ON CONSTRUCTING MATRICES WITH PRESCRIBED SINGULAR VALUES AND DIAGONAL ELEMENTS

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Abstract. Similar to the well known Schur-Horn theorem that characterizes the relationship between the diagonal entries and the eigenvalues of a Hermitian matrix, the Sing-Thompson theorem characterizes the relationship between the diagonal entries and the singular values of an arbitrary matrix. It is noted in this paper that, based on the induction principle, such a matrix can be constructed numerically by a fast recursive algorithm, provided that the given singular values and diagonal elements satisfy the Sing-Thompson conditions.

1. Introduction. It has been observed that the main diagonal entries and the eigenvalues of any Hermitian matrix enjoy an interesting relationship. This relationship is completely characterized by what is now known as the Schur-Horn theorem [2]. For reference, we describe specifically the theorem in two parts. More details and related topics can be found, for example, in [3, Theorems 4.3.26, 4.3.32 and 4.3.33].

THEOREM 1.1. *(Schur-Horn Theorem)*

1. *Given an arbitrary Hermitian matrix H, let $\lambda = [\lambda_i] \in R^n$ and $a = [a_i] \in R^n$ denote the vectors of eigenvalues and main diagonal entries of H, respectively. If the entries are arranged in increasing order $a_{j_1} \leq \ldots \leq a_{j_n}$ and $\lambda_{m_1} \leq \ldots \leq \lambda_{m_n}$, then*

$$\sum_{i=1}^{k} a_{j_i} \geq \sum_{i=1}^{k} \lambda_{m_i} \tag{1.1}$$

for all $k = 1, 2, \ldots, n$, and the equality holds when $k = n$.

2. *Let $a, \lambda \in R^n$ be two given vectors that satisfy (1.1). Then there exists a Hermitian matrix H with eigenvalues λ and diagonal entries a.*

The notion of (1.1) is also known as *a majorizing λ*. The second part of the Schur-Horn theorem gives rise to an interesting *inverse eigenvalue problem*, namely, to construct a Hermitian matrix with the prescribed eigenvalues and diagonal entries. Numerical methods for such a construction were first proposed in [1]. Later an efficient recursive method was discussed in [9].

In the case of a general real matrix, it seems impossible that the main diagonal entries and the eigenvalues would continue to hold a connection

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as specific as the Schur-Horn theorem. Indeed, the most general result in this regard is as follows due to Mirsky [4]. Seeking any additional bearing would be too vague a problem to have any significant ramification.

THEOREM 1.2. *A necessary and sufficient condition for the existence of a matrix with eigenvalues* $\lambda_1, \ldots, \lambda_n$ *and main diagonal elements* $a_1, \ldots, a_n$ *is that*

$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} \lambda_i. \tag{1.2}$$

Naturally, the next level of question is to find any connection between the main diagonal entries and the singular values of a general matrix, as was posed by Mirsky in [5]. Such a relationship was discovered independently by Sing [6] and Thompson [7]. Similar to the notion of majorization, it turns out that the necessary and sufficient conditions for the existence of a matrix with prescribed main diagonal entries and prescribed singular values also involve a set of inequalities which we state as follows.

THEOREM 1.3. *(Sing-Thompson Theorem) Let* $d, s \in R^n$ *be two vectors with entries arranged in the order* $s_1 \geq s_2 \geq \ldots s_n$ *and* $|d_1| \geq |d_2| \geq \ldots |d_n|$*, respectively. Then there exists a real matrix* $A \in R^{n\times n}$ *with singular values* s *and main diagonal entries* d *(possibly in different order) if and only if*

$$\sum_{i=1}^{k} |d_i| \leq \sum_{i=1}^{k} s_i \tag{1.3}$$

for all $k = 1, 2, \ldots, n$ *and*

$$\left(\sum_{i=1}^{n-1} |d_i|\right) - |d_n| \leq \left(\sum_{i=1}^{n-1} s_i\right) - s_n. \tag{1.4}$$

In analogue to that in the Schur-Horn theorem, the sufficient condition in Sing-Thompson theorem gives rise to an *inverse singular value problem*, namely, to construct a real $n \times n$ matrix from the prescribed singular values and diagonal entries. This paper discusses one numerical procedures for solving such a problem. The idea is based on Thompson's original proof by induction and is inspired by the approach in [9].

For convenience, we shall denote the diagonal matrix whose main diagonal entries are the same as those of the matrix M as $\mathrm{diag}(M)$, and denote the diagonal matrix whose diagonal entries are formed from the vector v as $\mathrm{diag}(v)$. In this paper, we first demonstrate how the 2×2 case can be constructed by elementary algebra. We then use the 2×2 case a building block to construct the general matrix. Numerical experiments

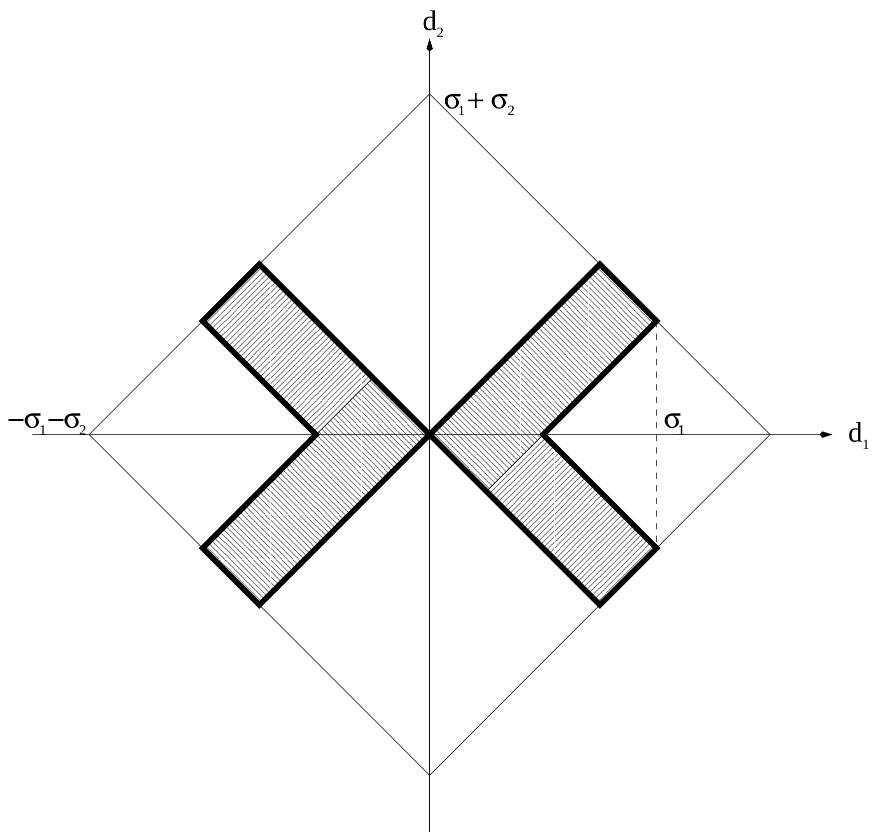

FIG. 2.1. *Domain of feasible d_1 and d_2, given σ_1 and σ_2.*

suggest that the construction of an $n \times n$ matrix requires approximately $O(n^2)$ flops.

2. The 2×2 Case. The conditions in Theorem 1.3 for the existence of a matrix $A \in R^{2\times 2}$ with singular values $\{\sigma_1, \sigma_2\}$ and main diagonal entries $\{d_1, d_2\}$ become that

$$\begin{aligned} \sigma_1 &\geq \sigma_2 \\ |d_1| &\geq |d_2| \\ |d_1| &\leq \sigma_1 \\ |d_1| + |d_2| &\leq \sigma_1 + \sigma_2 \\ |d_1| - |d_2| &\leq \sigma_1 - \sigma_2. \end{aligned} \tag{2.1}$$

A typical region for d_1 and d_2 satisfying the above conditions with fixed $\sigma_1 \geq \sigma_2 > 0$ is depicted by the shaded domain in Figure 2.1. In this section we study how such a matrix A can be constructed. Specifically, we want to determine the two orthogonal matrices U_1 and U_2 such that

$$\operatorname{diag}(U_1 \Sigma U_2^T) = \operatorname{diag}([d_1, d_2]) \tag{2.2}$$

where $\Sigma = \operatorname{diag}([\sigma_1, \sigma_2])$. This 2×2 construction will become a building block in our recursive algorithm.

There are only two classes of 2×2 orthogonal matrices, i.e.,

$$\begin{bmatrix} c & s \\ -s & c \end{bmatrix} \text{ or } \begin{bmatrix} c & s \\ s & -c \end{bmatrix}$$

where $c^2 + s^2 = 1$. The orthogonal matrices U_1 and U_2 can assumed any of these two forms. Let the entries corresponding to U_i be denoted as c_i and s_i. If both U_1 and U_2 belong to the same class of orthogonal matrices, then the equation (2.2) is equivalent to the system

$$\left\{ \begin{array}{rcl} (\sigma_1^2 - \sigma_2^2) c_1 c_2 & = & (\sigma_1 d_1 - \sigma_2 d_2) \\ (\sigma_1^2 - \sigma_2^2) s_1 s_2 & = & (\sigma_1 d_2 - \sigma_2 d_1). \end{array} \right. \tag{2.3}$$

If U_1 and U_2 belong to two different classes, then (2.2) is equivalent to the system

$$\left\{ \begin{array}{rcl} (\sigma_1^2 - \sigma_2^2) c_1 c_2 & = & (\sigma_1 d_1 + \sigma_2 d_2) \\ (\sigma_1^2 - \sigma_2^2) s_1 s_2 & = & -(\sigma_1 d_2 + \sigma_2 d_1). \end{array} \right. \tag{2.4}$$

We want to show that if conditions (2.2) are satisfied, then at least one of the two systems (2.3) and (2.4) will have real solutions for c_i and s_i, $i = 1, 2$, and we want to compute these values.

With the notation defined by

$$\begin{aligned} A &:= (\sigma_1 d_1 - \sigma_2 d_2)^2, \\ B &:= (\sigma_1 d_2 - \sigma_2 d_1)^2, \\ C &:= (\sigma_1 d_1 + \sigma_2 d_2)^2, \\ D &:= (\sigma_1 d_2 + \sigma_2 d_1)^2, \\ E &:= (\sigma_1^2 - \sigma_2^2)^2, \end{aligned}$$

we claim that the orthogonal matrices U_1 and U_2 can be constructed according to the following recipe.

LEMMA 2.1. *Suppose* $|\sigma_1 - \sigma_2| \geq |d_1 - d_2|$. *Then the two roots* r_i, $i = 1, 2$, *of the quadratic equation*

$$-Ex^2 + (E - B + A)x - A = 0 \tag{2.5}$$

are real and bounded between 0 and 1. Define $U_i = \begin{bmatrix} c_i & s_i \\ -s_i & c_i \end{bmatrix}$ *where*

$$\begin{aligned} c_1 &= r_1, \\ c_2 &= sgn\left(\frac{\sigma_1 d_1 - \sigma_2 d_2}{\sigma_1^2 - \sigma_2^2}\right) r_2, \\ s_1 &= \sqrt{1 - r_1^2}, \\ s_2 &= sgn\left(\frac{\sigma_1 d_2 - \sigma_2 d_1}{\sigma_1^2 - \sigma_2^2}\right) \sqrt{1 - r_2^2}. \end{aligned}$$

Then (2.2) is satisfied.

LEMMA 2.2. *Suppose* $|\sigma_1 - \sigma_2| < |d_1 - d_2|$. *Then the two roots* r_i, $i = 1, 2$, *of the quadratic equation*

$$-Ex^2 + (E - D + C)x - C = 0 \tag{2.6}$$

are real and bounded between 0 and 1. Define $U_1 = \begin{bmatrix} c_1 & s_1 \\ -s_1 & c_1 \end{bmatrix}$ *and* $U_2 = \begin{bmatrix} c_2 & s_2 \\ s_2 & -c_2 \end{bmatrix}$ *where*

$$\begin{aligned} c_1 &= r_1, \\ c_2 &= sgn\left(\frac{\sigma_1 d_1 + \sigma_2 d_2}{\sigma_1^2 - \sigma_2^2}\right) r_2, \\ s_1 &= \sqrt{1 - r_1^2}, \\ s_2 &= -sgn\left(\frac{\sigma_1 d_2 + \sigma_2 d_1}{\sigma_1^2 - \sigma_2^2}\right) \sqrt{1 - r_2^2}. \end{aligned}$$

Then (2.2) is satisfied.

Proof. We shall prove Lemma 2.1 only. The case in Lemma 2.2 can be argued similarly. For convenience, we write $X := c_1^2$ and $Y := c_2^2$. Upon squaring both sides of (2.3), we obtain

$$\left\{ \begin{array}{rcl} EXY & = & A \\ E(1-X)(1-Y) & = & B \end{array} \right. \tag{2.7}$$

because $s_1^2 = 1 - X$ and $s_2^2 = 1 - Y$. Upon further elimination and substitution, (2.7) is equivalent to the quadratic equation (2.5) in terms of X. The discriminant of (2.5) can be factorized as

$$\begin{aligned} (E - B + A)^2 - 4AE = (\sigma_1 - \sigma_2)^2(\sigma_1 + \sigma_2)^2 \\ [(\sigma_1 + \sigma_2)^2 - (d_1 + d_2)^2][(\sigma_1 - \sigma_2)^2 - (d_1 - d_2)^2]. \end{aligned} \tag{2.8}$$

Obviously, the first two factors in each of the above two expressions are obviously nonnegative. The Sing-Thompson conditions imply that the third factor in each expressions is also nonnegative. Since it is assumed that $|\sigma_1 - \sigma_2| \geq |d_1 - d_2|$, the fourth factor is also nonnegative. This proves that the equation (2.5) has real solutions.

Note specifically that the smaller root,

$$X_1 = \frac{(E - B + A) - \sqrt{(E - B + A)^2 - 4AE}}{2E}, \tag{2.9}$$

is positive because $E - B + A = (\sigma_1^2 - \sigma_2^2)(\sigma_1^2 - \sigma_2^2 + d_1^2 - d_2^2) \geq 0$ and and $AE \geq 0$. To show that the larger root,

$$X_2 = \frac{(E - B + A) + \sqrt{(E - B + A)^2 - 4AE}}{2E}, \tag{2.10}$$

is no greater than 1, it suffices to show that $\sqrt{(E - B + A)^2 - 4AE} \leq E + B - A$. The inequality follows directly from the factorization that $\left(\sqrt{(E - B + A)^2 - 4AE}\right)^2 - (E+B-A)^2 = -4(\sigma_1-\sigma_2)^2(\sigma_1+\sigma_2)^2(\sigma_1 d_2 - \sigma_2 d_1)^2 \leq 0$. and the observation that $E + B - A = (\sigma_1^2 - \sigma_2^2)[(\sigma_1^2 - \sigma_2^2) - (d_1^2 - d_2^2)] \geq 0$ because of the Sing-Thompson conditions in (2.2).

Symmetry of X and Y in the system (2.7) implies that X and Y are exactly the two roots of (2.5). The signs of c_1 and c_2 (and also those of s_1 and s_2) should be determined from (2.3). Indeed, it can be checked that only the signs of the products $c_1 c_2$ and $s_1 s_2$ matter in the construction. This proves Lemma 2.1. $\square$

It is interesting to note that the pair (d_1, d_2) described in Lemma 2.1 corresponds to the region in Figure 2.1 where the shaded lines are slanted from the northwest to the southeast. In the meantime, the pair (d_1, d_2) described in Lemma 2.2 must come from the region in Figure 2.1 where the shaded lines are slanted from the northeast to the southwest. In summary, the above argument can be implemented into an algorithm that is listed in Table 2.1.

3. A Recursive Algorithm. Assuming that the vectors d and s are arranged in non-increasing order and satisfy the Sing-Thompson conditions, Thompson carried out the proof of sufficient conditions in Theorem 1.3 by considering two mutually exclusive cases followed by the mathematical induction. In this note we want to point out that his induction argument is in fact implementable as a recursive algorithm.

In order to explain the recursive algorithm more plainly, we quickly review Thompson's proof below. Let

$$k := \max\{i \mid |d_1| \leq s_j \text{ for } j = 1, \ldots, i\}. \tag{3.1}$$

```
function [U,V]=two_by_two(d,s);
%
% Create 2 by 2 orthogonal matrices U and V such that
%
%     diag(U*diag(s)*V') = diag(d)
%
if abs(s(1)-s(2)) <= 2*s(1)*eps
  U = eye(2); V = eye(2);
  return
end
if abs(d(1)) <= abs(d(2))
  iflag = 1; d = flipud(d);
else
  iflag = 0;
end
temp1 = s(1)*d(1)-s(2)*d(2); A = temp1^2;
temp2 = s(1)*d(2)-s(2)*d(1); B = temp2^2;
temp3 = s(1)*d(1)+s(2)*d(2); C = temp3^2;
temp4 = s(1)*d(2)+s(2)*d(1); D = temp4^2;
temp5 = s(1)*s(1)-s(2)*s(2); E = temp5^2;
if abs(d(1)-d(2)) <= abs(s(1)-s(2))
  temp = sqrt(abs((E-B+A)^2-4*A*E));
  R = sqrt([(E-B+A)+temp;(E-B+A)-temp]/(2*E)).*[1;sign(temp1/temp5)];
  S = sqrt(1-R.^2).*[1;sign(temp2/temp5)];
  U = [R(1),S(1);-S(1),R(1)]; V = [R(2),S(2);-S(2),R(2)];
else
  temp = sqrt(abs((E-D+C)^2-4*E*C));
  R = sqrt([(E-D+C)+temp;(E-D+C)-temp]/(2*E)).*[1;sign(temp3/temp5)];
  S = sqrt(1-R.^2).*[1;-sign(temp4/temp5)];
  V = [R(2),S(2);S(2),-R(2)]; U = [R(1),S(1);-S(1),R(1)];
end
if iflag == 1
  U = flipud(U); V = flipud(V);
end
```

TABLE 2.1

A MATLAB program for 2×2 case.

Clearly, $k \geq 1$ by conditions in (1.3).

Case 1. Suppose $k \leq n-2$. Let

$$t := s_k + s_{k+1} - |d_1|. \tag{3.2}$$

Then $t > 0$. It is not definite which of $|d_1|$ and t is larger, but by the definition of k it is easy to see that $||d_1| - t| \leq s_k - s_{k+1}$. It follows that the two sets of numbers $\{d_1, t\}$ and $\{s_k, s_{k+1}\}$ satisfy the conditions in (2.2). For these 2×2 case, there exists orthogonal matrices $\tilde{U}_1$ and $\tilde{U}_2$ in $R^{2\times 2}$ such that

$$\operatorname{diag}\left(\tilde{U}_1 \operatorname{diag}([s_k, s_{k+1}])\tilde{U}_2^T\right) = \operatorname{diag}\left([d_1, t]\right).$$

Furthermore, using the definition of t, the two new sequences of numbers,

$$\left\{ \begin{array}{c} s_1 \geq s_2 \geq \ldots s_{k-1} \geq t \geq s_{k+1} \geq \ldots \geq s_n \\ |d_2| \geq |d_3| \geq \ldots |d_k| \geq |d_{k+1}| \geq |d_{k+2}| \ldots \geq |d_n| \end{array} \right. \tag{3.3}$$

satisfy the Sing-Thompson conditions. If the induction hypothesis holds, then there exist orthogonal matrices $\hat{U}_1$ and $\hat{U}_2$ in $R^{(n-1)\times(n-1)}$ such that

$$\operatorname{diag}\left(\hat{U}_1 \operatorname{diag}([s_1, \ldots, s_{k-1}, t, s_{k+1}, \ldots, s_n])\hat{U}_2^T\right) = \operatorname{diag}([d_2, \ldots, \ldots, d_n]).$$

If we decompose partition $\hat{U}_i$ into blocks

$$\hat{U}_i = \begin{bmatrix} \hat{U}_{i,11} & \hat{U}_{i,12} \\ \hat{U}_{i,21} & \hat{U}_{i,22} \end{bmatrix}, \quad i = 1, 2, \tag{3.4}$$

where $\hat{U}_{i,11}$ is of size $(k-1)\times(k-1)$ and define

$$U_i := \begin{bmatrix} \hat{U}_{i,11} & 0 & \hat{U}_{i,12} \\ 0 & 1 & 0 \\ \hat{U}_{i,21} & 0 & \hat{U}_{i,22} \end{bmatrix} \begin{bmatrix} I_{k-1} & 0 & 0 \\ 0 & \tilde{U}_i & 0 \\ 0 & 0 & I_{n-k-1} \end{bmatrix}, \quad i = 1, 2, \tag{3.5}$$

then it is readily seen that

$$\operatorname{diag}\left(U_1 \operatorname{diag}\{s_1, \ldots, s_n\} U_2^T\right) = \operatorname{diag}\{d_2, \ldots, d_{k-1}, d_1, d_{k+1}, \ldots, d_n\}.$$

Case 2. Suppose $n-1 \leq k \leq n$. Through an amazing insight, Thompson observed that all twelve inequalities involved in the following extremes

$$\left. \begin{array}{r} -s_{n-1} + s_n + |d_n| \\ 0 \\ \sum_{i=1}^{n-1} |d_i| - \sum_{i=1}^{n-2} s_i \end{array} \right\} \leq \left\{ \begin{array}{l} s_{n-1} \\ s_{n-1} + s_n - |d_n| \\ s_{n-1} - s_n + |d_n| \\ \sum_{i=1}^{n-2} s_i - \sum_{i=1}^{n-2} |d_i| + |d_{n-1}| \end{array} \right. \tag{3.6}$$

held. Let t be any value lying between the maximum of the left extremes and the minimum of the right extremes. Then it follows that the two set of numbers $\{t, d_n\}$ and $\{s_{n-1}, s_n\}$ satisfy the conditions in (2.2), regardless whichever t or $|d_n|$ is larger. It also follows that the two new sequences of numbers,

$$\left\{ \begin{array}{c} s_1 \geq s_2 \geq \ldots \geq s_{n-2} \geq t \\ |d_1| \geq |d_2| \geq \ldots \geq |d_{n-2}| \geq |d_{n-1}| \end{array} \right. \tag{3.7}$$

satisfy the Sing-Thompson conditions. We therefore can construct orthogonal matrices $\tilde{U}_1$ and $\tilde{U}_2$ in $R^{2\times 2}$ such that

$$\operatorname{diag}\left(\tilde{U}_1 \operatorname{diag}([s_{n-1}, s_n])\tilde{U}_2^T\right) = \operatorname{diag}\left([t, d_n]\right).$$

By the induction hypothesis, we also can construct orthogonal matrices $\hat{U}_1$ and $\hat{U}_2$ in $R^{(n-1)\times(n-1)}$ such that

$$\operatorname{diag}\left(\hat{U}_1 \operatorname{diag}([s_1, s_2, \ldots, s_{n-2}, t])\hat{U}_2^T\right) = \operatorname{diag}([d_1, d_2, \ldots, d_{n-1}]).$$

If we define

$$U_i := \left[\begin{array}{cc} \hat{U}_i & 0 \\ 0 & 1 \end{array}\right] \left[\begin{array}{cc} I_{n-2} & 0 \\ 0 & \tilde{U}_i \end{array}\right], \quad i = 1, 2, \tag{3.8}$$

then it is readily seen that

$$\operatorname{diag}\left(U_1 \operatorname{diag}\{s_1, \ldots, s_n\} U_2^T\right) = \operatorname{diag}\{d_1, d_2, \ldots, d_n\}.$$

In both cases, we notice that the original problem of size n is broken down to two subproblems of sizes 2 and $n-1$. The discussion in the previous section leads to a constructive solution for the subproblem of size 2. More significantly, the desired diagonal entries and singular values for the subproblem of size $n-1$ are given *explicitly* by either (3.3) or (3.7), which therefore can be further downsized to subproblems of sizes 2 and $n-2$. Repeating this argument, we find out that the construction of a matrix with prescribed diagonal entries and singular values can now be done by first *dividing* the original problem into a sequence of 2×2 subproblems and then by *conquering* these 2×2 subproblems to build up the original problem. This divide and conquer process is similar to that occurred in the radix-2 fast Fourier transform.

In MATLAB expressions, this recursive algorithm can be conveniently described as in Table 3.1. Note that the function **svd_diag** calls itself. The divide-and-conquer feature in our algorithm bring on fast computation. The cost should be in the order of $O(n^2)$. This estimation can be observed from numerical experiments. Using the diagonal entries

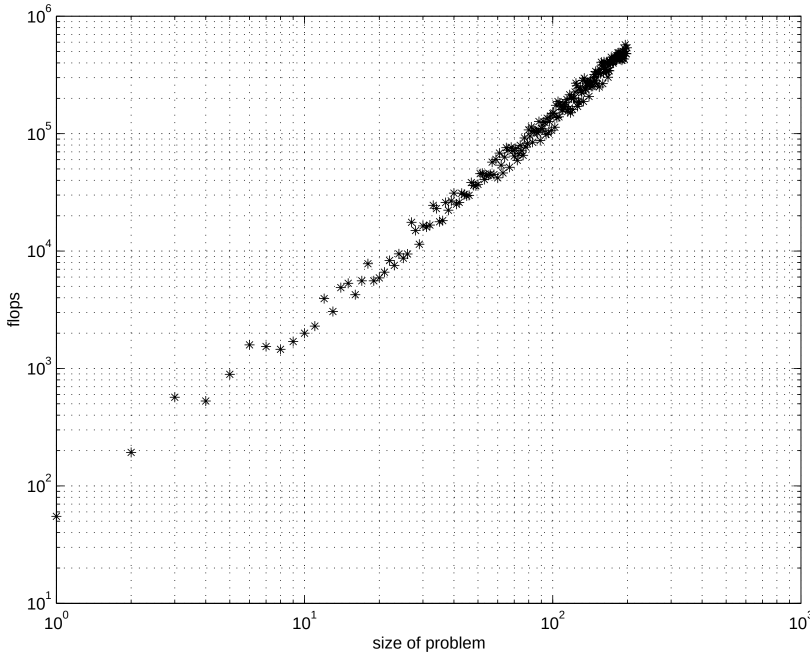

FIG. 3.1. *Computational cost versus size of problems.*

and singular values of 200 random matrices from size 2 to 200 as test data, we report the total floating point operations on log-log scale in Figure 3.1. The near linearity and its slop of the graph strongly suggests that the cost is approximately $O(n^2)$.

4. Conclusion. In an environment that allows a subprogram to invoke itself recursively, the existence of a solution proved by the mathematical induction can often be transformed into an recursive algorithm. In this note we have illustrated one particular application, by using MATLAB, to the construction of matrices with prescribed diagonal elements and singular values.

```
function [U,V]=svd_diag(d,s);
n = length(d);
if n == 2
  [U,V] = two_by_two(d,s);
else
  k = 1; i = 2;
  while (i <= n & abs(d(1)) <= s(i))
    k = k+1; i = i+1;
  end
  if k <= n-2
    t = s(k)+s(k+1)-abs(d(1));
    [U2,V2] = two_by_two([d(1);t],[s(k);s(k+1)]);
    s = [s(1:k-1);t;s(k+2:n)]; d = d(2:n);
    [U1,V1] = svd_diag(d,s);
    U = [U1(1:k-1,1:k-1),zeros(k-1,1),U1(1:k-1,k:n-1);
         zeros(1,k-1),1.0,zeros(1,n-k);
         U1(k:n-1,1:k-1),zeros(n-k,1),U1(k:n-1,k:n-1)];
    V = [V1(1:k-1,1:k-1),zeros(k-1,1),V1(1:k-1,k:n-1);
         zeros(1,k-1),1.0,zeros(1,n-k);
         V1(k:n-1,1:k-1),zeros(n-k,1),V1(k:n-1,k:n-1)];
    U(:,k:k+1)=U(:,k:k+1)*U2; V(:,k:k+1)=V(:,k:k+1)*V2;
  else
    temp1 = sum(abs(d(1:n-2))); temp2 = sum(s(1:n-2));
    temp3 = abs(d(n))-s(n-1); temp4 = s(n)-abs(d(n));
    temp5 = temp1-temp2;
    lower = max([s(n)+temp3,0,temp5+abs(d(n-1))]);
    upper = min([s(n-1),s(n-1)+temp4,s(n-1)-temp4,+abs(d(n-1))-temp5]);
    t = (lower+upper)/2;
    [U2,V2] = two_by_two([t;d(n)],[s(n-1);s(n)]);
    s = [s(1:n-2);t]; d = d(1:n-1);
    [U1,V1] = svd_diag(d,s);
    U = [U1,zeros(n-1,1); zeros(1,n-1),1];
    V = [V1,zeros(n-1,1); zeros(1,n-1),1];
    U(:,n-1:n)=U(:,n-1:n)*U2; V(:,n-1:n)=V(:,n-1:n)*V2;
  end
end
```

Table 3.1

A MATLAB program for the recursive algorithm.

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