STRUCTURED QUADRATIC INVERSE EIGENVALUE PROBLEM, I. SERIALLY LINKED SYSTEMS

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Abstract. Quadratic pencils arising from applications are often inherently structured. Factors contributing to the structure include the connectivity of elements within the underlying physical system and the mandatory nonnegativity of physical parameters. For physical feasibility, structural constraints must be respected. Consequently, they impose additional challenges on the inverse eigenvalue problems which intend to construct a structured quadratic pencil from prescribed eigeninformation. Knowledge of whether a structured quadratic inverse eigenvalue problem is solvable is interesting in both theory and applications. However, the issue of solvability is problem dependent and has to be addressed structure by structure. This paper considers one particular structure where the elements of the physical system, if modeled as a mass-spring system, are serially linked. The discussion recasts both undamped or damped problems in a framework of inequality systems that can be adapted for numerical computation. Some open questions are described.

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1. Introduction.

The time-invariant second order differential system

$$M\ddot{\mathbf{x}} + C\dot{\mathbf{x}} + K\mathbf{x} = f(t), \tag{1.1}$$

where $\mathbf{x} \in \mathbb{R}^n$ and M, C, $K \in \mathbb{R}^{n\times n}$, arises in many important applications, including applied mechanics, electrical oscillation, vibro-acoustics, fluid mechanics, signal processing, and finite element discretization of PDEs. In most cases, parameters characterizing the underlying physical system are embedded in the matrix coefficients M, C and K. It is well known that if

$$\mathbf{x}(t) = \mathbf{v}e^{\lambda t}$$

represents a fundamental solution to (1.1), then the scalar λ and the vector $\mathbf{v}$ must solve the *quadratic eigenvalue problem* (QEP)

$$(\lambda^2 M + \lambda C + K)\mathbf{v} = 0. \tag{1.2}$$

That is, the dynamical bearing of the differential system (1.1) usually can be interpreted via the eigenvalues and eigenvectors of the algebraic system (1.2). Because of this connection, considerable efforts have been devoted to the QEP in the literature. Readers are referred to the treatise by Tisseur and Meerbergen [28] for a good survey of many applications, mathematical properties, and a variety of numerical techniques for the QEP.

The eigenvalue problem associated with the model (1.1) can be considered from two different viewpoints. From *a priori* known physical parameters such as mass, length, elasticity, inductance, capacitance, and so on, the process of analyzing and deriving the spectral information so as to induce the dynamical behavior of a system is referred to as a *direct* problem. The *inverse* problem, in contrast, is to validate, determine, or estimate the parameters of the system according to its

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observed or expected behavior. The concern in the direct problem is to express the behavior in terms of the parameters whereas in the inverse problem the concern is to express the parameters in term of the behavior. Both problems are important in applications. This paper is concerned about the inverse problem.

The inverse eigenvalue problem is a diverse area full of research interests and activities. See the newly revised book by Gladwell [20] for mechanics applications, the review article [6] for linear problems, the recently completed monograph by Chu and Golub [8] for general theory, algorithms and applications, and the many references collected therein from various disciplines. Among current development, the quadratic inverse eigenvalue problem (QIEP) is particularly more important and challenging with many unanswered questions. This paper concentrates only on the issue of solvability for one specially structured QIEPs.

We have to stress that depending on the applications, the term QIEP has been used in the literature to mean a rather wide variety of formulations. For instance, the QIEP studied by Ram and Ehlay in [25] involves only symmetric tridiagonal matrix coefficients where *two* sets of eigenvalues are given. The QIEP studied by Starek and Inman in [27] is associated with nonproportional underdamped systems. Lancaster and Prells [23] considered the QIEP with symmetric and positive semi-definite damping C where *complete* information on eigenvalues and eigenvectors is given and all eigenvalues are simple and non-real. There are also works which utilize notions of feedback control to reassign the eigenstructure [11, 24]. The list goes on and on and can hardly be exhaustive. The QIEP considered in this article deals with yet another scenario which is quite common in practice — construct the quadratic pencil with only *a few* eigenvalues and their corresponding eigenvectors.

In vibration industries, including aerospace, automobile, and manufacturing, through vibration tests where the excitation and the response of the structure at selected points are measured experimentally, there are identification techniques to extract a portion of eigenpair information from the measurements. However, the size of the physical system can be so large and complicated that it is not always possible to attain knowledge of the entire spectrum. While there is no reasonable analytical tool available to evaluate the entire spectral information, it is simply unwise to use experimental values of high natural frequencies to reconstruct a model. Additionally, it is often demanded, especially in structural design, that certain eigenvectors should also satisfy some specific conditions. A finite-element generated model therefore needs to be updated using only a few measured eigenvalues and eigenvectors [14, 17]. Furthermore, quantities related to high frequency terms in a *finite model* generally are susceptible to measurement errors due to the finite bandwidth of measuring devices. Spectral information, therefore, should not be used at its full extent. For these reasons, it might be more sensible to consider an inverse eigenvalue problem where only a *portion* of eigenvalues and eigenvectors is available. The QIEP that is of interest to us therefore can be formulated as follows:

> **(QIEP)** Construct a nontrivial quadratic pencil $Q(\lambda) = \lambda^2 M + \lambda C + K$ so that its matrix coefficients (M, C, K) are of a specified structure and $Q(\lambda)$ has a specified set $\{(\lambda_i, \mathbf{v}_i)\}_{i=1}^k$ as its eigenpairs.

Since we are only interested in real matrices, it is natural to expect that the prescribed eigenpairs are closed under complex conjugation. Without loss of generality, we shall denote the prescribed eigenpairs in the matrix form (Λ, X) where $\Lambda \in \mathbb{R}^{k\times k}$ is block diagonal with at most 2×2 blocks along the diagonal wherever a complex-conjugate pair of eigenvalues appear in the prescribed spectrum and $X \in \mathbb{R}^{n\times k}$ represents the "eigenvector matrix" in the sense that each pair of column vectors associated with a 2×2 block in Λ retains the real and the imaginary part, respectively, of the original complex eigenvector. In this way, we may identify the given eigepairs (Λ, X) as an element in $\mathbb{R}^k \times \mathbb{R}^{n\times k}$. The QIEP therefore amounts to solving the algebraic equation

$$M X \Lambda^2 + C X \Lambda + K X = 0. \tag{1.3}$$

At the first glance, the relationship (1.3) is only a homogeneous linear system of nk algebraic equations. If there are no other constraints, the triplet (M, C, K) constitutes $3n^2$ unknowns. Since k is bounded above by $2n$, the system is well under-determined. It is intuitively true that the system should be solvable in general. The real challenge is to characterize the solution in terms of the given (Λ, X), when the matrix coefficients in the pencil are required to satisfy some structural constraints.

The particular constraints that will be studied in this paper are twofold — the connectivity constraint inherent to the serially linked physical structure and the nonnegative constraint required by the physical parameters. Despite the fact that some general results have been obtained for the quadratic inverse eigenvalue problems [21, 23, 25], we have not seen that the above-mentioned constraints being taken into account. Our contribution is innovative in that the structured QIEP is cast as a system of inequalities whose solvability can then be checked numerically.

2. Issue of Connectivity. An inverse eigenvalue problem without a structure often is trivial and meaningless [8]. This is particularly so for QIEPs arising from applications. The structure imposed on a QIEP depends inherently on the *connectivity* of the underlying physical system. For practical reasons, the physical feasibility must be respected. It is therefore necessary that the reconstructed pencil in the inverse problem staisfies the inherent connectivity structure. In other words, there is more to just solving the algebraic equation (1.3) for a QIEP. The matrices in the solution (M, C, K) must carry a certain specified structure. We give two examples to illustrate how the connectivity affects the structure.

Example 1. Assuming that the springs respond according to Hook's law and that the damping is negatively proportional to the velocity, the coefficient matrices corresponding to the four-degree-of-freedom mass-spring system depicted in Figure 2.1 should be structured as follows:

$$M = \begin{bmatrix} m_1 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \\ 0 & 0 & m_3 & 0 \\ 0 & 0 & 0 & m_4 \end{bmatrix}, \; C = \begin{bmatrix} c_1 + c_2 & 0 & -c_2 & 0 \\ 0 & 0 & 0 & 0 \\ -c_2 & 0 & c_2 + c_3 & -c_3 \\ 0 & 0 & -c_3 & c_3 \end{bmatrix}, \; K = \begin{bmatrix} k_1 + k_2 + k_3 & -k_2 & -k_5 & 0 \\ -k_2 & k_2 + k_3 & -k_3 & 0 \\ -k_5 & -k_3 & k_3 + k_4 + k_5 & -k_4 \\ 0 & 0 & -k_4 & k_4 \end{bmatrix}.$$

A typical way in the literature to describe the structure of a general mass-spring system is that the mass matrix M is diagonal, both the damping matrix C and the stiffness matrix K are symmetric and banded, M is positive definite and K is positive semi-definite. But now we see that such a characterization indeed is not fine enough for physical feasibility. One must be specific in that the internal connectivity of the masses and springs defines how the diagonal entries of C and K are related to their off-diagonal entries. A different configuration of the connectivity in the system will lead to a different structure. It is extremely important to point out an additional inherent constraint, that is, all physical parameters are expected to be nonnegative.

Example 2. Assuming Ohm's law and Kirchoff's law in the circuit, the matrix coefficients for the governing equation of the current in the RLC network depicted in Figure 2.2 should have the following structure:

$$M = \begin{bmatrix} -L_2 & L_2 & 0 & 0 \\ L_2 & -L_2 & 0 & 0 \\ 0 & 0 & L_3 & 0 \\ 0 & 0 & 0 & L_4 \end{bmatrix}, \; C = \begin{bmatrix} 0 & R_2 & -R_2 & 0 \\ R_1 + R_4 & 0 & 0 & -R_4 \\ 0 & -R_2 & R_2 + R_3 & 0 \\ -R_4 & 0 & 0 & R_4 \end{bmatrix}, \; K = \begin{bmatrix} 0 & \frac{1}{C_2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{C_3} & -\frac{1}{C_3} \\ 0 & 0 & -\frac{1}{C_3} & \frac{1}{C_3} + \frac{1}{C_4} \end{bmatrix}.$$

In contrast to the structure associated with a mass-spring system, note that the structure associated with an electronic circuit generally may not be definite or even symmetric.

Since structured problems often results in special interrelationship within its eigenstructure, the observed measurement which in many cases is contaminated with random noise may not be consistent with that innate structure. The structural constraint therefore severely limits whether a QIEP is solvable. It would be desirable to have a general theory by which one can obtain an affirmative answer on whether a specifically configured system could be build or not and, if yes,

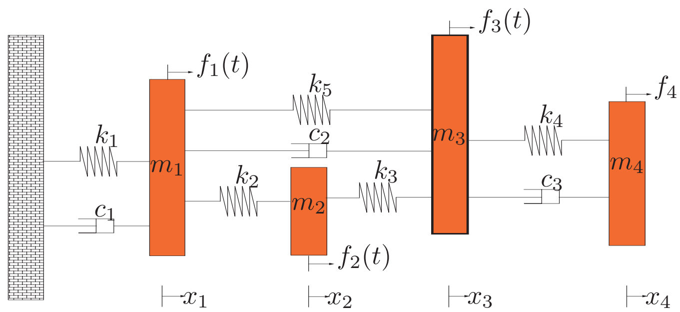

FIG. 2.1. *A four-degree-of-freedom mass-spring system.*

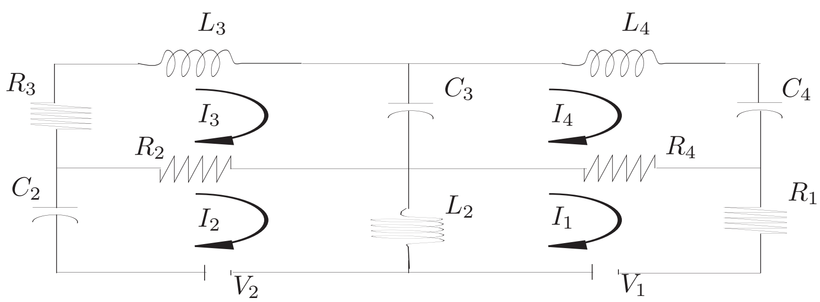

FIG. 2.2. *An RLC electronic network.*

how the parameters should be valued. To our knowledge, such a theory does not exist. Most of the current QIEP techniques consider only the algebraic solvability, that is, solving the QIEP without taking into account the nonnegativity of the solution. The main thrust of this paper is to discuss the (physical) solvability of one particular structure. This is only one step forward with the hope of stimulating further studies toward solving the general structured QIEPs.

3. Symmetric structure. One of the simplest structural constraint in many applications is symmetry. If all matrix coefficients involved in a pencil are symmetric, we say that we have a *self-adjoint* pencil. The theory of solvability for self-adjoint QIEPs has been established recently by Chu, Datta, Lin and Xu in [9]. To make this note more self-contained, it might be informative to summarize some of the results in this section without repeating the details. Define

$$k_{max} = \begin{cases} 3\ell + 1, & \text{if } n = 2\ell, \\ 3\ell + 2, & \text{if } n = 2\ell + 1. \end{cases} \tag{3.1}$$

The following theorem characterizes the general solvability [9].

THEOREM 3.1. *Given any positive integer $k \le k_{max}$, let (Λ, X) represent k arbitrarily prescribed eigenpairs which are closed under complex conjugation. Then*

1. *The self-adjoint QIEP associated with (Λ, X) is always solvable.*
2. *For almost all k prescribed eigenpairs (Λ, X), the solutions to the corresponding self-adjoint QIEP form a subspace of dimensionality $\frac{3n(n+1)}{2} - nk$.*

It might shed some additional insight if we elaborate a little further upon the solvability indicated in the above theorem. In the event that $k \ge n$, partition the eigenvalues and eigenvectors as

$$\Lambda = \mathrm{diag}\{\Upsilon_1, \Upsilon_2\},$$
$$X = [Z_1, Z_2],$$

with $Z_1 \in \mathbb{R}^{n\times n}$, $Z_2 \in \mathbb{R}^{n\times(k-n)}$, $\Upsilon_1 \in \mathbb{R}^{n\times n}$ and $\Upsilon_2 \in \mathbb{R}^{(k-n)\times(k-n)}$, respectively. Define

$$\Xi := Z_1^{-1} Z_2. \tag{3.2}$$

Then there exists a nontrivial self-adjoint quadratic pencil $Q(\lambda)$ with partial eigenstructure (Λ, X) if and only if the system of matrix equations [8],

$$\left\{ \begin{array}{rcl} \Upsilon_1^\top B - B\Upsilon_1 &=& A\Upsilon_1^2 - \Upsilon_1^{2\top} A, \\ \Upsilon_1^\top B\Xi - B\Xi\Upsilon_2 &=& A\Xi\Upsilon_2^2 - \Upsilon_1^{2\top} A\Xi, \\ A &=& A^\top, \\ B &=& B^\top, \end{array} \right. \tag{3.3}$$

has a nontrivial solution (A, B) in $\mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}$. In that case, the solution to the QIEP is given by

$$\begin{aligned} M &= Z_1^{-\top} A Z_1^{-1}, \\ C &= Z_1^{-\top} B Z_1^{-1}, \\ K &= Z_1^{-\top}(-\Upsilon_1^\top B - \Upsilon_1^{2\top} A) Z_1^{-1}. \end{aligned}$$

To better grasp the linear system (3.3), let $\mathbf{vec}(A)$ represent the matrix A by stacking the columns of A into one long vector. Let $\mathcal{K} \in \mathbb{R}^{\frac{n(n-1)}{2}\times n^2}$ denote the matrix representation of the projection of a square matrix onto the subspace of skew-symmetric matrices. Then (3.3) can be expressed as

$$\underbrace{\begin{bmatrix} \Upsilon_1^{2\top} \otimes I_n - I_n \otimes \Upsilon_1^{2\top} & \Upsilon_1^\top \otimes I_n - I_n \otimes \Upsilon_1^\top \\ (\Xi\Upsilon_2^2)^\top \otimes I_n - \Xi^\top \otimes \Upsilon_1^{2\top} & (\Xi\Upsilon_2)^\top \otimes I_n - \Xi^\top \otimes \Upsilon_1^\top \\ \mathcal{K} & O_{\frac{n(n-1)}{2}\times n^2} \\ O_{\frac{n(n-1)}{2}\times n^2} & \mathcal{K} \end{bmatrix}}_{\mathcal{T}} \begin{bmatrix} \mathbf{vec}(A) \\ \mathbf{vec}(B) \end{bmatrix} = 0. \tag{3.4}$$

In other words, in order to have nontrivial solution (A, B) for (3.3), the prescribed eigenpairs (Λ, X) must be such that the $n(n + k - 1) \times 2n^2$ coefficient matrix $\mathcal{T}$ has column rank less than $2n^2$.

Corollary 3.2. *Given $k \geq n$ arbitrarily prescribed eigenpairs (Λ, X) which are closed under complex conjugation, define the matrix $\mathcal{T}$ as in (3.4).*

1. *If $n \leq k \leq k_{max}$, then the null space of $\mathcal{T}$ has dimensionality at least $\frac{3n(n+1)}{2} - nk$ and is generically of that dimensionality.*
2. *If $k > k_{max}$, then the matrix $\mathcal{T}$ is generically of full column rank. The collection of (Λ, X) for which (1.3) has a nontrivial self-adjoint solution has measure zero in the space $\mathbb{R}^n \times \mathbb{R}^{n\times k}$.*

In the event that $k < n$, we still can solve the QIEP by embedding this eigeninformation in a larger set of n eigenpairs. In particular, we expand $X \in \mathbb{R}^{n\times k}$ to

$$\widehat{X} := [X, \widetilde{X}] \in \mathbb{R}^{n\times n}, \tag{3.5}$$

where $\widetilde{X} \in \mathbb{R}^{n\times(n-k)}$ is arbitrary while making $\hat{X}$ nonsingular and expand $\Lambda \in \mathbb{R}^{k\times k}$ to

$$\widehat{\Lambda} := \mathrm{diag}\{\Lambda, \widetilde{\Lambda}\}, \tag{3.6}$$

where $\widehat{\Lambda}$ is a diagonal matrix with distinct eigenvalues. Taking into account the freedom of this expansion, it is not difficult to prove that the solutions to the QIEP with $k < n$ form a subspace of dimensionality $\frac{n(n+3)}{2} + n(n - k)$.

Example 3. Suppose n distinct eigenvalues Λ and n linearly independent eigenvectors X, both of which are closed under conjugation, are given. Let $A \in \mathbb{R}^{n\times n}$ be an *arbitrary* symmetric matrix. The system (3.3) is reduced to the equation

$$\Lambda^\top B - B\Lambda = A\Lambda^2 - \Lambda^{2\top} A. \tag{3.7}$$

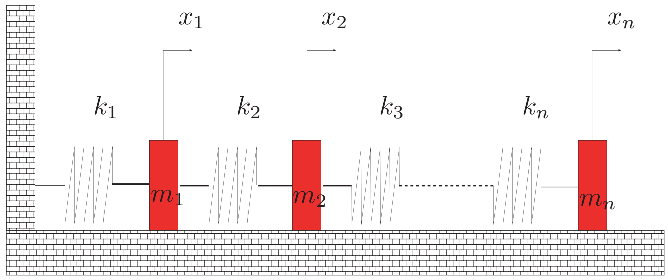

FIG. 4.1. *An undamped mass-spring system.*

for the unknown symmetric matrix B. From the block diagonal structure of Λ, it follows that B is determined up to n free parameters. Using A and B as parameters, the general solution to the self-adjoint QIEP with the prescribed pair (X, Λ) as part of its eigenstructure is given by

$$M = X^{-\top} A X^{-1}, \tag{3.8}$$
$$C = X^{-\top} B X^{-1}, \tag{3.9}$$
$$K = -X^{-\top} \Lambda^{\top} (B + \Lambda^{\top} A) X^{-1}. \tag{3.10}$$

In all cases, so long as the self-adjoint QIEP is solvable, we do have a way to obtain a parametric representation of the solution (M, C, K) of (1.3). Despite the fact that we know how to solve a self-adjoint QIEP algebraically, however, it must be stressed that algebraic solvability does not necessarily imply physical feasibility. Physical feasibility means, for example, that the special matrix structure resulting from the underlying connectivity must hold and that the physical parameters must be nonnegative. These added conditions make the QIEP much more interesting but harder. We are not aware of many research results in this regard. In what follows, we consider a special structure resulted from serially linked mass-spring systems.

4. Serially linked structure without damping. Our experience seems to indicate that whether a structured QIEP is solvable probably is problem dependent. The issue of solvability needs to be analyzed structure by structure. One special setting, the serially linked, undamped mass-spring system depicted in Figure 4.1, is considered in this section. Such a setting can be used to model many other physical systems, including a vibrating beam, a composite pendulum, or a string with beads [20].

Matrices corresponding to the quadratic pencil $\lambda^2 M + K$ for the system in Figure 4.1 have the structure,

$$M = \begin{bmatrix} m_1 & 0 & \dots & 0 \\ 0 & m_2 & & \\ \vdots & & \ddots & \\ 0 & & & m_n \end{bmatrix}, \quad K = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 & \dots & 0 & 0 \\ -k_2 & k_2 + k_3 & -k_3 & & & 0 \\ 0 & -k_3 & k_3 + k_4 & & & \\ \vdots & & & \ddots & & \vdots \\ 0 & & & & k_{n-1} + k_n & -k_n \\ 0 & & & & -k_n & k_n \end{bmatrix}. \tag{4.1}$$

It is natural to assume that the masses and spring constants are all positive. In this case, eigenvalues associated with this kind of system are necessarily pure imaginary. The inverse eigenvalue problem means to find positive values for the masses $m_1, \dots, m_n$ and spring constants $k_1, \dots, k_n$ from prescribed eigeninformation.

4.1. Sign Condition. A typical approach in the literature for this type of inverse problem has been to recast the quadratic pencil as a linear pencil $\mu I + J$ with a Jacobi matrix $J = M^{-1/2}KM^{-1/2}$. A classical theory has been that *two sets of eigenvalues* can uniquely solve the corresponding Jacobi inverse eigenvalue problem [8, Section 4.2], but that result does not guarantee the nonnegativity of M and K. Furthermore, what can be said if the system is to be reconstructed from eigenpairs?

A total of $2n$ physical parameters are to be determined in (4.1). Each given eigenpair provides n equations. Imposing two eigenpairs generally will lead to the trivial algebraic solution for such a system, unless the prescribed eigenpairs satisfy some additional internal relationship. So it is reasonable to ask a more fundamental question of constructing the system with *one* prescribed eigenpair $(i\beta, \mathbf{x})$ where $i = \sqrt{-1}$, $\beta \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$.

Denote $\mathbf{x} = [x_1, \ldots, x_n]^\top$ and $x_0 = 0$. It is not difficult to see from (1.3) under the structure (4.1) that the recursive relationship,

$$k_n = \frac{\beta^2 m_n x_n}{x_n - x_{n-1}}, \tag{4.2}$$

$$k_i = \frac{\beta^2 m_i x_i + k_{i+1}(x_{i+1} - x_i)}{x_i - x_{i-1}}, \quad i = n-1, \ldots, 1, \tag{4.3}$$

must hold. Our goal is to choose a positive value for m_i so as to define a positive value for k_i. In order to achieve this goal, the entries of the eigenvector $\mathbf{x}$ must satisfy some sign properties. More specifically, we observe the following fact directly from (4.2) and (4.3).

THEOREM 4.1. *The necessary condition for a given vector $\mathbf{x} = [x_1, \ldots, x_n]^\top \in \mathbb{R}^n$ with distinct entries to be an eigenvector of a quadratic pencil $\lambda^2 M + K$ with structure as is specified in (4.1) where masses m_i and spring constants are positive is that $x_n(x_n - x_{n-1}) > 0$ and the signs of the triplets $(x_{i+1} - x_i, x_i, x_i - x_{i-1})$ for $i = 2, \ldots, n-1$ are neither $(+,+,-)$ nor $(-,-,+)$.*

4.2. Numerical Algorithm. In the event that the given vector $\mathbf{x}$ does satisfy the necessary condition in Theorem 4.1, there are infinitely many ways to choose positive values for m_i which are then used to define k_i. The following is a *make-or-break* algorithm for the construction where we specify only one particular way of selecting m_i.

ALGORITHM 4.1. *Given an arbitrary eigenpair $(i\beta, \mathbf{x})$, $\beta \neq 0$ and $\mathbf{x} \neq 0$, assume the normalization $x_n = 1$ and $m_n = 1$. The following steps either construct the masses $m_1, \ldots, m_{n-1}$ and spring constants $k_1, \ldots, k_n$, all positive, for the pencil $\lambda^2 M + K$, or determine that such a system with the prescribed eigenpair does not exist.*

1. *initialization:*
    - $s_n = 0.9$; *(mass decreasing factor)*
    - $s_p = 1.1$; *(mass increasing factor)*
    - $\eta = \beta^2$;
2. *if* $x_{n-1} < 1$,
    - $k_n = \frac{\beta^2}{1 - x_{n-1}}$; *(use formula (4.2))*

    else
    - *return* *(inconsistent eigenvector)*

    end
3. *for i from* $n-1$ *to* 2, *do*
    - (a) $\rho = \frac{\eta}{x_i}$;
    - (b) *if* $x_{i-1} < x_i$,
        - *if* $\rho > 0$,
            - *if* $x_i < 0$,
                - *return* *(inconsistent eigenvector)*

else
$m_i = 1$; *(any $m_i > 0$ will be fine)*
end
else
if $x_i < 0$,
$m_i = -\frac{s_n\rho}{\beta^2}$; *(need $0 < m_i < -\frac{\rho}{\beta^2}$)*
else
$m_i = -\frac{s_p\rho}{\beta^2}$; *(need $m_i > -\frac{\rho}{\beta^2}$)*
end
end
else
if $\rho > 0$,
if $x_i < 0$,
$m_i = 1$;
else
return *(inconsistent eigenvector)*
end
else
if $x_i < 0$,
$m_i = -\frac{s_p\rho}{\beta^2}$;
else
$m_i = -\frac{s_n\rho}{\beta^2}$;
end
end
end
(c) $\eta = \eta + \beta^2 m_i x_i$;
(d) $k_i = \frac{\eta}{x_i - x_{i-1}}$; *(use formula (4.3))*

4. finale:
(a) $\rho = \frac{\eta}{x_1}$;
(b) if $\rho > 0$,
$m_1 = 1$;
else
$m_1 = -\frac{s_p\rho}{\beta^2}$;
end
(c) $\eta = \eta + \beta^2 m_1 x_1$;
(d) $k_1 = \frac{\eta}{x_1}$; *(use formula (4.3))*

The working of the algorithm, based on Theorem 4.1, appears naive enough in just checking a few signs. It is precisely this simplicity of the necessary and sufficient conditions on the given eigenvector for ensuring a *positive* solution that we think is gratifying and elegant.

Note that in the above construction no restriction on the magnitude of β is imposed. This is quite curious in that the physical system can have one arbitrary natural frequency. One might thus ask that, after the pencil $\lambda^2 M + K$ is determined, what the remaining eigenstructure should look like. To explore this important question, it is interesting to recall the classical Courant Nodal Line Theorem [31]. Roughly speaking, it has been known that critical information about a vibration system can be recovered from the places where *nothing happens*. These places are referred to as the *nodal lines*. In our discrete model, the incident of "nothing happens" means that natural mode crosses the zero equilibrium. Courant's theorem gives an exact count of the numbers of nodal lines. We shall not elaborate the particulars here, but refer readers to the treatises [10, 31] for more detailed discussion. Its simplest form related to our application can be stated as follows:

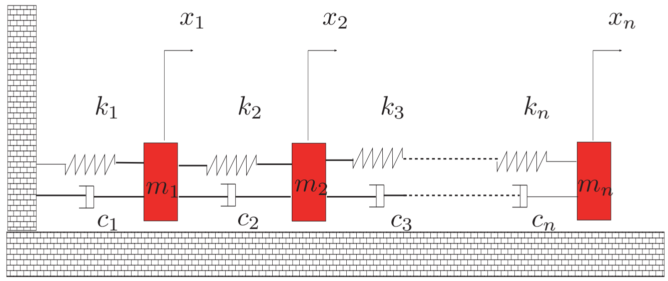

FIG. 5.1. *A damped mass-spring system.*

THEOREM 4.2. *If the given eigenvector* $\mathbf{x}$ *is feasible and if there are* τ *changes of signs in the sequence when going through* x_1 *to* x_n*, then* $\mathbf{x}$ *is the* $(\tau+1)$*-th eigenvector (in the series where the corresponding frequencies are arranged in the ascending order) of the pencil regardless how the masses* m_i *are defined.*

In other words, from the changes of signs in the prescribed eigenvector $\mathbf{x}$, we know precisely how many other natural frequencies are above β and how many are below, although the values of these other frequencies can be quite arbitrary. The normalization $m_n = 1$ is immaterial. It only indicates the unit of mass used relative to m_n. It is the choice of values for other m_i's, which in turn determines values for other k_i's, that affects the total spectrum. How to select values for m_i when there are multiple alternatives so as to satisfy some additional spectral information is a very different but important question. At present, we have no knowledge about how this problem should be tackled. For other types of connectivity including damped systems or RLC configurations, the resulting pencil matrix structure will be different. We are not aware of any similar nodal line theorem. It is very likely that the condition of solvability will also vary. Again, this is a widely open area for further research.

5. Serially linked structure with damping.

When damping is present in the mass-spring system in the way depicted in Figure 5.1, the QIEP becomes much more complicated. The associated matrices in the pencil $\lambda^2 M + \lambda C + K$ have the same structure of M and K as in (4.1) and, additionally, that of

$$C = \begin{bmatrix} c_1+c_2 & -c_2 & 0 & \dots & 0 & 0 \\ -c_2 & c_2+c_3 & -c_3 & & & 0 \\ 0 & -c_3 & c_3+c_4 & & & \\ \vdots & & & \ddots & & \vdots \\ 0 & & & & c_{n-1}+c_n & -c_n \\ 0 & & & & -c_n & c_n \end{bmatrix}. \tag{5.1}$$

There are a total of $3n$ physical parameters to be determined, all of which need to be positive real numbers. It is clear that specifying more than three eigenpairs will make the QIEP over-determined. In the same spirit as before, we shall consider that two pairs of eigenpairs are given and want to determine $c_1, \dots, c_n$ and $k_1, \dots, k_n$ in terms of $m_1, \dots, m_n$. Depending upon whether the specified eigenvalues are complex or not, we divide our discussion into two cases.

5.1. Complex eigenpairs. Assume that the given eigenpair are complex conjugate to each other. This eigeninformation is given through the matrices

$$\Lambda = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}, \quad X = \begin{bmatrix} x_1 & x_2 & \dots & x_n \\ y_1 & y_2 & \dots & y_n \end{bmatrix}^\top, \tag{5.2}$$

where the 2×2 matrix Λ represents the complex-conjugate eigenvalues $\alpha \pm i\beta$, the first and the second columns of X represent the real and the imaginary parts of the corresponding eigenvector, respectively.

For $i = 1, \dots, n$, denote the differences

$$X_i := x_i - x_{i-1}, \tag{5.3}$$

$$Y_i := y_i - y_{i-1}, \tag{5.4}$$

with the notation $x_0 = y_0 = 0$. The algebraic system (1.3) characterizing the QIEP can be expressed as a system of n pairs of equations:

$$\begin{cases} \left((\alpha^2-\beta^2)x_n - 2\alpha\beta y_n\right) m_n + (\alpha X_n - \beta Y_n)c_n + X_n k_n &= 0 \\ \left(2\alpha\beta x_n + (\alpha^2-\beta^2)y_n\right) m_n + (\alpha Y_n + \beta X_n)c_n + Y_n k_n &= 0, \end{cases} \tag{5.5}$$

and

$$\begin{cases} \left((\alpha^2-\beta^2)x_i - 2\alpha\beta y_i\right) m_i + (\alpha X_i - \beta Y_i)c_i - (\alpha X_{i+1} - \beta Y_{i+1})c_{i+1} + X_i k_i - X_{i+1}k_{i+1} = 0 \\ \left(2\alpha\beta x_i + (\alpha^2-\beta^2)y_i\right) m_i + (\alpha Y_i + \beta X_i)c_i - (\alpha Y_{i+1} + \beta X_{i+1})c_{i+1} + Y_i k_i - Y_{i+1}k_{i+1} = 0, \end{cases} \tag{5.6}$$

for $i = n-1, \dots, 1$. Our goal is to seek *positive* solutions $c_1, \dots, c_n$ and $k_1, \dots, k_n$ in terms of *positive* values of $m_1, \dots, m_n$. Toward that end, we find it convenient to rewrite the above equations in matrix form. Define

$$\Gamma := -\begin{bmatrix} \alpha^2-\beta^2 & -2\alpha\beta \\ 2\alpha\beta & \alpha^2-\beta^2 \end{bmatrix} = -\Lambda^{2\top}, \tag{5.7}$$

and

$$\Theta_i := \begin{bmatrix} \alpha X_i - \beta Y_i & X_i \\ \beta X_i + \alpha Y_i & Y_i \end{bmatrix}, \tag{5.8}$$

$$\mathbf{z}_i := \begin{bmatrix} x_i \\ y_i \end{bmatrix}, \tag{5.9}$$

for $i = 1, \dots, n$. It is readily seen that equations (5.5) and (5.6) can be conveniently expressed in the compact form:

$$\Theta_n \begin{bmatrix} c_n \\ k_n \end{bmatrix} = m_n \Gamma \mathbf{z}_n, \tag{5.10}$$

$$\Theta_i \begin{bmatrix} c_i \\ k_i \end{bmatrix} = \Theta_{i+1} \begin{bmatrix} c_{i+1} \\ k_{i+1} \end{bmatrix} + m_i \Gamma \mathbf{z}_i, \quad i = n-1, \dots, 1. \tag{5.11}$$

Clearly, $\det(\Theta_i) = -\beta(X_i^2 + Y_i^2)$. By assuming the generic condition that no two consecutive entries of the given eigenvectors are the same, all matrices Θ_i are invertible. We thus obtain the following recursive relationship:

$$\begin{bmatrix} c_n \\ k_n \end{bmatrix} = m_n \Theta_n^{-1} \Gamma \mathbf{z}_n, \tag{5.12}$$

$$\begin{bmatrix} c_i \\ k_i \end{bmatrix} = \Theta_i^{-1} \Gamma \left[m_n \mathbf{z}_n + m_{n-1}\mathbf{z}_{n-1} + \dots + m_i \mathbf{z}_i \right], \qquad i = n-1, \dots, 1, \tag{5.13}$$

where we can express Θ_i^{-1} explicitly as

$$\Theta_i^{-1} = \frac{1}{\beta(X_i^2+Y_i^2)} \begin{bmatrix} -Y_i & X_i \\ \beta X_i + \alpha Y_i & -\alpha X_i + \beta Y_i \end{bmatrix}.$$

It remains to analyze conditions under which the physical parameters are positive.

Observe that all entries in Γ, Θ_i^{-1} and $\mathbf{z}_i$, $i = 1, \ldots, n$, are known and fixed from the prescribed eigenpairs. For each $i = 1, \ldots n$ and $j = i, \ldots n$, write

$$\Theta_i^{-1}\Gamma \mathbf{z}_j = \begin{bmatrix} a_{ij} \\ b_{ij} \end{bmatrix}. \tag{5.14}$$

Assemble these entries into upper triangular matrices

$$A := \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ 0 & a_{22} & \ldots & a_{2n} \\ \vdots & & & \vdots \\ 0 & 0 & \ldots & a_{nn} \end{bmatrix}, \quad B := \begin{bmatrix} b_{11} & b_{12} & \ldots & b_{1n} \\ 0 & b_{22} & \ldots & b_{2n} \\ \vdots & & & \vdots \\ 0 & 0 & \ldots & b_{nn} \end{bmatrix},$$

respectively. From (5.12) and (5.13), we now recast the QIEP as a system of inequalities.

Theorem 5.1. *Solving for positive physical parameters c_i and k_i in terms of positive masses m_i, $i = 1, \ldots, n$, for the QIEP amounts to showing that the system of inequalities*

$$\begin{bmatrix} A \\ B \end{bmatrix} \mathbf{m} > 0, \tag{5.15}$$

is consistent *for some $\mathbf{m} = [m_1, \ldots, m_n]^\top > 0$.*

5.2. Real eigenpairs. Assume that the given pair of eigenvalues are real and distinct. We denote the prescribed eigenpairs via the matrices

$$\Lambda = \begin{bmatrix} \alpha & 0 \\ 0 & \gamma \end{bmatrix}, \qquad X = \begin{bmatrix} x_1 & x_2 & \ldots & x_n \\ y_1 & y_2 & \ldots & y_n \end{bmatrix}^\top, \tag{5.16}$$

where the two columns of X are the two eigenvectors associated with eigenvalues α and γ, respectively. The corresponding algebraic equation (1.3) can be formulated as the following n pairs of equations:

$$\left\{ \begin{array}{lcl} \alpha^2 m_n x_n + \alpha X_n c_n + X_n k_n & = & 0 \\ \gamma^2 m_n y_n + \gamma Y_n c_n + Y_n k_n & = & 0, \end{array} \right. \tag{5.17}$$

and

$$\left\{ \begin{array}{lcl} \alpha^2 m_i x_i + \alpha X_i c_i - \alpha X_{i+1} c_{i+1} + X_i k_i - X_{i+1} k_{i+1} & = & 0 \\ \gamma^2 m_i y_i + \gamma Y_i c_i - \gamma Y_{i+1} c_{i+1} + Y_i k_i - Y_{i+1} k_{i+1} & = & 0, \end{array} \right. \tag{5.18}$$

for $i = 1, \ldots, n-1$. Defining the 2×2 matrices,

$$\Pi_i := -\begin{bmatrix} \alpha X_i & X_i \\ \gamma Y_i & Y_i \end{bmatrix}, \qquad i = 1, \ldots, n, \tag{5.19}$$

we obtain equivalent equations in compact form:

$$\Pi_n \begin{bmatrix} c_n \\ k_n \end{bmatrix} = m_n \Lambda^2 \mathbf{z}_n \tag{5.20}$$

$$\Pi_i \begin{bmatrix} c_i \\ k_i \end{bmatrix} = \Pi_{i+1} \begin{bmatrix} c_{i+1} \\ k_{i+1} \end{bmatrix} + m_i \Lambda^2 \mathbf{z}_i, \qquad i = n-1, \ldots, 1 \tag{5.21}$$

Assuming that all matrices Π_i's are nonsingular, again we obtain a recursive relationship,

$$\begin{bmatrix} c_n \\ k_n \end{bmatrix} = m_n \Pi_n^{-1} \Lambda^2 \mathbf{z}_n, \tag{5.22}$$

$$\begin{bmatrix} c_i \\ k_i \end{bmatrix} = \Pi_i^{-1} \Lambda^2 \left[m_n \mathbf{z}_n + m_{n-1} \mathbf{z}_{n-1} + \ldots + m_i \mathbf{z}_i \right], \qquad i = n-1, \ldots, 1, \tag{5.23}$$

which is in the same format as that in (5.12) and (5.13). If we write

$$\Pi_i^{-1} \Lambda^2 \mathbf{z}_j = \begin{bmatrix} a_{ij} \\ b_{ij} \end{bmatrix}, \tag{5.24}$$

then to determine positive c_i and k_i in terms of positive m_i would end up with a system of inequalities similar to (5.15).

5.3. Numerical algorithm. We have shown in the above that solving a QIEP with damping from two given eigenpairs, both of which are either real-valued or complex-conjugate, is equivalent to solving a linear (strict) inequality system of the form (5.15). Without causing any ambiguity, we shall use the same notation A and B to represent the coefficient matrices involved in the inequality systems for both cases. Specifically, recall that both A and B are upper triangular matrices whose entries are defined by either (5.14) or (5.24). The question now is to determine the conditions on the prescribed eigenpairs under which the QIEP is solvable. In contrast to the QIEP with no damping in the preceding section, we are not aware of any analytic way to characterize the solvability of the QIEP with damping. To check the feasibility, we will have to resort to some numerical procedures.

Solving inequality systems of convex functions is a classical problem that has been discussed extensively in the literature, including the Farkas Lemma [15], the von Neumann Theorem [29, pp.138-143], and the Ky Fan Theorem [16, Theorem 1]. As an example, we state without proof the Strong Alternative Theorem [4, pp.260-262] as follows.

THEOREM 5.2. *The system of strictly linear inequalities $\mathcal{A}\mathbf{x} < \mathbf{b}$ is infeasible if and only if there exists a vector $\boldsymbol{\omega} \geq 0$, $\boldsymbol{\omega} \neq \mathbf{0}$ such that $\mathcal{A}^\top \boldsymbol{\omega} = \mathbf{0}$ and $\mathbf{b}^\top \boldsymbol{\omega} \leq 0$.*

For our application, we embed the need of $\mathbf{m} > 0$ in the inequalities and consider the augmented system $\mathcal{A}\mathbf{m} > 0$ with

$$\mathcal{A} = \begin{bmatrix} A \\ B \\ I \end{bmatrix}.$$

By Theorem 5.2, we see that $\mathcal{A}\mathbf{m} > 0$ is infeasible and, hence, our (serially linked with damping) structured QIEP is not solvable if and only if there exist nonnegative vectors $\boldsymbol{\omega}_1, \boldsymbol{\omega}_2$ and $\boldsymbol{\omega}_3$, not all zero, such that $A^\top \boldsymbol{\omega}_1 + B^\top \boldsymbol{\omega}_2 + \boldsymbol{\omega}_3 = \mathbf{0}$. This is equivalent to the statement that $A^\top \boldsymbol{\omega}_1 + B^\top \boldsymbol{\omega}_2 \leq \mathbf{0}$ for some nonnegative, but not both zero, vectors $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$. Such a result of duality, though interesting in theory, might not be easy to realize in computation. Instead, we propose a much simpler numerical approach.

Observe that the inequality system (5.15) is solvable for some $\mathbf{m} > 0$ if and only if it is solvable for $c\mathbf{m}$ with any $c > 0$. By scaling the variable $\mathbf{m}$ if necessary, we can recast our problem of solving an inequality system into the problem of finding a minimax solution of a multi-variable function. More specifically, we consider the problem

$$\max_{0 \leq \mathbf{m} \leq 1} \min \left\{ A\mathbf{m}, B\mathbf{m} \right\}, \tag{5.25}$$

where for each given $\mathbf{m} \in \mathbb{R}^n$ the minimum is taken over the $2n$ entries of the vectors $A\mathbf{m}$ and $B\mathbf{m}$. It is not difficult to observe the following fact.

THEOREM 5.3. *If the inequality system (5.15) is consistent, then the maximal objective value in (5.25) is positive. Conversely, if (5.15) is inconsistent, then the optimal objective value to (5.25) is necessarily zero with* $\mathbf{m} = 0$.

Note that the function $\min\{A\mathbf{m}, B\mathbf{m}\}$ is concave in $\mathbf{m}$. Thus its maximal value over the convex domain $\{\mathbf{m} \in \mathbb{R}^n | 0 \le m_i \le 1,\ i = 1, \dots n\}$ is unique. We can easily convert (5.25) into a convex programming problem and there are readily available routines to solve (5.25). For example, the Matlab routine `fminimax` implements a sequential quadratic programming method and seems capable of handling our problem reasonably well. With the equivalence of (5.15) and (5.25) in mind, we propose the following algorithm.

ALGORITHM 5.1. *Given two arbitrary complex-conjugate (or real-valued) eigenpairs, the following steps generally will either construct the masses* $m_1, \dots, m_n$*, the damping constants* $c_1, \dots, c_n$ *and the spring constants* $k_1, \dots, k_n$*, all positive and* $m_i \le 1$*, for the pencil* $\lambda^2 M + \lambda C + K$ *with the prescribed eigenpairs, where* M*,* C *and* K *are of the structure specified in (4.1) and (5.1), or determine that such a system does not exist. In the rare case where the algorithm is terminated prematurely, the prescribed eigenpairs are not generic enough in the sense that one of the matrices* Θ_i *defined in (5.8) (or* Π_i *defined in (5.19)) is singular for some* $i = 1, \dots, n$.

1. *Let* (Λ, X) *represent the given eigenpairs in the sense of (5.2) (or (5.16)).*
   (a) *For* $i = 1, \dots, n$*, form vectors* $\mathbf{z}_i$ *according to (5.9), and compute the differences* X_i *and* Y_i *according to (5.3) and (5.4), respectively.*
   (b) *Check to see that* Θ_i *(or* Π_i*) is nonsingular for all* $i = 1, \dots, n$*. Otherwise, report that the given spectral information is not generic and stop.*
2. *Form the upper triangular matrices* A *and* B *according to (5.14) (or (5.24)).*
3. *Solve the minimax problem (5.25).*
   (a) *If a solution* $\mathbf{m} > 0$ *exists, define* $\mathbf{c} = A\mathbf{m}$ *and* $\mathbf{k} = B\mathbf{m}$.
   (b) *If a solution* $\mathbf{m} > 0$ *does not exist, the given eigenpairs are not feasible.*

6. Numerical Experiment. In this section we demonstrate the above ideas by some numerical examples.

Example 4. We first generate randomly the data

$$\mathbf{m} = [1.0660, 0.2593, 0.6809, 0.5453, 1.0000]^\top,$$
$$\mathbf{k} = [7.6210, 4.5647, 0.1850, 8.2141, 4.4470]^\top,$$

and create an undamped pencil $\lambda^2 M + K$ in the form (4.1). The resulting system has natural frequencies

$$5.7620, 4.9207, 2.5971, 2.3485, 0.2768,$$

with columns in the matrix

$$\begin{bmatrix} -0.0080 & -0.5440 & -0.1126 & 0.7932 & 0.0153 \\ 0.0405 & 1.6238 & -0.1232 & 1.0958 & 0.0405 \\ -0.6480 & -0.0019 & 0.7796 & 0.0902 & 0.6579 \\ 1.1197 & -0.0347 & 0.3641 & 0.0263 & 0.6677 \\ -0.1732 & 0.0078 & -0.7046 & -0.1095 & 0.6794 \end{bmatrix},$$

as the corresponding natural modes. We point out earlier that the order of frequencies corresponds precisely to the number of sign changes in the eigenvector. Suppose now we use the third eigenpair as the given eigenpair and apply Algorithm 4.1 to reconstruct the quadratic pencil. To illustrate the working of Algorithm 4.1, we allow the mass increasing factor s_p to vary from 1.001 to 10^5. The resulting allowable masses and the corresponding frequencies are plotted in the left and right graphs of Figure 6.1, respectively. Clearly, the solution to the QIEP is not unique. Note that in this particular example, only the second and the third masses are changed by our algorithm, whereas in

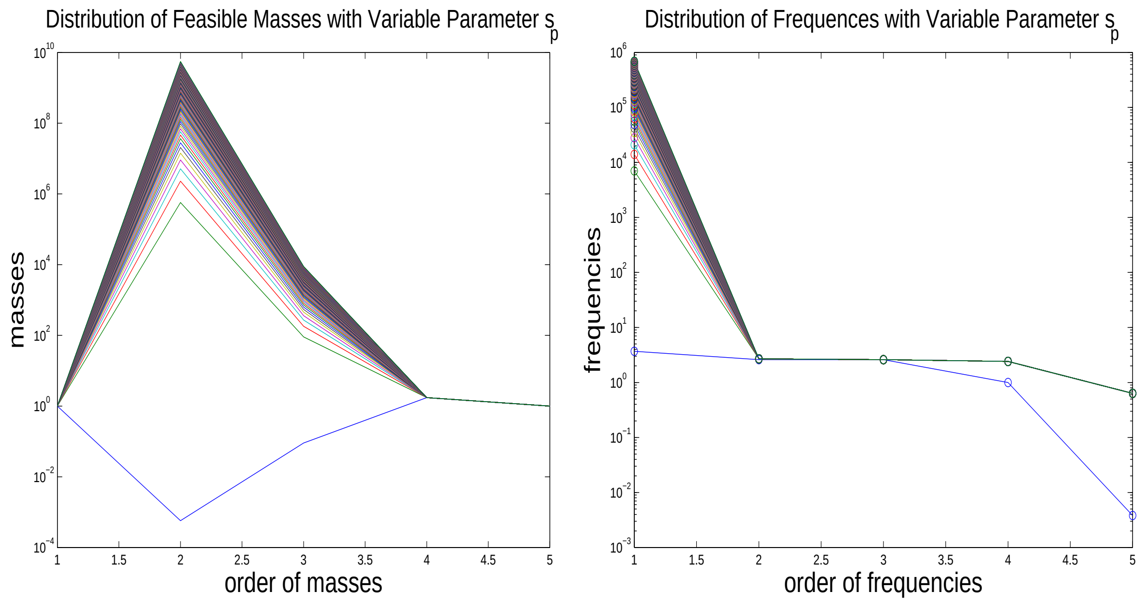

FIG. 6.1. *Allowable masses and frequency responses for undamped system.*

all cases the third frequency stays invariant. The experiment also suggests an interesting observtaion that an increase on the second and the third mass, while maintaining the third eigepair, has more effect on the first eigenvalue than on the fourth or the fifth eigenvalues.

Example 5. We randomly generate the data

$$\begin{aligned}\mathbf{m} &= [3.4915, 4.4928, 5.2297, 4.1880, 1.0000]^\top,\\ \mathbf{c} &= [0.0579, 0.3529, 0.8132, 0.0099, 0.1389]^\top,\\ \mathbf{k} &= [4.0571, 9.3547, 9.1690, 4.1027, 8.9365]^\top,\end{aligned}$$

and create a pencil $\lambda^2 M + \lambda C + K$ in the form (5.1). The resulting system has 5 pairs of complex conjugate eigenvalues,

$$-0.0863 \pm 3.3596i, -0.1792 \pm 2.6044i, -0.0792 \pm 1.7918i, -0.0079 \pm 1.0365i, -0.0019 \pm 0.3736i,$$

not necessarily in any particular order, and the corresponding eigenvectors

$$\begin{bmatrix} 0.0005 \mp 0.0005i & 0.0696 \mp 0.2457i & -0.2044 \pm 0.3474i & -0.2495 \pm 0.3822i & -0.3451 \pm 0.1371i \\ -0.0012 \pm 0.0015i & -0.1084 \pm 0.2648i & -0.0325 \pm 0.0894i & -0.2572 \pm 0.3949i & -0.4763 \pm 0.1907i \\ 0.0019 \mp 0.0056i & 0.0566 \mp 0.1061i & 0.1583 \mp 0.3303i & -0.1457 \pm 0.1913i & -0.5767 \pm 0.2341i \\ -0.0176 \pm 0.0597i & -0.0018 \pm 0.0095i & -0.0335 \pm 0.0987i & 0.3470 \mp 0.5006i & -0.7014 \pm 0.2830i \\ 0.0676 \mp 0.2260i & -0.0181 \pm 0.0339i & -0.0575 \pm 0.1522i & 0.3943 \mp 0.5691i & -0.7125 \pm 0.2875i \end{bmatrix}.$$

Suppose we use the first complex conjugate eigenpairs as the given eigenpairs and apply Algorithm 5.1 to reconstruct the quadratic pencil. We solve the minimax problem (5.25) by employing the Matlab code fminimax with tolerance TolFun = TolCon = 10^{-8}. We obtain a solution to the QIEP at

$$\begin{aligned}\mathbf{m} &= [0.0512, 0.2413, 0.5096, 0.1653, 0.0326]^\top,\\ \mathbf{c} &= [0.0045, 0.0045, 0.0588, 0.0045, 0.0045]^\top,\\ \mathbf{k} &= [0.0045, 0.1520, 0.6021, 0.4316, 0.2912]^\top.\end{aligned}$$

Note that $\mathbf{m}$ is limited to the range $[0,1]$, whereas we could have normalized m_5 to unit mass without effecting the spectrum. An examination of the remaining eigenvalues does not seem to suggest that there is anything analogous to the Courant Nodal Line theorem for the damped case.

7. Conclusion. We have studied the quadratic inverse eigenvalue problem arising from dynamical systems whose vibratory properties can be modeled after a serially linked mass-spring oscillator. The resulting QIEP must oblige a special matrix structure. In particular, the physical parameters must be positive. These constraints severely limit the solvability of the QIEP.

We show that for undamped problem, the order of frequencies corresponds precisely to the number of sign changes in the eigenvectors. A make-or-break algorithm is proposed in this work. For damped problem, we show that the solvability of the QIEP is equivalent to the consistency of a certain system of inequalities which can be solved via a minimax algorithm.

Thus far, the permissible eigenpair information is rather limited. For undamped problem only one eigenpair is allowed while for damped problem only two eigenpairs are allowed. In either case, the masses $\mathbf{m}$ are used as the free parameters to express the other two physical parameters $\mathbf{c}$ and $\mathbf{k}$. In certain applications, it is desirable to express $\mathbf{m}$ and $\mathbf{c}$ in terms of $\mathbf{k}$. However, such a task is much harder to analyze and we have not obtained significant results yet.

If the QIEP is solvable, usually there are multiple solutions. Although we are able to match the prescribed eigenpair information in our construction, our technique thus far cannot control the remaining eigenvalue or eigenvectors. It remains to be an open question on whether additional eigenvalue information, not eigenvector information, can help to further restrict the number of solutions or control the remaining eigenstructure.

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