

# Spillover Phenomenon in Quadratic Model Updating

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**Model updating concerns the modification of an existing but inaccurate model with measured data. For models characterized by quadratic pencils, the measured data usually involve incomplete knowledge of natural frequencies, mode shapes, or other spectral information. In conducting the updating, it is often desirable to match only the part of observed data without tampering with the other part of unmeasured or unknown eigenstructure inherent in the original model. Such an updating, if possible, is said to have no spillover. This paper studies the spillover phenomenon in the updating of quadratic pencils. In particular, it is shown that an updating with no spillover is always possible for undamped quadratic pencils, whereas spillover for damped quadratic pencils is generally unpreventable.**

## Nomenclature

$A, B$	= parameter matrices; Eq. (37)
$C$	= damping matrix in a pencil
$C_0$	= initial damping matrix in a pencil
$D$	= diagonal matrix; Eq. (25)
$f(t)$	= external force
$H, \hat{H}$	= intermediate matrices
$K$	= stiffness matrix in a pencil
$K_0$	= initial stiffness matrix in a pencil
$k_{\max}$	= maximal allowable number of prescribed eigenpairs; Eq. (28)
$M$	= mass matrix in a pencil
$M_0$	= initial mass matrix in a pencil
$Q(\lambda)$	= quadratic pencil in $\lambda$
$\hat{Q}$	= orthogonal matrix in the $QR$ decomposition
$\hat{R}$	= upper triangular matrix in the $QR$ decomposition
$\mathbb{R}^n$	= $n$ dimensional Euclidean space over real numbers
$\mathbb{R}^{n \times n}$	= vector space of all $n \times n$ real-valued matrices
$S, T, U$	= parameter matrices; Eq. (33)
$t$	= time variable
$V$	= orthogonal matrix; Eq. (25)
$v$	= eigenvector
$X$	= eigenvector matrix
$\hat{X}$	= extended matrix of $X$ ; Eq. (44)
$X_1$	= eigenvector matrix with eigenvectors $\{\mathbf{u}_i\}_{i=1}^k$
$X_2$	= eigenvector matrix of inert eigenvectors
$x$	= state variable
$\dot{x}$	= derivative of $x$ with respect to time $t$
$Y$	= eigenvector matrix with eigenvectors $\{\mathbf{y}_i\}_{i=1}^k$
$\Gamma$	= (diagonal) parameter matrix

$\Delta C$	= correction of damping matrix $C$
$\Delta K$	= correction of stiffness matrix $K$
$\Delta M$	= correction of mass matrix $M$
$\Delta Q(\lambda)$	= incremental pencil; Eq. (59)
$\hat{\Lambda}$	= (diagonal) eigenvalue matrix
$\hat{\Lambda}$	= expanded matrix of $\Lambda$ ; Eq. (45)
$\Lambda_2$	= (diagonal) eigenvalue matrix of inert eigenvalues
$\lambda$	= eigenvalue
$\{\lambda_i, \mathbf{u}_i\}_{i=1}^k$	= initial eigenpairs to be updated
$\lambda_j^{[2]}$	= $2 \times 2$ real-valued matrix with complex eigenvalues $\alpha_j \pm \beta_j i$ ; Eq. (38)
$\mu$	= $-\lambda^2$
$\Xi$	= parameter matrix; Eq. (53)
$\Sigma$	= (diagonal) eigenvalue matrix with eigenvalues $\{\sigma_i\}_{i=1}^k$
$\Phi$	= parameter matrix; Eq. (23)
$\{\sigma_i, \mathbf{y}_i\}_{i=1}^k$	= newly measured eigenpairs
$\Upsilon_1, \Upsilon_2$	= intermediate matrices
$\Psi, \Omega$	= intermediate matrices

## I. Introduction

**M**ODELING is one of the most fundamental tools used to simulate the complex world. The goal of modeling is to come up with a representation that is simple enough for mathematical manipulation yet powerful enough for describing, inducing, and reasoning complicated phenomena. Partially because of the inevitable disturbances to the measuring devices of an observation and partially because of the insufficient representation of the true attributes of a physical system, precise mathematical models are rarely available in practice. With gradual confidence built on improved technologies or repeated experiments, the measured data are often regarded as more realistic to the true natural phenomena than the predicted value from an existing model. It thus becomes necessary, when compared with realistic data, to update a primitive model to attain consistency with empirical results. This procedure of updating or revising an existing model is an essential step toward establishing an effective model. This paper concerns the model updating of quadratic pencils to reflect measured spectral information [1–3].

Quadratic pencils arise from the study of the second-order differential system

$$M\ddot{x} + C\dot{x} + Kx = f(t) \quad (1)$$

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where  $\mathbf{x} \in \mathbb{R}^n$  and  $M, C, K \in \mathbb{R}^{n \times n}$ . Such a differential system has a wide scope of important applications, including applied mechanics, electrical oscillation, vibroacoustics, fluid mechanics, signal processing, and finite element discretization of partial differential equations (PDEs). In most applications involving Eq. (1), specifications of the underlying physical system are embedded in the matrix coefficients  $M, C$ , and  $K$ . If a fundamental solution to Eq. (1) is represented by

$$\mathbf{x}(t) = \mathbf{v}e^{\lambda t}$$

then the scalar  $\lambda$  and the vector  $\mathbf{v}$  must solve the quadratic eigenvalue problem (QEP):

$$(\lambda^2 M + \lambda C + K)\mathbf{v} = 0 \tag{2}$$

In this way, the bearing of the dynamic system (1) can be largely interpreted via the eigenvalues and eigenvectors of the algebraic system (2). Because of this connection and applications to other disciplines, considerable efforts have been devoted to the QEP in the literature. A collection of applications, mathematical properties, and a variety of numerical techniques for the QEP can be found in the survey treatise by Tisseur and Meerbergen [4]. For convenience, the quadratic pencil  $Q(\lambda) = \lambda^2 M + \lambda C + K$  will be identified by the triplet  $(M, C, K)$  of matrices henceforth.

The eigenvalue problem associated with the model (1) can be studied from two different aspects. The process of analyzing and deriving the spectral information and, hence, inferring the dynamic behavior of a system from a priori known physical parameters such as mass, length, elasticity, inductance, capacitance, and so on is referred to as a direct problem. The inverse problem, in contrast, is to validate, determine, or estimate the parameters of the system according to its observed or expected behavior. In other words, the concern in the direct problem is to express the behavior in terms of the parameters, whereas in the inverse problem, the concern is to express the parameters in terms of the behavior. The inverse problem is just as important as the direct problem in applications. The model updating problem considered in this paper is a special case of the inverse eigenvalue problem.

The inverse eigenvalue problem is a diverse area full of research interests and activities. See the newly revised book by Gladwell [5], the review article [6], and the recently completed monograph by Chu and Golub [7] in which more than 460 references are collected. At present, the theory and algorithms for the quadratic inverse eigenvalue problem (QIEP), that is, finding  $(M, C, K)$  from given eigeninformation, are far from being complete. Conceivably, the quadratic problem is more challenging than the linear problems with many unanswered questions.

There are various ways to formulate a QIEP, differing mainly in the desirable structure of the matrix coefficients  $M, C$ , and  $K$  and the available eigeninformation. The focus of this paper is on the updating of self-adjoint QIEPs in which all matrix coefficients are symmetric. Be cautioned that this is just an important first step toward more sophisticated structures. For example, in vibration modeling, often the mass matrix  $M$  is diagonal and positive, both the damping matrix  $C$  and the stiffness matrix  $K$  are symmetric and banded, and  $K$  is positive semidefinite. Indeed, this paper is centered around one specific scenario in which only a few eigenvalue and the corresponding eigenvectors (measured at full degree of freedom) are available. There are multiple reasons why such a scenario is justifiable, namely, in vibration industries, including aerospace, automobile, and manufacturing, through vibration tests in which the excitation and the response of the structure at selected points are measured experimentally, there are identification techniques to extract a portion of eigenpair information from the measurements. However, quantities related to high frequency terms in a finite-dimensional model generally are susceptible to measurement errors because of the finite bandwidth of measuring devices. It is simply unwise to use experimental values of high natural frequencies to reconstruct a model. In fact, in a large and complicated physical system, it is often impossible to acquire knowledge of the entire

spectral information. Although there is no reasonable analytical tool available to evaluate the entire spectral information, only partial information through experiments is attainable. Additionally, it is often demanded, especially in structural design, that certain eigenvectors should also satisfy some specific conditions. For these reasons, it might be more sensible to consider a model updating using only a few measured eigenvalues and eigenvectors [1,8,9]. Note that in practice the eigenvectors are measured only at finite degree of freedom because of hardware limitations. There are ways to deal with incomplete measured data, such as model reduction and model expansion techniques. See the discussion in the book by Friswell and Mottershead [1] and some algorithmic approaches developed in [10]. For the purpose of clarity, it is assumed that the eigenvectors in this paper have been measured to the full degree of freedom, or some measures have been taken so that a comparison with analytical eigenvectors is possible.

It is often desirable in updating an existent model that the newly measured parameters enter the system without altering other unrelated vibration parameters. The so-called no spillover phenomenon in the engineering literature imposes an additional challenge in model updating. No spillover is required either because these unrelated parameters are proven acceptable in the previous model, and engineers do not wish to introduce new vibrations via updating, and engineers simply do not know any information about these parameters. The quadratic model updating problem (MUP) with no spillover, therefore, can be stated as follows.

MUP: Given a structured quadratic pencil  $(M_0, C_0, K_0)$  and a few of its associated eigenpairs  $\{(\lambda_i, \mathbf{u}_i)\}_{i=1}^k$  with  $k \ll 2n$ , assume that newly measured eigenpairs  $\{(\sigma_i, \mathbf{y}_i)\}_{i=1}^k$  have been obtained. Update the pencil  $(M_0, C_0, K_0)$  to  $(M, C, K)$  of the same structure such that

- 1) The subset  $\{(\lambda_i, \mathbf{u}_i)\}_{i=1}^k$  is replaced by  $\{(\sigma_i, \mathbf{y}_i)\}_{i=1}^k$  as  $k$  eigenpairs of  $(M, C, K)$ .
- 2) The remaining (unknown)  $2n - k$  eigenpairs of  $(M, C, K)$  are the same as those of the original  $(M_0, C_0, K_0)$ .

The MUP, as stated before, is of immense practical importance. Similar problems have been studied in the literature. The work by Friswell et al. [11] updates the model by minimal adjustment of only the damping and the stiffness matrices. Baruch [12], Bermann and Nagy [13], and Wei [14,15] consider only undamped systems. Minas and Inman [16,17] correct the finite element model with measured modal data. The team of Datta, Elhay, Ram, and Sarkissian [18–21] and the team of Lin and Wang [22] adopt the feedback control approach. Despite the many efforts, there does not seem to be a satisfactory theory or techniques thus far, even for the case in which the required structure in the MUP is self-adjoint only. Existing methods have severe computational and engineering limitations, which restrict their usefulness in real applications. One of the main concerns is that these methods “cannot guarantee that extra, spurious modes are not introduced into the range of the frequency range of interest.” [1] The purpose of this paper is to provide a systematic study toward this spillover phenomenon.

The main contribution of this paper is as follows. This paper offers a simple yet effective mathematical argument to reach a conclusion that is perhaps a rather surprising disappointment to engineering practitioners: for a damped system, the MUP as is described here generally is unsolvable. In other words, unless the newly measured eigenpairs  $\{(\sigma_i, \mathbf{y}_i)\}_{i=1}^k$  satisfy some fairly stringent conditions, an updating of a damped quadratic pencils will surely cause spillover. The characterization of those sufficient and necessary conditions for solvability is complicated enough that it warrants a separate paper [23] to address the details. This paper concentrates on the unsolvability, which offers a different view of the MUP.

The notation will prove to be convenient. The diagonal matrix  $\Lambda \in \mathbb{R}^{2n \times 2n}$  represents the “eigenvalue matrix” of the quadratic pencil (2) in the sense that  $\Lambda$  is in real diagonal form with  $2 \times 2$  blocks along the diagonal replacing the complex-conjugate pairs of eigenvalues originally there. Similarly, let  $X \in \mathbb{R}^{n \times 2n}$  represent the eigenvector matrix in the sense that each pair of column vectors associated with a  $2 \times 2$  block in  $\Lambda$  retains the real and the imaginary part of the original complex eigenvector. It is clear that the relationship

$$MX\Lambda^2 + CX\Lambda + KX = 0_{n \times 2n} \tag{3}$$

holds. Partition  $X$  and  $\Lambda$  as

$$X = [Y, X_2], \quad \Lambda = \text{diag}\{\Sigma, \Lambda_2\} \tag{4}$$

where the pair  $(Y, \Sigma) \in \mathbb{R}^{n \times k} \times \mathbb{R}^{k \times k}$  corresponds to the portion of eigenstructure that has been modified, and  $(X_2, \Lambda_2)$  corresponds to the inert portion of eigenstructure in the original model, which should not be changed. To answer whether a self-adjoint quadratic pencil can be updated with no spillover, a more fundamental question is whether self-adjoint quadratic pencils can have arbitrary spectral structure  $(X, \Lambda)$ . The main thrust in the paper is to answer the MUP from the QIEP point of view.

## II. Zero Damping

We first consider the self-adjoint pencil:

$$\lambda^2 M_0 + K_0$$

where  $M_0$  is assumed to be positive definite. It is known in this case that  $\lambda^2$  is real. Thus, by defining  $\mu := -\lambda^2$ , we can rewrite the quadratic pencil as a linear pencil

$$\mu M_0 - K_0 \tag{5}$$

effectively reducing the number of eigenvalues for the system (5) to  $n$ . We shall continue using the same notation in (4) to indicate the partition of eigenstructure for (5), except that we know for sure in this case that  $\Lambda \in \mathbb{R}^{n \times n}$  is truly diagonal and that no complex-valued eigenvectors are involved in  $X \in \mathbb{R}^{n \times n}$ . We shall make a practical assumption that all diagonal entries of  $\Lambda$  are distinct. Such an assumption can be deemed reasonable because multiple roots are sensitive to perturbations and, hence, are hardly observable in real applications.

### A. Two-Sided Updating

Given  $(X, \Lambda)$ , this updating problem concerns finding symmetric matrices  $\Delta M$  and  $\Delta K$  such that the following equations hold simultaneously:

$$(M_0 + \Delta M)X_2\Lambda_2 = (K_0 + \Delta K)X_2 \tag{6}$$

$$(M_0 + \Delta M)Y\Sigma = (K_0 + \Delta K)Y \tag{7}$$

Note that each eigenpair gives rise to  $n$  equations and that  $\Delta M$  and  $\Delta K$  involve only  $n(n + 1)$  unknown entries. Because there are  $n^2$  equations in  $n(n + 1)$  unknowns, it is likely that the system (6) and (7) is solvable for any given  $(X, \Lambda)$  where  $X \in \mathbb{R}^{n \times n}$  is nonsingular and  $\Lambda \in \mathbb{R}^{n \times n}$  is diagonal. In other words, not only the updating with no spillover is always possible, but there are  $n$  degrees of freedom in choosing the parameters. The question is how to find such a general solution. The following analysis is classical in the literature. We include somewhat details for completion.

Our answer comes from the observation that for the linear pencil  $\mu M - K$  to have eigenstructure  $(X, \Lambda)$ , it is necessary that

$$[-X^T, \Lambda^T X^T] \begin{bmatrix} K^T \\ M^T \end{bmatrix} = 0 \tag{8}$$

On the other hand, it is trivial that

$$[-I_n, \Lambda^T] \begin{bmatrix} \Lambda^T S \\ S \end{bmatrix} = 0 \tag{9}$$

for any  $S \in \mathbb{R}^{n \times n}$ . We thus obtain a parametric representation:

$$M = S^T X^{-1}, \quad K = S^T \Lambda X^{-1}$$

We are interested in selecting  $S$  so as to construct self-adjoint pencils. For  $M$  to be symmetric, the matrix  $S$  must be such that

$$S^T X^{-1} = X^{-T} S$$

implying that the matrix  $\Gamma$  defined by

$$\Gamma := SX = X^T S^T \tag{10}$$

is symmetric. For  $K$  to be symmetric, the matrix  $S$  must also be such that

$$S^T \Lambda X^{-1} = X^{-T} \Lambda^T S$$

implying the equality

$$\Lambda^T \Gamma = \Gamma \Lambda \tag{11}$$

Because  $\Lambda$  is of diagonal matrix with distinct entries, it follows that  $\Gamma$  is also a diagonal matrix containing exactly  $n$  free parameters. Upon choosing an arbitrary  $\Gamma$ , a substitution by  $S^T = X^{-T} \Gamma$  concludes that the linear pencil  $\mu M - K$  with

$$M = X^{-T} \Gamma X^{-1} \tag{12}$$

$$K = X^{-T} \Gamma \Lambda X^{-1} \tag{13}$$

is self-adjoint and has eigenstructure  $(X, \Lambda)$ . This is the parametric solution to the inverse eigenvalue problem associated with  $(X, \Lambda)$ . More important, if the parameter matrix  $\Gamma$  is positive definite, then so is the matrix  $M$ . We thus have proved the following fact.

*Theorem 2.1:* A self-adjoint linear pencil can have arbitrary eigenstructure with distinct eigenvalues and linearly independent eigenvectors. Indeed, given an eigenstructure  $(X, \Lambda)$ , the solutions  $(M, K)$  form a subspace of dimensionality  $n$  in the product space  $\mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n}$  and can be parameterized by the diagonal matrix  $\Gamma$  via the relationships (12) and (13).

The task of modifying a partial eigenstructure from  $(X_1, \Lambda_1)$  to  $(Y, \Sigma)$  while maintaining the remaining eigenstructure  $(X_2, \Lambda_2)$ , therefore is possible.

*Corollary 2.2:* Given any  $k \leq n$ , assume that the observed eigenvalues  $\Sigma$  and the original eigenvalues  $\Lambda_2$  are all distinct. Assume also that the corresponding observed eigenvectors  $Y$  and the original eigenvectors  $X_2$  form a nonsingular matrix. Then the model updating of (5) with no spillover is always possible.

With the parameterization (12) and (13) in hand, we can further refine the model updating problem by demanding that the changes  $\Delta M$  and  $\Delta K$  be kept at minimum with respect to some measurement. For instance, the model updating problem could be modified to the problem of finding the optimal solution to the minimization problem [24]:

$$\min_{\Gamma = \text{diagonal}} \|X^{-T} \Gamma X^{-1} - M_0\|_F^2 + \|X^{-T} \Gamma \Lambda X^{-1} - K_0\|_F^2 \tag{14}$$

We hastily point out that in the recipe (12) and (13) for constructing  $M$  and  $K$ , as well as in the minimal change formulation (14), knowledge of the full eigenstructure is required. This is precisely the scenario which we dismissed earlier as not feasible in practice. What we have proved is that the updating with no spillover is possible in theory. It remains a problem of practical importance to construct  $M$  and  $K$  without any a priori knowledge of  $(X_2, \Lambda_2)$ .

We demonstrate one possible way of constructing  $M$  and  $K$  without any prior knowledge of  $(X_2, \Lambda_2)$ . Recall that we may always assume without loss of generality that the eigenvectors  $(X_1, X_2) \in \mathbb{R}^{n \times k} \times \mathbb{R}^{n \times (n-k)}$  of the original pencil  $\mu M_0 - K_0$  are normalized in such a way that

$$\begin{bmatrix} X_1^T \\ X_2^T \end{bmatrix} M_0 [X_1, X_2] = I_n \tag{15}$$

$$\begin{bmatrix} X_1^T \\ X_2^T \end{bmatrix} K_0 [X_1, X_2] = \text{diag}\{\Lambda_1, \Lambda_2\} \tag{16}$$

Likewise, by choosing  $\Gamma = I_n$ , we see that

$$M = X^{-\top} X^{-1} \quad (17)$$

$$K = X^{-\top} \Lambda X^{-1} \quad (18)$$

If none of the diagonal entries is zero, then from the facts that

$$M^{-1} = XX^{\top} = YY^{\top} + X_2 X_2^{\top},$$

$$K^{-1} = X\Lambda^{-1}X^{\top} = Y\Sigma^{-1}Y^{\top} + X_2\Lambda_2^{-1}X_2^{\top},$$

$$M_0^{-1} = X_1 X_1^{\top} + X_2 X_2^{\top}, \quad K_0^{-1} = X_1 \Lambda_1^{-1} X_1^{\top} + X_2 \Lambda_2^{-1} X_2^{\top}$$

we can express  $M^{-1}$  and  $K^{-1}$  as

$$M^{-1} = M_0^{-1} + YY^{\top} - X_1 X_1^{\top} \quad (19)$$

$$K^{-1} = K_0^{-1} + Y\Sigma^{-1}Y^{\top} - X_1 \Lambda_1^{-1} X_1^{\top} \quad (20)$$

showing the construction of  $M$  and  $K$  without the knowledge of  $(X_2, \Lambda_2)$ . Note that the previous approach constructs  $M^{-1}$ , not  $M$ . Note also that the matrix coefficients  $M$  and  $K$  constructed in this way are uniquely determined, and we no longer have the freedom in selecting other  $\Gamma$ . Our construction demonstrates only the point that in theory  $M$  can be constructed without reference to  $X_2$ , but may not be of much value in real application. Developing a more practical way of constructing  $M$  without reference to  $X_2$  remains an interesting research topic. We do have in this case the inequalities

$$\|M_0 M^{-1} - I\| \leq \|M_0\| \|X_1 X_1^{\top} - YY^{\top}\|,$$

$$\|K_0 K^{-1} - I\| \leq \|K_0\| \|X_1 \Lambda_1^{-1} X_1^{\top} - Y\Sigma^{-1}Y^{\top}\|$$

which might be useful for estimating an upper bound for (14).

### B. One-Sided Updating

It might be worthy to briefly examine another scenario proposed in the dissertation by Carvalho [25]. The question is whether an updating can be accomplished by just modifying one single coefficient matrix, say  $K_0$ . That is, instead of (6) and (7), can a symmetric matrix  $\Delta K$  be found such that the equations

$$M_0 X_2 \Lambda_2 = (K_0 + \Delta K) X_2 \quad (21)$$

$$M_0 Y \Sigma = (K_0 + \Delta K) Y \quad (22)$$

are satisfied? A quick count shows that there are  $n^2$  equations in  $n(n+1)/2$  unknowns. We think that the updating problem as an overdetermined system cannot be solved unless the newly prescribed eigenstructure  $(Y, \Sigma)$  satisfies some consistency stipulations. To explore the necessary conditions, we first claim that any feasible candidate  $\Delta K$  must be parameterized as follows:

*Lemma 2.3:* Assume that the eigenvectors  $(X_1, X_2)$  of the pencil  $\mu M_0 - K_0$  have been normalized as in (15) and (16). A symmetric matrix  $\Delta K$  satisfies (21) if and only if there exists a symmetric matrix  $\Phi \in \mathbb{R}^{k \times k}$  such that

$$\Delta K = M_0 X_1 \Phi X_1^{\top} M_0 \quad (23)$$

*Proof:* The matrix  $\Delta K$  satisfies (21) if and only of  $\Delta K X_2 = 0$ , implying that the row space of  $\Delta K$  must be a left null space of  $X_2$ , which, by (15), is spanned by the rows of  $X_1^{\top} M_0$ . The formula (23) then follows from the requirement for the symmetry of  $\Delta K$ .

With  $\Delta K$  defined by (23), we now argue that the Eq. (22) holds only when a rather strict consistency condition is satisfied. The call made in [25] about updating (5) with no spillover on the single matrix  $K_0$  therefore can be achieved only when  $Y$  is of some very special form. We remark that a similar spirit holds for the damped problem, but the derivation of the corresponding form is much more complicated [23].  $\square$

*Lemma 2.4:* There exists a symmetric matrix  $\Phi \in \mathbb{R}^{k \times k}$  such that

$$M_0 Y \Sigma = (K_0 + M_0 X_1 \Phi X_1^{\top} M_0) Y \quad (24)$$

if and only if

$$Y = X_1 V D \quad (25)$$

for some orthogonal matrix  $V \in \mathbb{R}^{k \times k}$  and some nonsingular diagonal matrix  $D \in \mathbb{R}^{k \times k}$ .

*Proof:* The proof is based on the orthogonality assumed in (15) and (16) among the eigenvectors. Because  $Y^{\top} M_0 X_2 = 0$ ,  $Y$  must be of the form  $Y = X_1 L$  for some  $k \times k$  matrix  $L$ . Furthermore,  $Y^{\top} M_0 Y = L^{\top} X_1^{\top} M_0 X_1 L = L^{\top} L$  is a diagonal matrix. We thus can write  $L = VD$  as described.  $\square$

### III. Quadratic Inverse Eigenvalue Problem

We have seen in the preceding section that a self-adjoint pencil  $\mu M - K$  in  $\mathbb{R}^{n \times n}$  with positive definite  $M$  can have arbitrary  $n$  distinct real eigenvalues and  $n$  linearly independent real eigenvectors. Consequently, the model updating with no spillover for self-adjoint linear pencils is always possible. Before we explore whether this result can be extended to quadratic pencils, it is natural to ask how much partial eigenpair information is allowable for constructing a self-adjoint quadratic pencil  $\lambda^2 M + \lambda C + K$ . A special case related to this question can be found in an earlier paper [26]. A more detailed analysis partially addressing this issue is given in the recent paper by Kuo, Lin, and Xu [27]. Our main contribution in this section is a complete characterization of the general solution.

Without causing ambiguity, we shall use the same notation  $(X, \Lambda) \in \mathbb{R}^{n \times k} \times \mathbb{R}^{k \times k}$  to denote  $k$  given eigenpairs. For the moment,  $k$  can be any integer between 1 and  $2n$ . We shall assume that  $\Lambda$  is closed under complex conjugation. Thus  $\Lambda$  is of diagonal form with  $2 \times 2$  blocks along the diagonal wherever a complex-conjugate pair of eigenvalues appear in the prescribed spectrum.

Consider the algebraic system

$$MX\Lambda^2 + CX\Lambda + KX = 0_{n \times k} \quad (26)$$

for the triplet  $(M, C, K)$ . There are  $nk$  equations in  $\frac{3n(n+1)}{2}$  unknowns in this homogeneous equation. It is intuitively true that if the number  $k$  of prescribed eigenpair is capped by the bound

$$k < 3(n+1)/2 \quad (27)$$

then the system (26) is underdetermined and the solutions form a subspace of dimensionality  $[3n(n+1)/2] - nk$ ; otherwise, the algebraic system and, hence, the QIEP, will have only a trivial solution. In what follows, we prove that this conjecture is indeed true. More important, in our proof, we provide a parametric representation of the solution. For solvability, we thus see that the maximal allowable number  $k_{\max}$  of prescribed eigenpairs is given by

$$k_{\max} = \begin{cases} 3\ell + 1, & \text{if } n = 2\ell \\ 3\ell + 2, & \text{if } n = 2\ell + 1 \end{cases} \quad (28)$$

These bounds of  $k_{\max}$  have the consequence that, in contrast to the linear pencil, the remaining  $2n - k_{\max}$  eigenpairs of a quadratic pencil cannot be arbitrarily assigned anymore. In other words, when  $n \geq 3$ , there will be no room to maintain no spillover if the updating intends to replace  $k_{\max}$  original eigenpairs by newly measured data.

We first analyze the necessary condition for the self-adjoint quadratic pencil  $(M, C, K)$  to have eigenstructure

$$(X, \Lambda) \in \mathbb{R}^{n \times k} \times \mathbb{R}^{k \times k}$$

For now, we have no restriction on  $k$ . Consider the matrix

$$\Omega := [I_k, \Lambda^{\top}, \Lambda^{2\top}] \in \mathbb{R}^{k \times 3k} \quad (29)$$

and let a basis of its null space be partitioned into three blocks so that we can write

$$\Omega \begin{bmatrix} U \\ T \\ S \end{bmatrix} = 0_{k \times 2k} \quad (30)$$

where  $S, T,$  and  $U$  are matrices in  $\mathbb{R}^{k \times 2k}$ . Obviously,

$$U = -\Lambda^\top T - \Lambda^{2\top} S \quad (31)$$

is determined once  $S$  and  $T$  are specified. Any triplet  $(M, C, K)$  satisfying (26) must be such that

$$\Omega \begin{bmatrix} X^\top K \\ X^\top C \\ X^\top M \end{bmatrix} = 0_{k \times n} \quad (32)$$

There must exist a matrix  $\Psi \in \mathbb{R}^{2k \times n}$  such that

$$\begin{bmatrix} U \\ T \\ S \end{bmatrix} \Psi = \begin{bmatrix} X^\top K \\ X^\top C \\ X^\top M \end{bmatrix} \quad (33)$$

Because  $M, C,$  and  $K$  are symmetric, the three matrices

$$A := S\Psi X \quad (34)$$

$$B := T\Psi X \quad (35)$$

$$F := U\Psi X \quad (36)$$

must also be symmetric in  $\mathbb{R}^{k \times k}$ . Substituting (31) into (36) and using the fact that  $F = F^\top$ , we obtain a critical relationship between  $A$  and  $B$ :

$$\Lambda^\top B - B\Lambda = A\Lambda^2 - \Lambda^{2\top} A \quad (37)$$

Observe that the difference on either side of (37) is a skew-symmetric matrix.

The previously detailed necessary condition can also be used to construct a solutions  $(M, C, K)$  in terms of  $A$  and  $B$ . Because of the constraint (37), not all entries in  $A$  or  $B$  are free. We shall exploit those free parameters and establish a parametric relationship. For clarity, we divide our discussion into three cases.

**A. Case  $k = n$**

This is the most important case, which plays a pivotal role in the other two cases. Suppose that a symmetric matrix  $A \in \mathbb{R}^{n \times n}$  is given. Denote

$$\Theta := A\Lambda^2 - \Lambda^{2\top} A$$

We need to see how  $B$  can be determined from the equation

$$\Lambda^\top B - B\Lambda = \Theta$$

For convenience, we may assume without loss of generality that  $\Lambda$  is of the diagonal form

$$\Lambda = \text{diag}\{\lambda_1^{[2]}, \dots, \lambda_v^{[2]}, \lambda_{v+1}, \dots, \lambda_\ell\}, \quad (38)$$

where

$$\lambda_j^{[2]} = \begin{bmatrix} \alpha_j & \beta_j \\ -\beta_j & \alpha_j \end{bmatrix} \in \mathbb{R}^{2 \times 2}$$

$\beta_j \neq 0$ , if  $j = 1, \dots, v$ ;  $\lambda_j \in \mathbb{R}$  if  $j = v + 1, \dots, \ell$ , and  $\ell + v = n$ . Partition  $B$  into  $\ell \times \ell$  blocks, denoted by  $B = [B_{ij}]$ , in such a way that  $\text{diag}\{B_{11}, \dots, B_{\ell\ell}\}$  has exactly the same structure as  $\Lambda$ . Then the  $(i, j)$  block of the skew-symmetric matrix  $\Lambda^\top B - B\Lambda$  is given by one of the following three possibilities:

$$\begin{cases} \lambda_i^\top B_{ij} - B_{ij} \lambda_j, & \text{if } v + 1 \leq i, j \leq \ell \\ (\lambda_i^{[2]})^\top B_{ij} - B_{ij} \lambda_j, & \text{if } 1 \leq i \leq v \text{ and } v + 1 \leq j \leq \ell \\ (\lambda_i^{[2]})^\top B_{ij} - B_{ij} (\lambda_j^{[2]}), & \text{if } 1 \leq i, j \leq v \end{cases} \quad (39)$$

In the first case,  $B_{ij}$  is a scalar. Upon comparing with the corresponding blocks in  $\Theta$ , we find that  $B_{ij}$  is uniquely determined except that  $B_{ii}$  is free. Likewise,  $B_{ij}$  in the second case is a  $2 \times 1$  block and all its entries are uniquely determined. In the third case, if we write

$$B_{ij} = \begin{bmatrix} x & y \\ y & z \end{bmatrix}$$

then

$$(\lambda_i^{[2]})^\top B_{ij} - B_{ij} (\lambda_j^{[2]}) = \begin{bmatrix} x(\alpha_i - \alpha_j) - y(\beta_i - \beta_j) & -z\beta_i - x\beta_j \\ x\beta_i + y(\alpha_i - \alpha_j) + z\beta_j & y(\beta_i - \beta_j) \end{bmatrix}$$

It is clear that if  $i = j$ , then  $y$  is free and  $x + z$  is a fixed constant, still giving rise to 2 degrees of freedom. If  $i \neq j$ , then all entries of  $B_{ij}$  are uniquely determined. In all, we conclude that the symmetric matrix  $A \in \mathbb{R}^{n \times n}$  can be totally arbitrary, whereas  $B$  is determined up to  $n$  free parameters. We thus declare the following theorem.

*Theorem 3.1:* Given  $n$  distinct eigenvalues  $\Lambda$  and  $n$  linearly independent eigenvectors  $X$ , both of which are closed under conjugation, let  $A \in \mathbb{R}^{n \times n}$  be an arbitrary symmetric matrix and let  $B$  be a solution to the Eq. (37). Then the self-adjoint quadratic pencil with its coefficients  $(M, C, K)$  defined by

$$M = X^{-\top} A X^{-1} \quad (40)$$

$$C = X^{-\top} B X^{-1} \quad (41)$$

$$K = -X^{-\top} \Lambda^\top (B + \Lambda^\top A) X^{-1} \quad (42)$$

and has the prescribed pair  $(X, \Lambda)$  as part of its eigenstructure.

*Proof:* The proof is already contained in our construction mentioned here, except that we need to remove the reference to the intermediate parameters  $\Psi, S,$  and  $T$ . The relationship (33) implies that  $M = X^{-\top} S\Psi$  for some  $\Psi \in \mathbb{R}^{2n \times n}$ . We also know from (34) that  $A = S\Psi X$ . Together, we can express  $M$  as  $M = X^{-\top} A X^{-1}$ . Similar arguments can be applied to  $C$  and  $K$ .  $\square$

It is worth mentioning that if  $A$  is selected to be symmetric and positive definite, then so is the leading coefficient  $M$ . Indeed, the previously stated construction parameterizes all possible solutions.

*Corollary 3.2:* The solutions  $(M, C, K)$  to the quadratic inverse eigenvalue problem with eigenstructure  $(X, \Lambda)$  as described in Theorem 3.1 form a subspace of dimensionality  $n(n + 3)/2$  in the product space  $\mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n}$ .

*Corollary 3.3:* With  $(M, C, K)$  being defined in Theorem 3.1, the corresponding quadratic pencil can be factorized as

$$\begin{aligned} \lambda^2 M + \lambda C + K &= X^{-\top} (\lambda I_n - \Lambda^\top) [B + (\lambda I_n + \Lambda^\top) A] X^{-1} \\ &= X^{-\top} [(B + A(\lambda I_n + \Lambda))] (\lambda I_n - \Lambda) X^{-1} \end{aligned} \quad (43)$$

Based on Corollary 3.3, the remaining eigenvalues therefore are determined by those of the linear pencil  $\lambda A + B + A\Lambda$ . Because the entire  $A$  and part of  $B$  are free, there is room to impose additional eigeninformation to the pencil. In [27], for instance, it was argued that additional  $n$  eigenvalues could be specified. In our context, we ask how many more eigenpairs can be prescribed. We shall study the general case in Sec. III.C.

**B. Case  $k < n$**

If less than  $n$  eigenpairs  $(X, \Lambda)$  are given, we can solve the QIEP by embedding this given eigeninformation in a larger set of  $n$  eigenpairs, giving leeway of more free parameters. In particular, we expand  $X \in \mathbb{R}^{n \times k}$  to

$$\widehat{X} := [X, \widetilde{X}] \in \mathbb{R}^{n \times n} \tag{44}$$

where  $\widetilde{X} \in \mathbb{R}^{n \times (n-k)}$  is arbitrary while making  $\widehat{X}$  nonsingular. A caution should be taken when counting the degrees of freedom; the columns in  $\widetilde{X}$  should be considered as being normalized because otherwise any normalization factor would have been included in the arbitrariness of  $A$ . With this normalization in mind, this expansion of eigenvectors involves additional  $(n - 1)(n - k)$  degrees of freedom. We then expand  $\Lambda \in \mathbb{R}^{k \times k}$  to

$$\widehat{\Lambda} := \text{diag}\{\Lambda, \widetilde{\Lambda}\} \tag{45}$$

where  $\widehat{\Lambda}$  is a diagonal matrix with distinct eigenvalues. This expansion of eigenvalues gives rise to another  $n - k$  degrees of freedom. With  $(\widehat{X}, \widehat{\Lambda})$  playing the role of  $(X, \Lambda)$  in Theorem 3.1, we can now construct the coefficient matrices  $M, C,$  and  $K$  according to the formulas (40–42), respectively, whereas  $A$  is taken as an arbitrary symmetric matrix in  $\mathbb{R}^{n \times n}$ , and  $B$ , depending on  $\widehat{\Lambda}$  through the relationship (37), maintains  $n$  degrees of freedom. Note that the parametrization of  $(M, C, K)$  in this embedding approach is nonlinear in  $A, B, \widehat{X}$  and  $\widehat{\Lambda}$ . We summarize this construction as follows:

*Theorem 3.4:* The quadratic inverse eigenvalue problem with  $k, k < n$  prescribed eigenpairs is always solvable. For almost all prescribed eigenstructure  $(X, \Lambda)$ , the solutions form a subspace of dimensionality  $[n(n + 3)/2] + n(n - k)$ .

*Proof:* We only need to justify the dimensionality. It is clear that the solutions  $(M, C, K)$  to the homogeneous system (26) with  $k$  prescribed eigenpairs form a subspace of dimensionality at least  $[3n(n + 1)/2] - nk$ , but in the previous equation, we have just found a parametric representation of  $(M, C, K)$ , which involves precisely  $n(n + 1)/2$  free parameters in  $A, n$  in  $B, n - k$  in  $\widetilde{\Lambda}$ , and  $(n - 1)(n - k)$  in  $\widetilde{X}$ , giving a total of  $[3n(n + 1)/2] - nk$  free parameters.  $\square$

**C. Case  $k > n$**

This case is rather involved and, to our knowledge, has never been discussed in the literature. At a first glance, we know from the relationships (33–35) that

$$A = S\Psi X = X^T M X \tag{46}$$

$$B = T\Psi X = X^T C X \tag{47}$$

$$F = U\Psi X = X^T K X \tag{48}$$

remain symmetric even in the case  $k > n$ . However, we cannot obtain a parametric representation of  $M, C,$  and  $K$  from  $A$  and  $B$  directly because  $X \in \mathbb{R}^{n \times k}$  is no longer an injection transformation. The challenge is to retrieve  $M, C,$  and  $K \in \mathbb{R}^{n \times n}$  from the seemingly overspecified  $A$  and  $B$ .

Rewrite the eigenvectors as

$$X = [Z_1, Z_2]$$

where  $Z_1 \in \mathbb{R}^{n \times n}$  and  $Z_2 \in \mathbb{R}^{n \times (k-n)}$ . Then we see that

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{bmatrix} = \begin{bmatrix} Z_1^T M Z_1 & Z_1^T M Z_2 \\ Z_2^T M Z_1 & Z_2^T M Z_2 \end{bmatrix} \tag{49}$$

where  $A_{ij}, i, j = 1, 2,$  are blocks with appropriate sizes. This relationship suggests that we may choose a symmetric matrix  $A_{11} \in \mathbb{R}^{n \times n}$  arbitrarily and define

$$M = Z_1^{-T} A_{11} Z_1^{-1} \tag{50}$$

This selection give rises to  $n(n + 1)/2$  degrees of freedom. Once  $M \in \mathbb{R}^{n \times n}$  is determined, the matrix  $A \in \mathbb{R}^{k \times k}$  is completely specified. There is no additional freedom in the choice of  $A$ . With

$A \in \mathbb{R}^{k \times k}$  given, we need to determine the matrix  $B \in \mathbb{R}^{k \times k}$  so that the necessary condition (37) is satisfied. Note that

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{12}^T & B_{22} \end{bmatrix} = \begin{bmatrix} Z_1^T C Z_1 & Z_1^T C Z_2 \\ Z_2^T C Z_1 & Z_2^T C Z_2 \end{bmatrix} \tag{51}$$

Consider the  $(1, 1)$  block of  $B$  first. With  $A_{11}$  given, using an argument similar to that made in Sec. III.A [see (39)], we see that the submatrix  $B_{11}$  is completely determined up to  $n$  free parameters. These would have determined a symmetric matrix

$$C = Z_1^{-T} B_{11} Z_1^{-1} \tag{52}$$

and, hence, the matrix  $B$  up to  $n$  free parameters. However, the very same  $C$  should also equate the two sides of Eq. (37) at the  $(1, 2)$  and  $(2, 2)$  blocks, respectively. These blocks involve more than  $n$  equations to be satisfied. We have no choice but to go back to modify the selection of  $A_{11}$  and sacrifice some of the freedom. In other words, the  $n$  free parameters in  $B_{11}$  and the matrix  $A_{11}$  must be further restricted so that the remaining part of  $B$  also satisfies (37). Toward that end, we reexamine the relationship between  $A$  and  $B$  block by block. If we define

$$\Xi := Z_1^{-1} Z_2 \tag{53}$$

then it follows that

$$A = \begin{bmatrix} A_{11} & A_{11} \Xi \\ \Xi^T A_{11} & \Xi^T A_{11} \Xi \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & B_{11} \Xi \\ \Xi^T B_{11} & \Xi^T B_{11} \Xi \end{bmatrix}$$

If we partition the given eigenvalues as

$$\Lambda = \text{diag}\{\Upsilon_1, \Upsilon_2\}$$

where  $\Upsilon_1 \in \mathbb{R}^{n \times n}$  and  $\Upsilon_2 \in \mathbb{R}^{(k-n) \times (k-n)}$ , then the critical condition (37) to be satisfied can be expressed as three equations:

$$\Upsilon_1^T B_{11} - B_{11} \Upsilon_1 = A_{11} \Upsilon_1^2 - \Upsilon_1^2 A_{11} \tag{54}$$

$$\Upsilon_1^T B_{11} \Xi - B_{11} \Xi \Upsilon_2 = A_{11} \Xi \Upsilon_2^2 - \Upsilon_1^2 A_{11} \Xi \tag{55}$$

$$\Upsilon_2^T \Xi^T B_{11} \Xi - \Xi^T B_{11} \Xi \Upsilon_1 = \Xi^T A_{11} \Xi \Upsilon_2^2 - \Upsilon_2^2 \Xi^T A_{11} \Xi \tag{56}$$

Postmultiplying (54) by  $\Xi$  and subtracting (55), we obtain that

$$A_{11} \Upsilon_1^2 \Xi + B_{11} \Upsilon_1 \Xi = A_{11} \Xi \Upsilon_2^2 + B_{11} \Xi \Upsilon_2$$

It follows that

$$\begin{aligned} \Xi^T (A_{11} \Xi \Upsilon_2^2 + B_{11} \Xi \Upsilon_2) &= \Xi^T (A_{11} \Upsilon_1^2 \Xi + B_{11} \Upsilon_1 \Xi) \\ &= (\Xi^T A_{11} \Upsilon_1^2 + \Xi^T B_{11} \Upsilon_1) \Xi = (\Upsilon_2^T \Xi^T B_{11} + \Upsilon_2^2 \Xi^T A_{11}) \Xi \end{aligned}$$

which is precisely (56). In the previous equation, the last equality follows from taking the transpose of Eq. (55). What we have just proved is that if we can solve Eqs. (54) and (55), then (56) is automatically solved. We have indicated earlier that any given  $A_{11}$  will determine  $B_{11}$  through (54) up to  $n$  free parameters. Thus, it only remains to choose the  $n$  free parameters in  $B_{11}$  and the  $n \times n$  symmetric matrix  $A_{11}$  to satisfy the  $n(k - n)$  linear equations imposed by (55). Totally, only

$$\frac{n(n + 1)}{2} + n - n(k - n) = \frac{3n(n + 1)}{2} - nk$$

degrees of freedom remain.

Together with results proved in the preceding sections, we have now established the following results.

*Theorem 3.5:* Given any  $1 \leq k < 3(n + 1)/2$ , the quadratic inverse eigenvalues with  $k$  prescribed eigenpairs is always solvable. For almost all prescribed eigenstructure  $(X, \Lambda)$ , the solutions form a

subspace of dimensionality precisely  $[3n(n + 1)/2] - nk$ . The maximal allowable number of prescribed eigenpairs is given by (28).

We demonstrate the occurrence of the spillover phenomenon when  $k$  is too large.

*Example 1.* Suppose that the prescribed eigenstructure is given by

$$X = \begin{bmatrix} 1 & 0 & 0 & 2 & -1 \\ 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & 2 & 2 \end{bmatrix}$$

and

$$\Lambda = \text{diag}\{1, 2, 3, 5, 8\}$$

This is a case in which  $n = 3$  and  $k = k_{\max} = 5$ . By our theory, the solution has 3 degrees of freedom. Solving the linear system (54) and (55), we find that the general solution to the QIEP can be represented as

$$M = \begin{bmatrix} s & -s + 4u & u \\ -s + 4u & w & -\frac{7}{10}s + \frac{14}{5}u \\ u & -\frac{7}{10}s + \frac{14}{5}u & -\frac{3}{10}u + \frac{7}{10}s \end{bmatrix}$$

$$C = \begin{bmatrix} -9s + 10u & 3s - 12u & -4u \\ 3s - 12u & -\frac{27}{5}s + \frac{108}{5}u - 7w & \frac{7}{5}s - 14u \\ -4u & \frac{7}{2}s - 14u & \frac{54}{5}u - \frac{77}{10}s \end{bmatrix}$$

$$K = \begin{bmatrix} 8s - 10u & -2s + 8u & 3u \\ -2s + 8u & \frac{54}{5}s - \frac{216}{5}u + 10w & -\frac{21}{5}s + \frac{84}{5}u \\ 3u & -\frac{21}{5}s + \frac{84}{5}u & -\frac{177}{10}u + \frac{84}{5}s \end{bmatrix}$$

It can be computed that

$$\det(M) = -\frac{1}{100}(7s - 10u)(272u^2 - 136su - 10wu - 10ws + 17s^2)$$

Obviously, the parameters  $s$ ,  $u$  and  $w$  can be chosen so that  $M$  is positive definite. We also find that the sixth eigenvalue is given by

$$\lambda_6 = -2 \frac{52u^2 + 37s^2 - 161su + 40st - 35tu}{17s^2 - 136su - 10wu + 272u^2 - 10sw}$$

with its corresponding eigenvector given by

$$\mathbf{x}_6 = \left[ \frac{29s - 36u + 5w}{5}, 1, \frac{29s - 36u + 5w}{5} \right]^T$$

It is clear that  $\mathbf{x}_6$  cannot be arbitrarily assigned and, hence, no spillover cannot be maintained.

#### IV. Case Study of No Spillover

We have argued in the preceding section that in the process of updating a quadratic pencil, the phenomenon of spillover is

inevitable in general. Thus it is natural and critical to ask under what stringent conditions we can maintain no spillover. To answer this question requires laborious and careful analysis, which is accomplished in the paper [23]. It might be informative if we demonstrate in this section by a numerical example on how specific the prescribed eigenstructure  $(Y, \Sigma)$  must be to maintain the no spillover in the updating.

Given the original self-adjoint quadratic pencil  $Q(\lambda) = \lambda^2 M_0 + \lambda C_0 + K_0$ , let its eigenvector and eigenvalue matrices be expressed in real-value form as we have described before. Partition the eigenstructure as  $[X_1, X_2] \in \mathbb{R}^{n \times 2n}$  and  $\text{diag}\{\Lambda_1, \Lambda_2\} \in \mathbb{R}^{2n \times 2n}$ , respectively, where the portion  $(X_1, \Lambda_1) \in \mathbb{R}^{n \times k} \times \mathbb{R}^{k \times k}$  is to be updated by newly measured eigenpair  $(Y, \Sigma)$ . Recall that the updating with no spillover means to find symmetric matrices  $\Delta M$ ,  $\Delta C$ , and  $\Delta K$  such that the equations

$$(M_0 + \Delta M)X_2\Lambda_2^2 + (C_0 + \Delta C)X_2\Lambda_2 + (K_0 + \Delta K)X_2 = 0 \quad (57)$$

$$(M_0 + \Delta M)Y\Sigma^2 + (C_0 + \Delta C)Y\Sigma + (K_0 + \Delta K)Y = 0 \quad (58)$$

are satisfied simultaneously. Considering this updating problem as a QIEP with prescribed eigenvectors  $[Y, X_2]$  and eigenvalues  $\text{diag}\{\Sigma, \Lambda_2\}$ , we are facing a homogeneous system with  $2n^2$  equations in  $3n(n + 1)/2$  unknowns. If  $n > 3$ , the system is overdetermined. To have a nontrivial solution,  $(Y, \Sigma)$  must satisfy some consistency conditions. This is in contrast to the undamped case studied in Sec. II.

We first explain the subtlety involved in the generic nature of the eigenpair  $(X_2, \Lambda_2)$ . Observe that if (57) holds, then the incremental pencil

$$\Delta Q(\lambda) := \lambda^2 \Delta M + \lambda \Delta C + \Delta K \quad (59)$$

necessarily has the  $\hat{k} = 2n - k$  eigenpairs  $(X_2, \Lambda_2)$  as part of its eigenstructure. If  $\hat{k} < 3(n + 1)/2$ , that is, if  $k > (n - 3)/2$ , then our theory asserts that there are nontrivial solutions  $(\Delta M, \Delta C, \Delta K)$  which, for almost all  $(X_2, \Lambda_2)$ , form a subspace of dimension  $n(2k - n + 3)/2$ . The pencil  $(\Delta M, \Delta C, \Delta K)$  can be characterized by the procedures described in Sec. III.C. If  $\hat{k} \geq 3(n + 1)/2$ , that is, if  $k \leq (n - 3)/2$ , then our theory implies that the solution space to the QIEP with generic eigenpairs  $(X_2, \Lambda_2)$  should be made of the trivial solution only. However, because we have already assumed that  $(M_0, C_0, K_0)$  has  $(X_2, \Lambda_2)$  as part of its eigenstructure, we have to conclude that  $(X_2, \Lambda_2)$  is not generic in the sense that the seemingly overdetermined algebraic system

$$\Delta M X_2 \Lambda_2^2 + \Delta C X_2 \Lambda_2 + \Delta K X_2 = 0 \quad (60)$$

is not overly determined at all and in fact has nontrivial solutions. The following numerical experiment serves to shed some insight into this situation. For more complete theory, readers are encouraged to read through [23].

*Example 2.* Consider the case  $n = 5$  and  $k = 1$  with

$$M_0 = \begin{bmatrix} 3.3308 & 1.9508 & 2.0792 & 1.0873 & 2.3424 \\ 1.9508 & 1.6595 & 1.3898 & 0.6036 & 1.5318 \\ 2.0792 & 1.3898 & 1.7062 & 0.8195 & 1.5197 \\ 1.0873 & 0.6036 & 0.8195 & 0.5217 & 0.7819 \\ 2.3424 & 1.5318 & 1.5197 & 0.7819 & 1.7472 \end{bmatrix}, C_0 = \begin{bmatrix} 1.0454 & 0.8031 & 1.1669 & 1.0143 & 0.7795 \\ 0.8031 & 1.3832 & 0.6174 & 1.3404 & 0.8307 \\ 1.1669 & 0.6174 & 1.6762 & 0.6650 & 1.0423 \\ 1.0143 & 1.3404 & 0.6650 & 0.9317 & 1.2889 \\ 0.7795 & 0.8307 & 1.0423 & 1.2889 & 0.5037 \end{bmatrix}$$

$$K_0 = \begin{bmatrix} 2.6981 & 2.2257 & 1.5499 & 1.6738 & 1.5832 \\ 2.2257 & 2.2472 & 1.4826 & 1.6162 & 1.3072 \\ 1.5499 & 1.4826 & 1.2197 & 1.0846 & 1.1743 \\ 1.6738 & 1.6162 & 1.0846 & 1.5889 & 0.8304 \\ 1.5832 & 1.3072 & 1.1743 & 0.8304 & 1.4532 \end{bmatrix}$$

The pencil  $(M_0, C_0, K_0)$  has eigenvectors

$$\underbrace{\begin{bmatrix} -0.0101 \\ 0.0395 \\ -0.0537 \\ 0.0805 \\ -0.0073 \end{bmatrix}}_{X_1}, \underbrace{\begin{bmatrix} 0.1148 & -0.0410 & -0.5947 & -0.2308 & -0.3380 \\ 0.0625 & -0.0001 & -0.2369 & -0.0038 & 0.4726 \\ 0.0243 & -0.0536 & 0.6170 & 0.1379 & -1.0000 \\ 0.0927 & -0.0737 & 0.4292 & 0.2892 & 0.2785 \\ -0.2859 & 0.1619 & 0.5020 & 0.1943 & 0.5821 \end{bmatrix}}_{Z_1}, \underbrace{\begin{bmatrix} 0.4771 & -0.0893 & 0.4573 & -0.2609 \\ 0.5463 & 0.0699 & -0.6302 & 0.3698 \\ -0.3110 & -0.3052 & -0.4527 & 0.1497 \\ -0.5810 & -0.4190 & 0.4052 & -0.2067 \\ -0.1483 & 0.6559 & 0.2904 & -0.1039 \end{bmatrix}}_{Z_2}$$

and eigenvalues

$$\underbrace{-12.4263}_{\Lambda_1}, \underbrace{\begin{bmatrix} 2.4975 & 1.5414 \\ -1.5414 & 2.4975 \end{bmatrix}}_{\Upsilon_1} \begin{bmatrix} -1.0079 & 0.6851 \\ -0.6851 & -1.0079 \end{bmatrix}, \underbrace{-0.0603}_{\Lambda_2}, \underbrace{\begin{bmatrix} 0.3444 & 0.9859 \\ -0.9859 & 0.3444 \end{bmatrix}}_{\Upsilon_2} \begin{bmatrix} -0.1218 & 0.7665 \\ -0.7665 & -0.1218 \end{bmatrix}$$

which we have partitioned into  $[X_1, Z_1, Z_2]$  and  $[\Lambda_1, \Upsilon_1, \Upsilon_2]$  with  $X_1 \in \mathbb{R}^{5 \times 1}$ ,  $Z_1 \in \mathbb{R}^{5 \times 5}$ ,  $Z_2 \in \mathbb{R}^{5 \times 4}$ ,  $\Lambda_1 \in \mathbb{R}$ ,  $\Upsilon_1 \in \mathbb{R}^{5 \times 5}$ , and  $\Upsilon_2 \in \mathbb{R}^{4 \times 4}$ , respectively. According to the theory developed earlier, the general solution  $(\Delta M, \Delta C, \Delta K)$  to (60) can be expressed in the form

$$\Delta M = Z_1^{-T} A_{11} Z_1^{-1}, \quad \Delta C = Z_1^{-T} B_{11} Z_1^{-1},$$

$$\Delta K = -Z_1^{-T} (A_{11} \Upsilon_1^2 + B_{11} \Upsilon_1) Z_1^{-1}$$

where the symmetric matrices  $A_{11}$  and  $B_{11}$  must satisfy Eqs. (54) and (55) simultaneously. Using our data listed previously, we find that

$$A_{11} = \alpha A^{[1]} + \beta A^{[2]} \tag{61}$$

$$B_{11} = \alpha B^{[1]} + \beta B^{[2]} \tag{62}$$

with arbitrary  $\alpha, \beta \in \mathbb{R}$ , where

$$A^{[1]} = \begin{bmatrix} 0.0010 & -0.0015 & 0.0005 & -0.0044 & -0.0297 \\ -0.0015 & 0.0025 & -0.0008 & 0.0070 & 0.0472 \\ 0.0005 & -0.0008 & 0.0006 & -0.0022 & -0.0161 \\ -0.0044 & 0.0070 & -0.0022 & 0.0201 & 0.1351 \\ -0.0297 & 0.0472 & -0.0161 & 0.1351 & 0.9106 \end{bmatrix}$$

$$A^{[2]} = \begin{bmatrix} -0.0026 & 0.0020 & -0.0047 & -0.0023 & 0.0050 \\ 0.0020 & -0.0021 & 0.0068 & 0.0031 & 0.0016 \\ -0.0047 & 0.0068 & -0.1916 & -0.0800 & 0.1530 \\ -0.0023 & 0.0031 & -0.0800 & -0.0295 & 0.0994 \\ 0.0050 & 0.0016 & 0.1530 & 0.0994 & 0.0379 \end{bmatrix}$$

$$B^{[1]} = \begin{bmatrix} -0.0096 & 0.0100 & -0.0051 & 0.0170 & 0.1452 \\ 0.0100 & -0.0075 & 0.0052 & -0.0031 & -0.0693 \\ -0.0051 & 0.0052 & -0.0020 & 0.0091 & 0.0754 \\ 0.0170 & -0.0031 & 0.0091 & 0.0439 & 0.1554 \\ 0.1452 & -0.0693 & 0.0754 & 0.1554 & 0.1111 \end{bmatrix}$$

$$B^{[2]} = \begin{bmatrix} 0.0159 & -0.0113 & 0.0159 & 0.0116 & -0.0097 \\ -0.0113 & 0.0075 & -0.0007 & -0.0057 & -0.0115 \\ 0.0159 & -0.0007 & -0.3738 & -0.1494 & 0.2315 \\ 0.0116 & -0.0057 & -0.1494 & -0.0719 & 0.0013 \\ -0.0097 & -0.0115 & 0.2315 & 0.0013 & -0.7634 \end{bmatrix}$$

serve as a basis. In other words, the solutions to (60), including the original pencil  $(M_0, C_0, K_0)$  in which corresponds to  $\alpha_0 = 0.5820$  and  $\beta_0 = -1.8046$ , form a two-dimensional subspace.

It is now clear that to update the eigenpair  $(X_1, \Lambda_1)$  of the pencil  $(M_0, C_0, K_0)$  to newly measured  $(Y, \Sigma)$  while maintaining no spillover to  $(X_2, \Lambda_2)$ , the newly measured eigenpair  $(Y, \Sigma)$  must satisfy the algebraic system

$$(\alpha A^{[1]} + \beta A^{[2]}) Z_1^{-1} Y \Sigma^2 + (\alpha B^{[1]} + \beta B^{[2]}) Z_1^{-1} Y \Sigma$$

$$= [(\alpha A^{[1]} + \beta A^{[2]}) \Upsilon_1^2 + (\alpha B^{[1]} + \beta B^{[2]}) \Upsilon_1] Z_1^{-1} Y$$

for some  $\alpha, \beta \in \mathbb{R}$ . At first glance, this is a system of five polynomials in eight unknowns whose real solutions form an algebraic variety of dimension three. Two degrees of this freedom come from the choice of  $\alpha$  and  $\beta$ , and the third degree of freedom comes from the scaling of the eigenvector. More specifically, the system by construction already has one real eigenvalue and four pairs of complex-conjugate eigenvalues. So the remaining eigenvalue  $\Sigma$  and the associated eigenvector  $Y$  must be real and are completely determined by  $\alpha$  and  $\beta$ . In fact, in the case when  $\beta \neq 0$ , then  $\Sigma$  is determined by the ratio  $\rho = \frac{\alpha}{\beta}$  through the characteristic polynomial

$$\det\{\lambda^2[\rho A^{[1]} + A^{[2]}] + \lambda[\rho B^{[1]} + B^{[2]}] - [(\rho A^{[1]} + A^{[2]}) \Upsilon_1^2 + (\rho B^{[1]} + B^{[2]}) \Upsilon_1]\} = 0$$

In Fig. 1 we sketch the admissible values of  $\Sigma$  as a function  $\rho$ . Note that the quadratic pencil becomes singular when  $\rho$  is an eigenvalue of the linear pencil  $\rho A^{[1]} + A^{[2]}$ . In Fig. 1, this happens at approximately  $\rho \approx -0.2388$ . The other  $-\infty$  to  $\infty$  jump depicted in the lower drawing of Fig. 1 indicates  $\Sigma = 0$  at approximately  $\rho \approx 0.9050$ . It is seen empirically that  $\Sigma$  can be arbitrary real numbers. However, to keep the eigenstructure  $(X_2, \Lambda_2)$  in the updated model, the admissible eigenvectors  $Y$  corresponding to  $\Sigma$  form at most a two-dimensional manifold in  $\mathbb{R}^5$ .

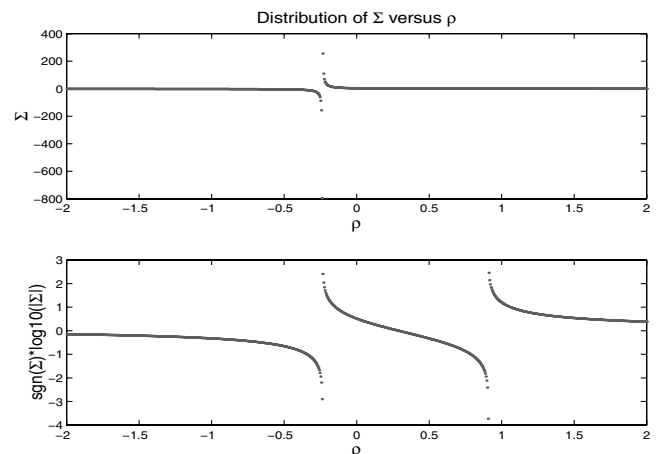


Fig. 1 Admissible values of  $\Sigma$  as a function of  $\rho$ .



## V. Conclusions

To mend the discrepancy between a mathematical model and the corresponding real-world system, one common procedure is to modify the model parameters so as to achieve a good correspondence between the analytic solution and the real data. In this paper, one such model updating of self-adjoint quadratic pencils using a few measured natural frequencies and mode shapes is considered. The model updating problem is cast as a quadratic inverse eigenvalue problem with prescribed eigenpairs.

Constructive proofs are given to show that the QIEP with no damping can be solved with any number of arbitrarily assigned eigenpairs, whereas the QIEP with damping can be solved with up to maximal allowable  $k_{\max}$  arbitrarily assigned eigenpairs. Consequently, updating with no spillover is entirely possible for undamped quadratic pencils, whereas spillover for damped quadratic pencils generally is unpreventable. Examples are given to demonstrate both the phenomenon of spillover and the conditions under which no spillover might be maintained. For the latter, the more complicated analytic conditions are presented in a separate paper [23]. In short, unless the newly measured eigenpairs satisfy some fairly stringent conditions, an updating of a damped quadratic pencils will surely cause spillover.

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