^\top).$$

It follows that

$$N_j(U + V) - (U + V)N_j^\top = 2\beta(W - W^\top), \quad (3.19)$$

$$N_j(W - W^\top) - (W - W^\top)N_j^\top = -2\beta(U + V). \quad (3.20)$$

Upon substituting (3.20) into (3.19), we obtain the linear equation,

$$N_j^2(W - W^\top) - 2N_j(W - W^\top)N_j^\top + (W - W^\top)N_j^{2\top} + 4\beta(W - W^\top) = 0,$$

which can be rewritten as

$$(I_{n_j} \otimes N_j^2 - 2N_j \otimes N_j + N_j^2 \otimes I_{n_j} + 4\beta I_{n_j} \otimes I_{n_j}) \mathbf{vec}(W - W^\top) = 0, \quad (3.21)$$

where  $\otimes$  stands for the Kronecker product and  $\mathbf{vec}$  the column vectorization of a matrix. By the structure of  $N_j$ , we see that the coefficient matrix in (3.21) is upper triangular with constant  $4\beta$

along its diagonal. It follows that  $W$  must be symmetric and, hence,  $V = -U$ . The equations (3.17) and (3.18) are reduced to the form (3.14). Their solutions  $U$  and  $W$  must be block upper triangular Henkel as shown in (3.15).  $\square$

It is interesting to take a special note of the more generic case when all eigenvalues are semi-simple, that is, when the algebraic multiplicity  $n_j$  is the same as the geometric multiplicity  $m_j$  for all  $j = 1, \dots, k$ . Under such an assumption, the upper triangular Hankel structure no longer shows up. The parameter matrix  $S$  is sufficiently and necessarily of the form

$$S = \text{diag} \left\{ \begin{bmatrix} U_1 & W_1 \\ W_1 & -U_1 \end{bmatrix}, \dots, \begin{bmatrix} U_\ell & W_\ell \\ W_\ell & -U_\ell \end{bmatrix}, S_{\ell+1}, \dots, S_k \right\}, \quad (3.22)$$

where  $U_j, W_j \in \mathbb{R}^{n_j \times n_j}$  are symmetric for  $j = 1, \dots, k$ , and  $S_j \in \mathbb{R}^{n_j \times n_j}$  is symmetric for  $j = \ell + 1, \dots, k$ . In particular, if all eigenvalues are simple, that is, if  $n_j = 1$  for all  $j = 1, \dots, k$  in (3.1), then the parameter matrix  $S$  is sufficiently and necessarily of the form

$$S = \text{diag} \left\{ \begin{bmatrix} s_1 & t_1 \\ t_1 & -s_1 \end{bmatrix}, \dots, \begin{bmatrix} s_\ell & t_\ell \\ t_\ell & -s_\ell \end{bmatrix}, s_{\ell+1}, \dots, s_k \right\}, \quad (3.23)$$

where  $s_j, t_j \in \mathbb{R}$  for  $j = 1, \dots, k$ ,  $s_j t_j \neq 0$  for  $j = 1, \dots, \ell$ , and  $s_j \neq 0$  for  $j = \ell + 1, \dots, k$ .

It is perhaps somewhat surprising to declare that, when the eigenvalues are semi-simple, that is, when  $N_j = 0$  for all  $j = 1, \dots, k$  in (3.2), the parameter matrix  $S$  in (3.22) actually can reveal a similar simple structure as in (3.23). The precise statement and the justification are given in the following section.

**3.2. Semi-simple Structure.** A real-valued Hamiltonian matrix  $H \in \mathbb{R}^{2n \times 2n}$  has the block structure that

$$H = \begin{bmatrix} E & F \\ G & -E^\top \end{bmatrix}, \quad (3.24)$$

where  $F$  and  $G$  are symmetric matrices in  $\mathbb{R}^{n \times n}$ . It is known that if  $H$  has no nonzero purely imaginary eigenvalues, then there exists an orthogonal matrix of the form

$$Q = \begin{bmatrix} Q_1 & Q_2 \\ -Q_2 & Q_1 \end{bmatrix}, \quad Q_1, Q_2 \in \mathbb{R}^{n \times n}, \quad (3.25)$$

such that

$$Q^\top H Q = \begin{bmatrix} T & R \\ 0 & -T^\top \end{bmatrix}, \quad (3.26)$$

where  $T$  is upper quasi-triangular and  $R = R^\top$ . In fact,  $Q$  can be chosen such that the eigenvalues of  $T$  are in the left half plane and such that each  $2 \times 2$  block of the diagonal of  $T$  is associated with a complex conjugate pair of eigenvalues. The factorization in (3.26) is known as the real Schur-Hamiltonian decomposition of  $H$  [21, Theorem 5.1].

Observe from (3.16) that each symmetric matrices  $S_j$ ,  $j = 1, \dots, \ell$ , is in fact symmetric Hamiltonian. Consequently, corresponding to each  $S_j$ ,  $j = 1, \dots, \ell$ , there exists an orthogonal matrix

$$Q_j = \begin{bmatrix} Q_{j1} & Q_{j2} \\ -Q_{j2} & Q_{j1} \end{bmatrix}, \quad Q_{j1}, Q_{j2} \in \mathbb{R}^{n_j \times n_j}, \quad (3.27)$$

such that

$$Q_j^\top S_j Q_j = \begin{bmatrix} T_j & 0 \\ 0 & -T_j^\top \end{bmatrix}, \quad (3.28)$$



where, due to the symmetry of  $S_j$ ,  $T_j$  is a diagonal matrix in  $\mathbb{R}^{n_j \times n_j}$ . We may assume further that

$$T_j = \text{diag}\{t_{j1}, \dots, t_{jn_j}\} \quad (3.29)$$

with  $t_{j\rho} < 0$  for  $\rho = 1, \dots, n_j$ . With this transformation in mind, we now specify the structure of  $S$  when all eigenvalues of  $\mathcal{Q}(\lambda)$  are semi-simple.

**THEOREM 3.3.** *Suppose that all eigenvalues of  $\mathcal{Q}(\lambda)$  are semi-simple. Then there exists a standard pair such that the corresponding parameter matrix  $S = \text{diag}\{S_1, \dots, S_k\}$  has the structure*

$$S_j = \begin{cases} \text{diag}\{-I_{n_j}, I_{n_j}\}, & \text{if } 1 \leq j \leq \ell, \\ \text{diag}\{\epsilon_{j1}, \dots, \epsilon_{jn_j}\}, & \text{if } \ell \leq j \leq k \end{cases} \quad (3.30)$$

where  $\epsilon_{j\rho} = \pm 1$  depending on the sign of eigenvalues of  $S_j$ .

*Proof.* Let  $S$  initially denote the parameter matrix defined via (2.10) in correspondence to the standard pair  $(\mathfrak{X}, \mathfrak{T})$  defined by (3.8) and (3.9). Our goal is to modify the standard pair to produce the desirable parameter matrix specified in (3.30).

For  $j = 1, \dots, \ell$ , let  $Q_j$  be the Hamiltonian structure-preserving orthogonal matrix characterized in (3.28). Note that  $Q_{j1} + \iota Q_{j2}$  is a unitary matrix in  $\mathbb{C}^{n_j \times n_j}$ . Columns of the complex-valued  $n \times n_j$  matrix

$$\tilde{X}_j := X_j(Q_{j1} + \iota Q_{j2})|T_j|^{\frac{1}{2}}, \quad (3.31)$$

therefore, remain to represent eigenvectors of  $\mathcal{Q}(\lambda)$  with corresponding eigenvalue  $\lambda_j$ . Based on (3.5), we can identify the real and the imaginary parts of  $\tilde{X}_j$  as

$$\begin{aligned} \tilde{X}_j &= \tilde{X}_{jR} + \iota \tilde{X}_{jI} \\ &\equiv [(X_{jR}Q_{j1} - X_{jI}Q_{j2})|T_j|^{\frac{1}{2}}, (X_{jR}Q_{j2} + X_{jI}Q_{j1})|T_j|^{\frac{1}{2}}] \\ &= [X_{jR}, X_{jI}]Q_j \text{diag}\{|T_j|^{\frac{1}{2}}, |T_j|^{\frac{1}{2}}\}. \end{aligned} \quad (3.32)$$

Similarly, for  $j = \ell + 1, \dots, k$ , let  $Q_j$  be the orthogonal matrix of eigenvectors of the symmetric matrix  $S_j$  so that  $Q_j^\top S_j Q_j = T_j := \text{diag}\{t_{j1}, \dots, t_{jn_j}\}$ . Define

$$\tilde{X}_j := X_j Q_j |T_j|^{\frac{1}{2}}, \quad (3.33)$$

which again represents eigenvectors of  $\mathcal{Q}(\lambda)$  with corresponding eigenvalue  $\lambda_j$ . It follows that  $(\tilde{\mathfrak{X}}, \mathfrak{T})$ , where  $\mathfrak{T}$  is the same as that given by (3.9) and

$$\tilde{\mathfrak{X}} := [\tilde{X}_{1R}, \tilde{X}_{1I}, \dots, \tilde{X}_{\ell R}, \tilde{X}_{\ell I}, \tilde{X}_{\ell+1}, \dots, \tilde{X}_k], \quad (3.34)$$

is a standard pair. Since  $N_j = 0$  by assumption, it is easy to verify that  $Q_j^\top J_r(\lambda_j) Q_j = J_r(\lambda_j)$  for  $j = 1, \dots, \ell$  and  $Q_j^\top J(\lambda_j) Q_j = J(\lambda_j)$  for  $j = \ell + 1, \dots, k$ . Through the transformation,

$$\begin{aligned} Q &:= \text{diag}\{Q_1, \dots, Q_\ell, Q_{\ell+1}, \dots, Q_k\}, \\ D &:= \text{diag}\{T_1^{\frac{1}{2}}, T_1^{\frac{1}{2}}, \dots, T_\ell^{\frac{1}{2}}, T_\ell^{\frac{1}{2}}, T_{\ell+1}^{\frac{1}{2}}, \dots, T_k^{\frac{1}{2}}\}, \end{aligned}$$

observe that the matrix

$$\tilde{S} := D^{-1} Q^\top S Q D^{-1} = \left( D Q^\top [\mathfrak{X}^\top, (\mathfrak{X} \mathfrak{T})^\top] \begin{bmatrix} C & M \\ M & 0 \end{bmatrix} \begin{bmatrix} \mathfrak{X} \\ \mathfrak{X} \mathfrak{T} \end{bmatrix} Q D \right)^{-1}, \quad (3.35)$$

has the structure specified in (3.30). Observe further that

$$\begin{aligned}\tilde{\mathfrak{X}} &= \mathfrak{X}QD, \\ \mathfrak{X}\mathfrak{T}QD &= \mathfrak{X}Q\mathfrak{T}D = \mathfrak{X}QD\mathfrak{T} = \tilde{\mathfrak{X}}\mathfrak{T},\end{aligned}$$

justifying that  $\tilde{S}$  is indeed the parameter matrix corresponding to the standard pair  $(\tilde{\mathfrak{X}}, \mathfrak{T})$ .  $\square$

**COROLLARY 3.4.** *Suppose that all eigenvalues of  $Q(\lambda)$  are semi-simple. Then the parameter matrix  $S$  characterized in Theorem 3.3 has trace zero.*

*Proof.* The congruence transformation

$$\begin{bmatrix} I_n & -\frac{1}{2}CM^{-1} \\ 0 & I_n \end{bmatrix} \begin{bmatrix} C & M \\ M & 0 \end{bmatrix} \begin{bmatrix} I_n & -\frac{1}{2}CM^{-1} \\ 0 & I_n \end{bmatrix}^\top = \begin{bmatrix} 0 & M \\ M & 0 \end{bmatrix}$$

asserts that the matrix  $\begin{bmatrix} C & M \\ M & 0 \end{bmatrix}$  has equal numbers of positive and negative eigenvalues. By Sylvester's law of inertia, it follows that the parameter matrix  $\tilde{S}$  defined in (3.35) has equal numbers of positive and negative 1's along its diagonal.  $\square$

In contrast to (1.7) and (1.8) for eigenvectors of a self-adjoint linear pencil with positive definite leading matrix coefficient, we conclude this section with a most general orthogonality property for eigenvectors of a self-adjoint quadratic pencil  $Q(\lambda)$ .

**COROLLARY 3.5.** *Suppose that all eigenvalues of  $Q(\lambda)$  are semi-simple. Then there exists a standard pair,*

$$\begin{aligned}\mathfrak{X} &= [\mathbf{x}_{1R}, \mathbf{x}_{1I}, \dots, \mathbf{x}_{\tau R}, \mathbf{x}_{\tau I}, \mathbf{x}_{2\tau+1}, \dots, \mathbf{x}_{2n}], \\ \mathfrak{T} &= \text{diag} \left\{ \begin{bmatrix} \alpha_1 & \beta_1 \\ -\beta_1 & \alpha_1 \end{bmatrix}, \dots, \begin{bmatrix} \alpha_\tau & \beta_\tau \\ -\beta_\tau & \alpha_\tau \end{bmatrix}, \lambda_{2\tau+1}, \dots, \lambda_{2n} \right\},\end{aligned}$$

where  $\mathbf{x}_{jR} \pm i\mathbf{x}_{jI}$  are complex conjugate eigenvectors associated with complex conjugate eigenvalues  $\alpha_j \pm i\beta_j$ ,  $j = 1, \dots, \tau$ , and  $\mathbf{x}_j$  is a real-valued eigenvector associated with real-valued eigenvalue  $\lambda_j$ ,  $j = 2\tau + 1, \dots, 2n$ , not necessarily all eigenvalues are distinct, such that

$$\begin{bmatrix} \mathfrak{X} \\ \mathfrak{X}\mathfrak{T} \end{bmatrix}^\top \begin{bmatrix} C & M \\ M & 0 \end{bmatrix} \begin{bmatrix} \mathfrak{X} \\ \mathfrak{X}\mathfrak{T} \end{bmatrix} = \Gamma := \text{diag} \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \dots, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\}, \quad (3.36)$$

$$\begin{bmatrix} \mathfrak{X} \\ \mathfrak{X}\mathfrak{T} \end{bmatrix}^\top \begin{bmatrix} -K & 0 \\ 0 & M \end{bmatrix} \begin{bmatrix} \mathfrak{X} \\ \mathfrak{X}\mathfrak{T} \end{bmatrix} = \Gamma\mathfrak{T}. \quad (3.37)$$

*Proof.* The expression (3.36) follows from rearranging the eigenvalues properly, if necessary, and applying Theorem 3.3 and Corollary 3.4. The expression (3.37) follows from the relationship (2.9).  $\square$

It is important to note that for a quadratic pencil the matrix  $\Gamma$  necessarily has equal numbers of 1's and -1's along its diagonal. This is in sharp contrast to (1.7) where eigenvectors are  $B$ -orthogonal.

**4. Applications.** Spectral decomposition of a linear transformation is so important that it has become a classic in the literature. What we have done in the above is to develop the theory of spectral decomposition for quadratic pencil. In particular, we realize in Theorem 2.3 that there is a parameter matrix  $S$  floating around in the decomposition. The parameter  $S$  plays a role of a normalization factor and, in general cases, its structure is well understood. Such a knowledge sometimes can shed considerable insights into difficult problems. In this section, we describe a few applications of this theory. Some of the problems below have been discussed elsewhere by lengthy papers, but our approach significantly simplifies the arguments.

**4.1. Inverse Eigenvalue Problem.** Generally speaking, an inverse eigenvalue problem is to reconstruct the coefficient matrices of a quadratic pencil from some known information of its eigenvalues and eigenvectors. Some general discussion for linear problems can be found in the book [5]. The quadratic inverse eigenvalue problems are much harder. There is quite a long list of studies on this subject. See, for example, [1, 13, 14, 16, 17, 18, 19, 20, 22] and the references contained therein. In this demonstration, we consider the special QIEP where the entire eigeninformation is given:

**(QIEP)** Given  $2n$  eigenpairs  $\{(\lambda_j, \mathbf{x}_j)\}_{j=1}^{2n}$  with

$$\begin{aligned} \lambda_{2j-1} &= \bar{\lambda}_{2j} = \alpha_j + \imath\beta_j, & \alpha_j &\in \mathbb{R}, \beta_j > 0, \\ \mathbf{x}_{2j-1} &= \bar{\mathbf{x}}_{2j} = \mathbf{x}_{jR} + \imath\mathbf{x}_{jI}, & \mathbf{x}_{jR}, \mathbf{x}_{jI} &\in \mathbb{R}^n, \quad j = 1, 2, \dots, \ell, \end{aligned}$$

and

$$\lambda_j \in \mathbb{R}, \quad \mathbf{x}_j \in \mathbb{R}^n, \quad j = 2\ell + 1, \dots, 2n,$$

construct a self-adjoint quadratic pencil  $\mathcal{Q}(\lambda)$  in the form of (1.10) so that the equations

$$\lambda_j^2 M \mathbf{x}_j + \lambda_j C \mathbf{x}_j + K \mathbf{x}_j = 0, \quad (4.1)$$

are satisfied for all  $j = 1, 2, \dots, 2n$ . That is,  $\mathcal{Q}(\lambda)$  has the prescribed set  $\{(\lambda_j, \mathbf{x}_j)\}_{j=1}^{2n}$  as its eigenpairs.

The system of equations in (4.1) can be written as

$$MX\Lambda^2 + CX\Lambda + KX = 0, \quad (4.2)$$

where

$$\begin{aligned} X &:= [\mathbf{x}_{1R}, \mathbf{x}_{1I}, \dots, \mathbf{x}_{\ell R}, \mathbf{x}_{\ell I}, \mathbf{x}_{2\ell+1}, \dots, \mathbf{x}_{2n}], \\ \Lambda &:= \text{diag} \left\{ \begin{bmatrix} \alpha_1 & \beta_1 \\ -\beta_1 & \alpha_1 \end{bmatrix}, \dots, \begin{bmatrix} \alpha_\ell & \beta_\ell \\ -\beta_\ell & \alpha_\ell \end{bmatrix}, \lambda_{2\ell+1}, \dots, \lambda_{2n} \right\}. \end{aligned} \quad (4.3)$$

We can immediately solve this inverse problem for all most all generically prescribed eigeninformation by Theorems 2.3 and 2.4. Specifically, using (3.23), we have the following result.

**THEOREM 4.1.** *Assume that the matrix  $\Lambda$  has only simple eigenvalues. Then the QIEP has a solution if and only if there is a nonsingular symmetric matrix  $S$  of the form*

$$S = \text{diag} \left\{ \begin{bmatrix} s_1 & t_1 \\ t_1 & -s_1 \end{bmatrix}, \dots, \begin{bmatrix} s_\ell & t_\ell \\ t_\ell & -s_\ell \end{bmatrix}, s_{2\ell+1}, \dots, s_{2n} \right\},$$

such that  $XSX^\top = 0$  and the product  $X\Lambda SX^\top$  is nonsingular. In this case, the matrix coefficients  $M$ ,  $C$  and  $K$  of the solution  $\mathcal{Q}(\lambda)$  are given by

$$M = (X\Lambda SX^\top)^{-1}, \quad C = -MX\Lambda^2 SX^\top M, \quad K = -MX\Lambda^3 SX^\top M + CM^{-1}C.$$

Now we can be more specific by taking into account some physical properties.

**Example 1.** For a damped vibrating system, we may assume that all the given eigenvalues in the QIEP are non-real, that is,  $\ell = n$ . In this case, it might be ‘‘tempting’’ to try a parameter matrix  $S$  that is of simpler form, say,

$$S = \text{diag} \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \dots, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\}.$$

The condition that  $XSX^\top = 0$  on eigenvectors then becomes the equation

$$X_R X_R^\top = X_I X_I^\top, \quad (4.4)$$

whereas the condition that  $X\Lambda SX^\top$  be nonsingular becomes that the matrix

$$[X_R, X_I] \begin{bmatrix} -\Omega & U \\ U & \Omega \end{bmatrix} [X_R, X_I]^\top, \quad (4.5)$$

be nonsingular with the notation

$$\begin{aligned} X_R &:= [\mathbf{x}_{1R}, \dots, \mathbf{x}_{nR}], \\ X_I &:= [\mathbf{x}_{1I}, \dots, \mathbf{x}_{nI}], \end{aligned}$$

and

$$\begin{aligned} \Omega &:= \text{diag}\{\alpha_1, \dots, \alpha_n\}, \\ U &:= \text{diag}\{\beta_1, \dots, \beta_n\}. \end{aligned}$$

We want to stress that the above very brief argument gives rise to precisely the sufficient conditions discussed in [20] for the solvability of the QIEP. Furthermore, the matrix coefficients  $M$ ,  $C$  and  $K$  of the *particular* solution  $\mathcal{Q}(\lambda)$  to the QIEP can be expressed as

$$\begin{aligned} M^{-1} &= -[X_R, X_I] \begin{bmatrix} -\Omega & U \\ U & \Omega \end{bmatrix} [X_R, X_I]^\top, \\ C &= M[X_R, X_I] \begin{bmatrix} U^2 - \Omega^2 & 2U\Omega \\ 2U\Omega & \Omega^2 - U^2 \end{bmatrix} [X_R, X_I]^\top M, \\ K &= M[X_R, X_I] \begin{bmatrix} 3\Omega U^2 - \Omega^3 & 3U\Omega^2 - U^3 \\ 3U\Omega^2 - U^3 & \Omega^3 - 3\Omega U^2 \end{bmatrix} [X_R, X_I]^\top M + CM^{-1}C. \end{aligned}$$

**Example 2.** For an over-damped system, it is well known that all the eigenvalues are real, non-positive, and semi-simple [11]. This is the case that all given eigenvalues in our QIEP are real, that is,  $\ell = 0$ . This inverse problem has been discussed in [19]. Again, if we limit ourselves to the specially selected parameter matrix

$$S = \text{diag}\{I, -I\},$$

then the condition  $XSX^\top = 0$  becomes

$$X_1 X_1^\top = X_2 X_2^\top,$$

and the condition  $X\Lambda SX^\top$  be nonsingular becomes

$$X_1 \Lambda_1 X_1^\top - X_2 \Lambda_2 X_2^\top$$

be nonsingular with the notation

$$\begin{aligned} X_1 &:= [\mathbf{x}_1, \dots, \mathbf{x}_n], \\ X_2 &:= [\mathbf{x}_{n+1}, \dots, \mathbf{x}_{2n}], \end{aligned}$$

and

$$\begin{aligned} \Lambda_1 &= \text{diag}\{\lambda_1, \dots, \lambda_n\}, \\ \Lambda_2 &= \text{diag}\{\lambda_{n+1}, \dots, \lambda_{2n}\}. \end{aligned}$$

The corresponding particular solution  $\mathcal{Q}(\lambda)$  to the QIEP is given by

$$\begin{aligned} M^{-1} &= X_1 \Lambda_1 X_1^\top - X_2 \Lambda_2 X_2^\top, \\ C &= M(X_2 \Lambda_2^2 X_2^\top - X_1 \Lambda_1^2 X_1^\top)M, \\ K &= M(X_2 \Lambda_2^3 X_2^\top - X_1 \Lambda_1^3 X_1^\top)M + CM^{-1}C. \end{aligned}$$

In all cases, we find it remarkable that Theorem 4.1 provides sufficient and necessary conditions for the general solution.

**4.2. Total Decoupling Problem.** It has been long desirable, yet with very limited success, to characterize the dynamical behavior of a complicated high-degree-of-freedom system in terms of the dynamics of some simpler low-degree-of-freedom subsystems. For linear pencils, there are usually modal coordinates under which the undamped quadratic eigenvalue problem can be represented by diagonal coefficient matrices. This amounts to the simultaneous diagonalization of two matrices by congruence or equivalence transformations [12, Section 4.5]. In other words, the undamped quadratic eigenvalue problem can be totally decoupled. In a more realistic environment where the pencil is quadratic and damped, it is commonly accepted that three general coefficient matrices  $M$ ,  $C$ , and  $K$  can hardly be diagonalized simultaneously by equivalence or congruence coordinate transformations. In the literature, engineers and practitioners have to turn to the so called *proportionally* or *classically damped* systems for the purpose of simultaneous diagonalization.

Recently Garvey, Friswell and Prells [7] proposed a notion that total decoupling of a system is *not* equivalent to simultaneous diagonalization. In particular, they argued that, under some mild assumptions, a general quadratic pencils can be converted by real-valued isospectral transformations into a totally decoupled system. That is, a complicated  $n$ -degree-of-freedom second order system can be reduced to  $n$  totally independent single-degree-of-freedom second order subsystems.

Given a self-adjoint quadratic pencil  $\mathcal{Q}(\lambda)$  in the form of (1.10), the principle idea is to seek a nonsingular matrix  $U \in \mathbb{R}^{2n \times 2n}$  such that the corresponding linearized system  $\mathcal{L}(\lambda)$  defined in (2.1) is transformed into

$$U^\top \begin{bmatrix} C & M \\ M & 0 \end{bmatrix} U = \begin{bmatrix} C_D & M_D \\ M_D & 0 \end{bmatrix}, \quad U^\top \begin{bmatrix} -K & 0 \\ 0 & M \end{bmatrix} U = \begin{bmatrix} -K_D & 0 \\ 0 & M_D \end{bmatrix}, \quad (4.6)$$

where  $M_D$ ,  $C_D$  and  $K_D$  are all diagonal matrices. Such a transformation, if exists, is isospectral so the original pencil (1.10) is equivalent to a totally decoupled system

$$\mathcal{Q}_D(\lambda) := M_D \lambda^2 + C_D \lambda + K_D, \quad (4.7)$$

in the sense that eigenvectors  $\mathbf{x}$  for  $\mathcal{Q}(\lambda)$  and  $\mathbf{y}$  for  $\mathcal{Q}_D(\lambda)$  are related by

$$\begin{bmatrix} \mathbf{x} \\ \lambda \mathbf{x} \end{bmatrix} = U \begin{bmatrix} \mathbf{y} \\ \lambda \mathbf{y} \end{bmatrix},$$

provided that  $M_D$  and  $M$  are nonsingular. The focus therefore is on the existence of  $U$ .

The original proof of this important result by Garvey, Friswell and Prells [7] contains some ambiguities which later were clarified and simplified by Chu and del Buono in [3]. A rather sophisticated algorithm was proposed in [4] for computing the transformation numerically without knowing a priori the eigeninformation. Here we offer a even simpler proof by employing the theory established in this paper. Furthermore, we stress that the techniques used here can be extended to high order systems.

For demonstration, assume that all the eigenvalues of  $Q(\lambda)$  are simple. By Corollary 3.5, we can find eigenvalues matrix  $\Lambda$  and eigenvector matrix  $X$  in the form of (4.3) such that

$$\begin{aligned} \begin{bmatrix} X \\ X\Lambda \end{bmatrix}^\top \begin{bmatrix} C & M \\ M & 0 \end{bmatrix} \begin{bmatrix} X \\ X\Lambda \end{bmatrix} &= S := \text{diag} \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \dots, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\}, \\ \begin{bmatrix} X \\ X\Lambda \end{bmatrix}^\top \begin{bmatrix} -K & 0 \\ 0 & M \end{bmatrix} \begin{bmatrix} X \\ X\Lambda \end{bmatrix} &= S\Lambda. \end{aligned} \quad (4.8)$$

From any given

$$M_D = \text{diag}\{m_1, m_2, \dots, m_n\}, \quad 0 \neq m_j \in \mathbb{R},$$

define

$$\begin{aligned} C_D &:= \text{diag}\{c_1, c_2, \dots, c_n\}, \\ K_D &:= \text{diag}\{k_1, k_2, \dots, k_n\} \end{aligned}$$

by

$$c_j := \begin{cases} -2m_j\beta_j, & j = 1, 2, \dots, \ell, \\ -m_j(\lambda_{2j-1} + \lambda_{2j}), & j = \ell + 1, \dots, n, \end{cases} \quad (4.9)$$

and

$$k_j := \begin{cases} m_j(\beta_j^2 + \alpha_j^2), & j = 1, 2, \dots, \ell, \\ m_j\lambda_{2j-1}\lambda_{2j}, & j = \ell + 1, \dots, n. \end{cases} \quad (4.10)$$

Then it is readily seen that the quadratic pencil

$$Q_D(\lambda) := M_D\lambda^2 + C_D\lambda + K_D$$

has the same spectrum as  $Q(\lambda)$ . Let  $\mathbf{e}_j$  denote the standard  $j$ th unit vector. Direct calculation shows that for  $1 \leq j \leq \ell$  we have

$$\begin{bmatrix} \mathbf{e}_j & 0 \\ \alpha_j\mathbf{e}_j & \beta_j\mathbf{e}_j \end{bmatrix}^\top \begin{bmatrix} C_D & M_D \\ M_D & 0 \end{bmatrix} \begin{bmatrix} \mathbf{e}_j & 0 \\ \alpha_j\mathbf{e}_j & \beta_j\mathbf{e}_j \end{bmatrix} = \begin{bmatrix} 0 & \beta_j m_j \\ \beta_j m_j & 0 \end{bmatrix},$$

and for  $\ell + 1 \leq j \leq n$  we have

$$\begin{aligned} \begin{bmatrix} \mathbf{e}_j \\ \lambda_{2j-1}\mathbf{e}_j \end{bmatrix}^\top \begin{bmatrix} C_D & M_D \\ M_D & 0 \end{bmatrix} \begin{bmatrix} \mathbf{e}_j \\ \lambda_{2j-1}\mathbf{e}_j \end{bmatrix} &= m_j(\lambda_{2j-1} - \lambda_{2j}), \\ \begin{bmatrix} \mathbf{e}_j \\ \lambda_{2j}\mathbf{e}_j \end{bmatrix}^\top \begin{bmatrix} C_D & M_D \\ M_D & 0 \end{bmatrix} \begin{bmatrix} \mathbf{e}_j \\ \lambda_{2j}\mathbf{e}_j \end{bmatrix} &= m_j(\lambda_{2j} - \lambda_{2j-1}). \end{aligned}$$

Obviously we also have

$$\frac{1}{2m_j\beta_j} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}^\top \begin{bmatrix} 0 & \beta_j m_j \\ \beta_j m_j & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Now select values of  $m_j$  such that

$$\begin{cases} m_j\beta_j > 0, & \text{if } 1 \leq j \leq \ell, \\ m_j(\lambda_{2j-1} - \lambda_{2j}) > 0, & \text{if } \ell + 1 \leq j \leq n, \end{cases}$$

and define vectors

$$\begin{cases} [\mathbf{y}_{jR}, \mathbf{y}_{jI}] := \frac{1}{\sqrt{2\beta_j m_j}} [\mathbf{e}_j, \mathbf{e}_j], & \text{if } 1 \leq j \leq \ell, \\ \mathbf{y}_{2j-1} = \mathbf{y}_{2j} := \frac{1}{\sqrt{m_j(\lambda_{2j-1} - \lambda_{2j})}} \mathbf{e}_j, & \text{if } \ell + 1 \leq j \leq n. \end{cases}$$

Then it follows that

$$\begin{aligned} \begin{bmatrix} Y \\ Y\Lambda \end{bmatrix}^\top \begin{bmatrix} C_D & M_D \\ M_D & 0 \end{bmatrix} \begin{bmatrix} Y \\ Y\Lambda \end{bmatrix} &= S, \\ \begin{bmatrix} Y \\ Y\Lambda \end{bmatrix}^\dagger \begin{bmatrix} -K_D & 0 \\ 0 & M_D \end{bmatrix} \begin{bmatrix} Y \\ Y\Lambda \end{bmatrix} &= S\Lambda, \end{aligned} \quad (4.11)$$

where

$$Y := [\mathbf{y}_{1R}, \mathbf{y}_{1I}, \dots, \mathbf{y}_{\ell R}, \mathbf{y}_{\ell I}, \mathbf{y}_{2\ell+1}, \dots, \mathbf{y}_{2n}]$$

is the eigenvector matrix of  $Q_D(\lambda)$ . Comparing (4.8) and (4.11), we conclude that the required nonsingular  $2n \times 2n$  matrix  $U$  can be taken as the matrix

$$U = \begin{bmatrix} X \\ X\Lambda \end{bmatrix} \begin{bmatrix} Y \\ Y\Lambda \end{bmatrix}^{-1}. \quad (4.12)$$

**4.3. Eigenvalue Embedding Problem.** Model updating concerns the modification of an existing but inaccurate model with measured data. For models characterized by quadratic pencils, the measured data usually involve incomplete knowledge of natural frequencies, mode shapes, or other spectral information. In conducting the updating, it is often desirable to match only the part of observed data without tampering with the other part of unmeasured or unknown eigenstructure inherent in the original model. Such an updating, if possible, is said to have no spill-over. It has been shown recently that model updating with no spill-over is entirely possible for undamped quadratic pencils, whereas spill-over for damped quadratic pencil generally is unpreventable [2]. Such a difficulty sometimes is compromised by considering the eigenvalue embedding problem (EEP), which we state as follows [1]:

**(EEP)** Given a self-adjoint quadratic pencil  $\mathcal{Q}_0(\lambda) = M_0\lambda^2 + C_0\lambda + K_0$  and a few of its associated eigenvalues  $\{\lambda_i\}_{i=1}^k$  with  $k < 2n$ , assume that new eigenvalues  $\{\mu_i\}_{i=1}^k$  have been measured. Update the quadratic pencil  $\mathcal{Q}_0(\lambda)$  to  $\mathcal{Q}(\lambda) = M\lambda^2 + C\lambda + K$  so that the subset  $\{\lambda_i\}_{i=1}^k$  is replaced by  $\{\mu_i\}_{i=1}^k$  as  $k$  eigenvalues of  $\mathcal{Q}(\lambda)$  while the remaining  $2n - k$  eigenpairs of  $\mathcal{Q}(\lambda)$ , which usually are unknown, are kept the same as those of the original  $\mathcal{Q}_0(\lambda)$ .

An iterative scheme that reassign one eigenvalue a time has been suggested in [1] as a possible numerical method for solving the EEP. That algorithm suffers from two shortcomings — that the calculation can break down prematurely and that not all desirable eigenvalues are guaranteed to be updated. Once again, using our theory, we offer a novel approach for solving the EEP which completely circumvents all inherent troubles of the algorithm proposed in [1].

For demonstration, assume that all eigenvalues of  $\mathcal{Q}_0(\lambda)$  are simple. Let the  $2n$  eigenpairs of  $\mathcal{Q}_0(\lambda)$  by  $\{(\lambda_i, \mathbf{x}_i)\}_{i=1}^{2n}$ . Denote the real-valued representations of  $\{(\lambda_i, \mathbf{x}_i)\}_{i=1}^k$  and  $\{(\lambda_i, \mathbf{x}_i)\}_{i=k}^{2n}$  by

$$\begin{aligned} \Lambda_1 &:= \text{diag} \left\{ \begin{bmatrix} \alpha_1 & \beta_1 \\ -\beta_1 & \alpha_1 \end{bmatrix}, \dots, \begin{bmatrix} \alpha_{\ell_1} & \beta_{\ell_1} \\ -\beta_{\ell_1} & \alpha_{\ell_1} \end{bmatrix}, \lambda_{2\ell_1+1}, \dots, \lambda_k \right\}, \\ X_1 &:= [\mathbf{x}_{1R}, \mathbf{x}_{1I}, \dots, \mathbf{x}_{\ell_1 R}, \mathbf{x}_{\ell_1 I}, \mathbf{x}_{2\ell_1+1}, \dots, \mathbf{x}_k] \end{aligned}$$

and

$$\Lambda_2 := \text{diag} \left\{ \begin{bmatrix} \alpha_{k+1} & \beta_{k+1} \\ -\beta_{k+1} & \alpha_{k+1} \end{bmatrix}, \dots, \begin{bmatrix} \alpha_{k+\ell_2} & \beta_{k+\ell_2} \\ -\beta_{k+\ell_2} & \alpha_{k+\ell_2} \end{bmatrix}, \lambda_{k+2\ell_2+1}, \dots, \lambda_{2n} \right\},$$

$$X_2 := [\mathbf{x}^{(k+1)R}, \mathbf{x}^{(k+1)I}, \dots, \mathbf{x}^{(k+\ell_2)R}, \mathbf{x}^{(k+\ell_2)I}, \mathbf{x}_{k+2\ell_2+1}, \dots, \mathbf{x}_{2n}],$$

respectively. Denote further the two matrices,

$$S_1^{-1} := \begin{bmatrix} X_1 \\ X_1 \Lambda_1 \end{bmatrix}^\top \begin{bmatrix} C & M \\ M & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_1 \Lambda_1 \end{bmatrix},$$

$$S_2^{-1} := \begin{bmatrix} X_2 \\ X_2 \Lambda_2 \end{bmatrix}^\top \begin{bmatrix} C & M \\ M & 0 \end{bmatrix} \begin{bmatrix} X_2 \\ X_2 \Lambda_2 \end{bmatrix}.$$

Then, by Theorem 2.3, we know that

$$\begin{aligned} M_0^{-1} &= X_1 \Lambda_1 S_1 X_1^\top + X_2 \Lambda_2 S_2 X_2^\top, \\ C_0 &= -M_0 (X_1 \Lambda_1^2 S_1 X_1^\top + X_2 \Lambda_2^2 S_2 X_2^\top) M_0, \\ K_0 &= -M_0 (X_1 \Lambda_1^3 S_1 X_1^\top + X_2 \Lambda_2^3 S_2 X_2^\top) M_0 + C_0 M_0^{-1} C_0. \end{aligned} \quad (4.13)$$

Denote the real-valued representation of  $\{\mu_i\}_{i=1}^k$  be  $W$ . Assume that  $W$  has exactly the same block diagonal structure as that of  $\Lambda_1$ , that is, assume that  $W$  is of the form

$$W = \text{diag} \left\{ \begin{bmatrix} \gamma_1 & \delta_1 \\ -\delta_1 & \gamma_1 \end{bmatrix}, \dots, \begin{bmatrix} \gamma_{\ell_1} & \delta_{\ell_1} \\ -\delta_{\ell_1} & \gamma_{\ell_1} \end{bmatrix}, \mu_{2\ell_1+1}, \dots, \mu_k \right\}.$$

Since the eigenvectors are not modified, the condition  $X_1 S_1 X_1^\top + X_2 S_2 X_2^\top = 0$  automatically holds. Suppose that the matrix

$$X_1 W S_1 X_1^\top + X_2 \Lambda_2 S_2 X_2^\top \quad (4.14)$$

is nonsingular. Then by Theorem 2.4, we know right away that one particular solution to EEP is given by

$$\begin{aligned} M^{-1} &= X_1 W S_1 X_1^\top + X_2 \Lambda_2 S_2 X_2^\top, \\ C &= -M (X_1 W^2 S_1 X_1^\top + X_2 \Lambda_2^2 S_2 X_2^\top) M, \\ K &= -M (X_1 W^3 S_1 X_1^\top + X_2 \Lambda_2^3 S_2 X_2^\top) M + C M^{-1} C. \end{aligned} \quad (4.15)$$

Combining (4.13) with (4.15), we see that the update takes place in the following way:

$$\begin{aligned} M^{-1} &= M_0^{-1} + X_1 (W - \Lambda_1) S_1 X_1^\top, \\ C &= M [M_0^{-1} C_0 M_0^{-1} - X_1 (W^2 - \Lambda_1^2) S_1 X_1^\top] M, \\ K &= M [M_0^{-1} (K_0 - C_0 M_0^{-1} C_0) M_0^{-1} - X_1 (W^3 - \Lambda_1^3) S_1 X_1^\top] M + C M^{-1} C, \end{aligned} \quad (4.16)$$

provided that  $M_0^{-1} + X_1 (W - \Lambda_1) S_1 X_1^\top$  is nonsingular. It is critically important to note that the update formula (4.16) from  $\mathcal{Q}_0(\lambda)$  to  $\mathcal{Q}(\lambda)$  does not need the information about  $(\Lambda_2, X_2)$ .

We believe that this closed form solution for the EEP is of interest itself and is innovative in the literature.



**5. Conclusion.** The classical notion of spectral decomposition for linear operators has been generalized to quadratic systems. By constructing a standard pair in the way described in (3.8) and (3.9) which includes the most general case of arbitrary algebraic or geometric multiplicities, we characterize the spectral decomposition of a self-adjoint quadratic pencil in terms of the usual eigeninformation as well an additional parameter matrix  $S$ .

Three fundamental relationships between  $S \in \mathbb{R}^{2n \times 2n}$  and  $(\mathfrak{X}, \mathfrak{T}) \in \mathbb{R}^{n \times 2n} \times \mathbb{R}^{2n \times 2n}$ , i.e., that  $\mathfrak{X}\mathfrak{T}S\mathfrak{X}^\top$  is nonsingular and that the equalities (2.15) and (2.16) hold, determine a necessary and sufficient condition for the spectral decomposition. The structure of  $S$  is investigated under different assumptions of multiplicities. In the special case when all eigenvalues are semi-simple,  $S$  can be taken to the special form

$$S = \text{diag} \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \dots, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\},$$

which is the generalized notion of “orthogonality” among eigenvectors of a self-adjoint quadratic pencil.

The parameterized spectral decomposition developed for quadratic pencils has great potential of applications. It is demonstrated how three nontrivial problems, each of which has attracted considerable research efforts in the literature, can now be easily solved by exploiting our theory.

#### REFERENCES

- [1] J. Carvalho, B. N. Datta, W. W. Lin, and C. S. Wang, *Symmetric preserving eigenvalue embedding in finite element model updating of vibrating structures*, Journal of Sound and Vibration., 290(2006), 839–864.
- [2] M. T. Chu, B. Datta, W. W. Lin and S. F. Xu, *The spill-over phenomenon in quadratic model updating*, preprint, 2005.
- [3] M. T. Chu and N. Del Buono, *Total decoupling of a general quadratic pencil, Part I: Theory*, preprint, 2005.
- [4] M. T. Chu and N. Del Buono, *Total decoupling of a general quadratic pencil, Part II: Structure preserving isospectral flows*, preprint, 2005.
- [5] M. T. Chu and G. H. Golub, *Inverse Eigenvalue Problems; Theory, Algorithms, and Applications*, Oxford University Press, Oxford, 2005.
- [6] P. R. Halmos, *What does the spectral theorem say?* Amer. Math. Monthly, 70(1963), 241–247.
- [7] S. D. Garvey, M. I. Friswell, and U. Prells, *Coordinate transformations for general second order systems, Pt. I: General transformations*, Journal of Sound and Vibration., 258(2002), 885–909.
- [8] S. D. Garvey, M. I. Friswell, and U. Prells, *Coordinate transformations for general second order systems, Pt. II: Elementary structure preserving transformations*, Journal of Sound and Vibration., 258(2002), 911–930.
- [9] F. R. Gantmacher, *The Theory of Matrices*, Vol. 1, Translated from the Russian by K. A. Hirsch, AMS Chelsea Publishing, Providence, RI, 1998.
- [10] I. Gohberg, P. Lancaster, and L. Rodman, *Spectral analysis of selfadjoint matrix polynomials*, Annals of Mathematics, 112 (1980), 34–71.
- [11] I. Gohberg, P. Lancaster, and L. Rodman, *Matrix Polynomials*, Computer Science and Applied Mathematics. Academic Press, Inc., New York-London, 1982.
- [12] R. A. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge University Press, New York, 1991.
- [13] Y. C. Kuo, W. W. Lin, and S. F. Xu, *On the general solution of partially described inverse quadratic eigenvalue problem and its applications*, SIAM J. Matrix Analysis Appl, to appear.
- [14] P. Lancaster, *Expressions for damping matrices in linear vibration problems*, Journal of Aero-Space Sciences, 28(1961), pp. 256.
- [15] P. Lancaster, *Lambda-matrices and Vibrating Systems*, Dover Publications; Dover edition, 1966.
- [16] P. Lancaster and J. Maroulas, *Inverse problems for damped vibrating systems*, Journal of Math. Analysis and Applications, 123(1987), 238–261.
- [17] P. Lancaster and Q. Ye, *Inverse spectral problems for linear and quadratic matrix pencils*, Linear Algebra and its Applications, 107(1988), 293–309.
- [18] P. Lancaster, *Isospectral vibrating systems. part I: The spectral method*, Linear Algebra and its Applications, 409(2005), 51–69.
- [19] P. Lancaster, *Inverse spectral problems for semi-simple damped vibrating systems*, SIAM J. Matrix Anal. Appl., 29(2007), 279–301.

- [20] P. Lancaster and U. Prells, *Inverse problems for damped vibrating systems*, Journal of Sound and Vibration, 283(2005), 891–914.
- [21] C. Paige and C. Van Loan, *A Schur decomposition for Hamiltonian matrices*, Linear Alg. Appl., 41(1981), 11–32.
- [22] U. Prells and P. Lancaster, *Isospectral vibrating systems. part II: Structure preserving transformations*, Oper. Theory: Adv. Appl., 163(2006), 275–298.
- [23] F. Tisseur and K. Meerbergen, *The quadratic eigenvalue problem*, SIAM Rev., 43(2001), 253–286.