ON THE SEMIGROUP OF STANDARD SYMPLECTIC MATRICES AND ITS APPLICATIONS

M. CHU*, N. DEL BUONO†, F. DIELE‡, T. POLITI§, AND S. RAGNI¶

Abstract. A matrix $Z \in \mathbb{R}^{2n\times 2n}$ is said to be in the standard symplectic form if Z enjoys a block LU-decomposition in the sense of $\begin{bmatrix} A & 0 \\ -H & I \end{bmatrix} Z = \begin{bmatrix} I & G \\ 0 & A^\top \end{bmatrix}$, where A is nonsingular and both G and H are symmetric and positive definite in $\mathbb{R}^{n\times n}$. Such a structure arises naturally in the discrete algebraic Riccati equations. This note contains two results: First, by means of a parameter representation it is shown that the set of all $2n \times 2n$ standard symplectic matrices is closed under multiplication and, thus, forms a semigroup. Secondly, block LU-decompositions of powers of Z can be derived in closed form which, in turn, can be employed recursively to induce an effective structure-preserving algorithm for solving the Riccati equations. The computational cost of doubling and tripling of the powers is investigated. It is concluded that doubling is the better strategy.

Key words. standard symplectic form, discrete algebraic Riccati equation, structure preserving, power method, block LU decomposition, semigroup

1. Introduction.

The generalized discrete algebraic Riccati equation (DARE)

$$X = A^\top XA - A^\top XB(R + B^\top XB)^{-1}B^\top XA + H, \tag{1.1}$$

where R and H are symmetric and positive definite matrices in $\mathbb{R}^{n\times n}$, arises in many important applications. With the full-ranked decomposition of $H = CC^\top$ and under some mild conditions, such as the system-theoretic notion that the pair of matrices (A, B) be stabilizable and the pair (A, C) be detectable, it has been established that the DARE has a unique stabilizing solution X [5]. It is further known that this unique solution X is symmetric and positive definite. Employing the Sherman-Morrison-Woodbury formula, it is not difficult to see that the DARE is equivalent to the equation

$$X = A^\top X(I + GX)^{-1}A + H, \tag{1.2}$$

if the inverse matrix $(I + GX)^{-1}$ with $G := BR^{-1}B^\top$ exists. The equation (1.2) can further be formulated as

$$\begin{bmatrix} A & 0 \\ -H & I \end{bmatrix} \begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} I & G \\ 0 & A^\top \end{bmatrix} \begin{bmatrix} I \\ X \end{bmatrix} \Phi \tag{1.3}$$

for some $\Phi \in \mathbb{R}^{n\times n}$. Denote

$$L := \begin{bmatrix} A & 0 \\ -H & I \end{bmatrix}, \tag{1.4}$$

$$M := \begin{bmatrix} I & G \\ 0 & A^\top \end{bmatrix}. \tag{1.5}$$

If we assume that A is invertible, it is seen from (1.3) that the space spanned by the columns of $[I, X]^\top$ (note that it is expected that $X = X^\top$) are invariant under $L^{-1}M$. Indeed, it is known that

*Department of Mathematics, North Carolina State University, Raleigh, NC 27695-8205. (chu@math.ncsu.edu)
This research was supported in part by the National Science Foundation under grants DMS-0073056 and CCR-0204157.
†Dipartimento di Matematica, Università degli Studi di Bari Via E. Orabona 4, I-70125 Bari, Italy. (delbuono@dm.uniba.it)
‡Istituto per le Applicazioni del Calcolo M. Picone, CNR, Via Amendola 122, 70126 Bari, Italy. (f.diele@area.ba.cnr.it)
§Dipartimento di Matematica, Politecnico di Bari, via Amendola 126/B, I-70126 Bari, Italy. (politi@poliba.it)
¶Facoltà di Economia, Università di Bari, Via Camillo Rosalba 56, 70100 Bari, Italy. (irmasr18@area.ba.cnr.it)

$L^{-1}M$ has exactly n eigenvalues inside the unit disk and that the unique solution X can be obtained from the invariant subspace associated with these stable eigenvalues [7, 8, 9]. One of the numerical techniques used for solving the DARE therefore is to compute the matrix $[P_1^\top, P_2^\top]^\top$ whose columns form a basis of the stable eigenspace. The unique solution X is then given by $P_2P_1^{-1}$. See, for example, the routine DARE available in the MATLAB control toolbox [10].

So long as G and H are symmetric, the pair of matrices (L, M) is said to be symplectic because they satisfy the relationship

$$LJL^\top = MJM^\top, \tag{1.6}$$

where

$$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}.$$

Equivalently, we may write (1.6) as

$$J = (L^{-1}M)J(L^{-1}M)^\top, \tag{1.7}$$

showing that the matrix,

$$Z := L^{-1}M, \tag{1.8}$$

is symplectic in the conventional sense. We further say that the pair (L, M) as defined in (1.4) and (1.5), as well as the corresponding product Z defined in (1.8), are in *standard symplectic form* (SSF) when both H and G are positive definite.

Recently it has been reported that from a given SSF pair (L, M), if we define

$$a := A(I + GH)^{-1}A, \tag{1.9}$$
$$g := G + AG(I + HG)^{-1}A^\top, \tag{1.10}$$
$$h := H + A^\top(I + HG)^{-1}HA, \tag{1.11}$$

then the pair $(\hat{L}, \hat{M})$ defined by

$$\hat{L} := \begin{bmatrix} a & 0 \\ -h & I \end{bmatrix}, \tag{1.12}$$
$$\hat{M} := \begin{bmatrix} I & g \\ 0 & a^\top \end{bmatrix}, \tag{1.13}$$

remains to be SSF [6]. Furthermore, it can be shown that

$$\hat{L}^{-1}\hat{M} = (L^{-1}M)^2 = Z^2. \tag{1.14}$$

As such, by repeating the above process, a SSF structure-preserving numerical method that computes the block LU-decompositions of $Z^2, Z^4, \ldots, Z^{2^k}, \ldots$ according to the closed-form formulas (1.9) through (1.11) becomes readily accessible. The power of squaring while maintaining the SSF structure is quite significant. It is capable of *quickly* separating stable and unstable eigenvalues (and the associated eigenspaces) in just a few iterations, even if some eigenvalues are close to the unit circle. Theoretical details and extensive numerical test results of this structure-preserving doubling algorithm (SDA) as well as its generalization, the structure-preserving swap and collapse algorithm (SSCA), for the periodic DAREs can be found in [6].

In contrast to the SSF structure-preserving scheme by squaring outlined above, the popular MATLAB routine DARE that implements a QZ-like algorithm to compute the stable invariant subspace would return a failure diagnosis when some eigenvalues are too close to the unit circle [1]. A more serious drawback of the QZ-like algorithm is that in spite of its numerical backward stability, it cannot preserve the symplectic structure. The computed eigenvalues therefore will *not* come in reciprocal pairs, although the exact eigenvalues should have this property. Even worse, small perturbations may cause eigenvalues close to the unit circle to cross the boundary and, hence, the number of computed eigenvalues inside the unit disk is no longer n. To remedy the above-mentioned troubles, many other algorithms with the symplectic structure in mind have been proposed. The matrix disk function method [2], for example, represents one of the latest developments in this area. Still, this method can preserve only the symplectic structure but not the SSF structure. It has been reported in [6] that the SSF structure-preserving doubling algorithm outperforms several existing important algorithms, including the MATLAB routine DARE and the more recently developed QR-SWAP algorithm [3], against a large set of benchmark problems described in [2, 4].

The emphasis of this note is not so much on the numerical methods for solving the DARE. Rather, we find it interesting that the SSF structure is preserved under the operation of matrix squaring and we want to see whether this operation can be generalized. Our main contribution in this paper is to show that the set of all $2n \times 2n$ SSF matrices, in fact, forms a semigroup under the operation of matrix multiplication. We complete our proof by exploiting a simple parametric representation of SSF matrices. Some possible applications and complexity analysis are discussed.

2. Parametrization of SSF Matrices. Given any $2n \times 2n$ real matrix Z, let its partition into 2×2 blocks of $n \times n$ matrices be denoted as

$$Z := \begin{bmatrix} \alpha & x \\ \beta & y \end{bmatrix}, \tag{2.1}$$

where α, β, x and y are $n \times n$ real matrices. Assuming that the leading block α is invertible, we have a block LU-decomposition of Z as

$$\begin{bmatrix} \alpha^{-1} & 0 \\ -\beta\alpha^{-1} & I \end{bmatrix} \begin{bmatrix} \alpha & x \\ \beta & y \end{bmatrix} = \begin{bmatrix} I & \alpha^{-1}x \\ 0 & y - \beta\alpha^{-1}x \end{bmatrix}. \tag{2.2}$$

It thus followed that Z is SSF if and only if its block entries satisfy the following three conditions:

$$\beta = s\alpha, \tag{2.3}$$

$$x = \alpha t, \tag{2.4}$$

$$y = \alpha^{-\top} + \beta\alpha^{-1}x = \alpha^{-\top} + s\alpha t, \tag{2.5}$$

for some $n \times n$ symmetric and positive definite matrices s and t. In other words, we can now characterize a SSF matrix Z by the triplet parameters (α, s, t) where α is nonsingular and both s and t are symmetric and positive definite. For DARE applications where often the inverse α^{-1} is readily embedded in the problem, it is computationally more efficient to represent a SSF matrix in terms of the triplet parameters $\{\delta, s, t\}$ where $\delta = \alpha^{-1}$. We use $\{\}$ to indicate that α^{-1} is used.

Let Z_1 and Z_2 be two arbitrary SSF matrices. Denote their block partitions as

$$Z_i = \begin{bmatrix} \alpha_i & x_i \\ \beta_i & y_i \end{bmatrix}, \tag{2.6}$$

and let their corresponding parametric representations be (α_i, s_i, t_i), $i = 1, 2$, respectively. Consider the matrix product and its block partition,

$$Z_p := Z_1 Z_2 := \begin{bmatrix} \alpha_p & x_p \\ \beta_p & y_p \end{bmatrix}. \tag{2.7}$$

In terms of the parametric representations, it is easy to see that

$$\alpha_p = \alpha_1\alpha_2 + x_1\beta_2 = \alpha_1(I + t_1s_2)\alpha_2, \tag{2.8}$$
$$\beta_p = \beta_1\alpha_2 + y_1\beta_2 = \left(s_1\alpha_1(I + t_1s_2) + \alpha_1^{-\top}s_2\right)\alpha_2, \tag{2.9}$$
$$x_p = \alpha_1x_2 + x_1y_2 = \alpha_1\left((I + t_1s_2)\alpha_2t_2 + t_1\alpha_2^{-\top}\right), \tag{2.10}$$
$$y_p = \beta_1x_2 + y_1y_2 = s_1\alpha_1\alpha_2t_2 + \left(\alpha_1^{-\top} + s_1\alpha_1t_1\right)\left(\alpha_2^{-\top} + s_2\alpha_2t_2\right). \tag{2.11}$$

We want to show that these four $n \times n$ blocks of Z_p satisfy all three relationships (2.3) to (2.5). That is, we claim that the product Z_p of any two SSF matrices remains to be SSF.

Note first from the right-hand side of (2.8) that the leading block $\alpha_p = \alpha_1\alpha_2 + x_1\beta_2$ is clearly invertible. Secondly, we have

$$\begin{aligned}
\beta_p\alpha_p^{-1} &= (\beta_1\alpha_2 + y_1\beta_2)(\alpha_1\alpha_2 + x_1\beta_2)^{-1} \\
&= \left(s_1\alpha_1(I + t_1s_2) + \alpha_1^{-\top}s_2\right)\left((I + t_1s_2)^{-1}\alpha_1^{-1}\right) \\
&= s_1 + \alpha_1^{-\top}s_2(I + t_1s_2)^{-1}\alpha_1^{-1} \\
&= s_1 + \alpha_1^{-\top}\left(s_2^{-1} + t_1\right)^{-1}\alpha_1^{-1}.
\end{aligned}$$

Using the fact that s_1, t_1 and s_2 are all symmetric and positive definite, we obtain the second matrix parameter

$$s_p := s_1 + \alpha_1^{-\top}s_2(I + t_1s_2)^{-1}\alpha_1^{-1}, \tag{2.12}$$

which is symmetric and positive definite, for the product Z_p. Similarly,

$$t_p := (\alpha_1\alpha_2 + x_1\beta_2)^{-1}(\alpha_1x_2 + x_1y_2) = t_2 + \alpha_2^{-1}(I + t_1s_2)^{-1}t_1\alpha_2^{-\top} \tag{2.13}$$

is another symmetric and positive definite matrix parameter we are looking for. It remains to show that y_p satisfies the nonlinear relationship (2.5) in terms of (α_p, s_p, t_p). Toward that end, we simply carry out a direct substitution. After much algebraic manipulation, we obtain that

$$\begin{aligned}
(\beta_1x_2 + y_1y_2) - \Big((\alpha_1\alpha_2 + x_1\beta_2)^{-\top} + (\beta_1\alpha_2 + y_1\beta_2)(\alpha_1\alpha_2 + x_1\beta_2)^{-1}(\alpha_1x_2 + x_1y_2)\Big) \\
= \alpha_1^{-\top}\left(I - (I + s_2t_1)^{-1} - s_2(I + t_1s_2)^{-1}t_1\right)\alpha_2^{-\top}.
\end{aligned} \tag{2.14}$$

By using the Sherman-Morrison-Woodbury formula, the quantity $I - (I + s_2t_1)^{-1} - s_2(I + t_1s_2)^{-1}t_1$ inside the middle parentheses on the right-hand side of (2.14) is seen to be precisely zero. We have thus proved the following result that extends the action of *squaring of a single SSF matrix* observed in [6] to the action of *matrix multiplication of arbitrary SSF matrices.* Although our derivation appears relatively simple, we have to remind readers that the four blocks of a SSF matrix are not related in a trivial way. Our proof is easy to deal with only after the parametric representation. We think this generalization is new and is of theoretical interest in itself.

Theorem 2.1. *The set $\mathcal{Z}$ of all $2n \times 2n$ real SSF matrices is closed under multiplication. That is, $\mathcal{Z}$ is a semigroup.*

Corollary 2.2. *If the parameters s and t are required to be only symmetric and positive semi-definite, then $\mathcal{Z}$ is a semigroup with the identity matrix. That is, $\mathcal{Z}$ is a monoid.*

We remark that $\mathcal{Z}$ cannot be a group because, even in the case of $n = 1$, the inverse matrix

$$\begin{bmatrix} \alpha & x \\ \beta & y \end{bmatrix}^{-1} = \begin{bmatrix} y & -x \\ -\beta & \alpha \end{bmatrix} \tag{2.15}$$

is clearly not SSF.

It is worthy to point out the subtle difference between SSF matrices and symplectic matrices. A $2n \times 2n$ matrix Z in the form (2.1) is symplectic if and only if

$$\alpha x^\top = x\alpha^\top, \tag{2.16}$$
$$\beta y^\top = y\beta^\top, \tag{2.17}$$
$$\alpha y^\top - x\beta^\top = I. \tag{2.18}$$

From (2.16) and (2.17), the matrices $u := \alpha x^\top$ and $v := \beta y^\top$ are symmetric. Assuming α is invertible, it follows that

$$\alpha^{-1}x = \alpha^{-1}u\alpha^{-\top}, \tag{2.19}$$
$$\beta\alpha^{-1} = v - \beta(\alpha^{-1}x)\beta^\top, \tag{2.20}$$

are also symmetric. Together we find that

$$y = \alpha^{-\top} + \beta x^\top \alpha^{-\top} = \alpha^{-\top} + \beta\alpha^{-1}x. \tag{2.21}$$

In other words, conditions (2.3), (2.4), and (2.5) of Z being SSF follow almost immediately from Z being symplectic whereas the collection of symplectic matrices forms a group. In particular, the nonlinear relationship (2.5) is more a property of symplectic matrices than a property of SSF matrices. The proof of (2.14), therefore, is obvious from the symplectic group. However, be aware that the matrices $t = \alpha^{-1}x$ and $s = \beta\alpha^{-1}$ in SSF matrices are required to be both symmetric and *positive definite* whereas the matrices in (2.19) and (2.20) are guaranteed only to be symmetric. It is the positive definiteness that separate SSF matrices from general symplectic matrices as a semigroup.

3. Applications. The parametrization of a SSF matrix Z can facilitate the computation of powers of Z. For example, if Z is identified by the parameters $\{\delta, s, t\}$, then using the representations (2.8), (2.12) and (2.13), the structure preserving parametrization $\{\delta^{(2)}, s^{(2)}, t^{(2)}\}$ of Z^2 is readily available:

$$\left\{\begin{array}{ll} \delta^{(2)} & := \delta(I + ts)^{-1}\delta, \\ s^{(2)} & := s + \delta^\top s(I + ts)^{-1}\delta, \\ t^{(2)} & := t + \delta(I + ts)^{-1}t\delta^\top. \end{array}\right. \tag{3.1}$$

These formulas agree precisely with those, i.e., (1.9), (1.10) and (1.11), already derived in [6]. The only difference is that we obtain (3.1) in a much easier way than that done in [6]. The parametrization formula (3.1) for Z^2 can be repeatedly applied to produce an iterative scheme, constituting the following SSF structure-preserving doubling algorithm (SDA) described in [6] for solving the DARE.

ALGORITHM 3.1. *Given parameters $\{\delta, s, t\}$ and a tolerance ϵ, do the following until convergence:*

1. *Define $\Omega := I + ts$;*
2. *Solve the linear systems*

$$\Omega U = \delta,$$
$$V\Omega = \delta,$$

for U and V;

3. *Compute*

$$\Delta s := \delta^\top sU,$$
$$\Delta t := Vt\delta^\top;$$

4. *Update*

$$\begin{aligned} \delta &\leftarrow \delta U, \\ s &\leftarrow s + \Delta s, \\ t &\leftarrow t + \Delta t; \end{aligned}$$

5. *If $\|\Delta s\|_F > \epsilon \|s\|_F$, go to Step 1; else, set $X = s$.*

We note that the two linear systems in Step 2 of Algorithm 3.1 can be solved by using the same decomposition of Ω. The following results concerning the convergence behavior of the above SDA algorithm are of great interest. Details of the proof can be found in [6].

THEOREM 3.1. *Given parameters $\{A, H, G\}$, define the symplectic pencil $L - \lambda M$ according to (1.4) and (1.5). Arrange the eigenvalues λ_i, λ_i^{-1}, $i = 1, \ldots, n$, of the pencil $L - \lambda M$ in the ordering $|\lambda_1| \leq \ldots \leq |\lambda_n| < 1 < |\lambda_n|^{-1} \leq \ldots \leq |\lambda_1|^{-1}$. Then*

1. *The sequence of matrices δ generated by the SDA algorithm converges to zero.*
2. *The sequence of matrices s generated by the SDA algorithm converges to the unique stabilizing solution X of the DARE problem (1.2).*
3. *The sequence of matrices t generated by the SDA algorithm converges to the solution Y of the dual DARE problem*

$$Y = AY(I + HY)^{-1}A^\top + G. \tag{3.2}$$

4. *In each case, the rate of convergence is $O(|\lambda_n|^{2^k})$.*

We have pointed out earlier, and it is not difficult to see it now, that the fast convergence of the SDA algorithm is mainly due to its effective computation of the parameter representations of the squares Z^{2^k}, $k = 1, 2, \ldots$. Equipped with Theorem 2.1, it is natural to ask that, if the SDA is ever successful, might it be possible to generalize the SSF structure-preserving algorithm by effectively computing the cubes Z^{3^k} or even Z^{4^k}?

Using the formulas (2.8), (2.12) and (2.13) for the product of Z and Z^2, and substituting in the expressions (3.1), we obtain

$$\left\{ \begin{array}{ll} \delta^{(3)} & := \delta[(I + ts)\alpha(I + ts) + t\alpha^{-\top}s]^{-1}\delta, \\ s^{(3)} & := s + \delta^\top \left[s + \delta^\top s(I + ts)^{-1}\delta\right] \left[I + ts + t\delta^\top s(I + ts)^{-1}\delta\right]^{-1}\delta, \\ t^{(3)} & := t + \delta \left[I + ts + \delta(I + ts)^{-1}t\delta^\top s\right]^{-1} \left[t + \delta(I + ts)^{-1}t\delta^\top\right]\delta^\top. \end{array} \right.$$

The seemingly complicated parametrization for Z^3 can further be simplified by defining the matrix

$$\Psi := (I + ts)\delta^{-1}(I + ts) + t\delta^\top s. \tag{3.3}$$

The expression

$$\delta^{(3)} = \delta\Psi^{-1}\delta, \tag{3.4}$$

with Ψ playing a similar role of Ω, is analogous to that of $\delta^{(2)}$ in (3.1). The manipulation in reducing the expressions of $s^{(3)}$ and $t^{(3)}$ is somewhat tricky. For the sake of readers, we derive the steps as follows:

$$\begin{array}{ll} s^{(3)} & = s + \delta^\top \left[s + \delta^\top s(I + ts)^{-1}\delta\right] \left[I + ts + t\delta^\top s(I + ts)^{-1}\delta\right]^{-1}\delta \\ & = s + \delta^\top \left[s + \delta^\top s(I + ts)^{-1}\delta\right] \left\{\Psi[\delta^{-1}(I + ts)]^{-1}\right\}^{-1}\delta \\ & = s + \delta^\top \left[s + \delta^\top s(I + ts)^{-1}\delta\right] \delta^{-1}(I + ts)\Psi^{-1}\delta \\ & = s + \delta^\top \left[s\delta^{-1}(I + ts) + \delta^\top s\right]\Psi^{-1}\delta. \end{array} \tag{3.5}$$

Similarly, it can be shown that

$$t^{(3)} = t + \delta\Psi^{-1}\left[(I+ts)\delta^{-1}t + t\delta^{\top}\right]\delta^{\top}. \tag{3.6}$$

It is worth noting the symmetry in the expressions (3.5) and (3.6), and their similarity to that in (3.1) for Z^2. In particular, the SDA algorithm for Z^{2^k} can now be modified for Z^{3^k}.

ALGORITHM 3.2. *Given parameters $\{\delta, s, t\}$ and a tolerance ϵ, do the following until convergence:*

0. *Define $\Omega := I + ts$ and solve the linear systems*

$$u\delta = \Omega,$$
$$\delta v = \Omega,$$

for u and v.

1. *Define $\zeta := \delta^{\top}s$, $\eta := t\delta^{\top}$, $\Psi := u\Omega + t\zeta$;*
2. *Solve the linear systems*

$$\Psi U = \delta,$$
$$V\Psi = \delta,$$

for U and V;

3. *Compute*

$$\Delta s := (\zeta v + \delta^{\top}\zeta)U,$$
$$\Delta t := V(u\eta + \eta\delta^{\top});$$

4. *Update*

$$\delta \leftarrow \delta U,$$
$$s \leftarrow s + \Delta s,$$
$$t \leftarrow t + \Delta t;$$

5. *If $\|\Delta s\|_F > \epsilon\|s\|_F$, go to Step 1; else, set $X = s$.*

We note again that all linear equations in Step 0 and Step 2 of Algorithm 3.2 can be solved by using the same decomposition of δ and Ψ, respectively. To compare the cost effectiveness of Algorithm 3.2 and Algorithm 3.1, we summarize in Table 3.1 the principal term of flops involved in each step.

Step	Operation	Algorithm 3.1	Algorithm 3.2
0	Matrix multiplication		n^3
0	LU decomposition		$\frac{2}{3}n^3$
0	Triangular solvers		$2n^3$
1	Matrix multiplications	n^3	$4n^3$
2	LU decompositions	$\frac{2}{3}n^3$	$\frac{2}{3}n^3$
2	Triangular solvers	$2n^3$	$2n^3$
3	Matrix multiplications	$4n^3$	$6n^3$
4	Matrix multiplications	n^3	n^3
	Total	$\frac{26}{3}n^3$	$\frac{52}{3}n^3$

TABLE 3.1
Comparison of total cost between squaring and cubing.

It is seen that Algorithm 3.2 is twice as expensive as Algorithm 3.1 per iteration. Obviously, applying Algorithm 3.1 consecutively twice per iteration to produce the sequence Z^{4^k}, $k = 1, 2, \ldots$, is more economical than applying Algorithm 3.2 once per iteration to produce the sequence Z^{3^k}, $k = 1, 2, \ldots$.

Although our theory assures that all powers Z^k of a SSF matrix Z remain to be SSF. Our study above seems to suggest the task of producing Z^5 or higher powers would involve more computational complexity and degrade the cost effectiveness. It appears that the best strategy is either squaring or quadrupling.

4. Conclusion. Motivated by a recent discovery that the SSF structure is preserved under squaring, we generalize the result to show that the SSF structure is preserved under matrix multiplication. In particular, we show that the set of $2n \times 2n$ SSF matrices forms a semigroup, but not a group. The block LU-decomposition of the square of a SSF matrix can be described explicitly and hence the DARE can be solved by a SSF structure-preserving algorithm. We characterize every SSF matrix by three matrix parameters and derive the block LU-decomposition of the product of two arbitrary SSF matrices explicitly. We present some complexity analysis and conclude that squaring and quadrupling are the two most cost effective strategies for solving the DARE.

Acknowledgment. The first author wishes to thank Professor Wen-Wei Lin for bringing the work of [6] to his attention while he was visiting the National Center for Theoretical Sciences at Hsingchu, Taiwan. The authors also thank one of the referees for pointing out that (2.5) is a symplectic group property.

REFERENCES

[1] W. F. Arnold, III, and A. J. Laub, Generalized eigenproblem algorithms and software for algebraic Riccati equations, Proc. IEEE, 72(1984), 1746-1754,

[2] P. Benner, Contributions to the numerical solutions of algebraic Riccati equations and related eigenvalue problems, Ph.D. Dissertation, Fakultät für Mathematik, TU Chemnitz-Zwickau, Chemnitz, Germany, 1997.

[3] P. Benner, R. Byers, R. Mayo, E.S. Quintano-Orti, and V. Hernández, Parallel algorithms for LQ optial control of discrete-time periodic linear systems, J. Para. Distr. Comp., 62(2002), 306-325.

[4] P. Benner, A. J. Laub, and V. Mehrmann, A collection of benchmark examples for the numerical solution of algebraic Riccati equations II: Discrete-time case, Tech. Rep. SPC 95-23, Fakultät für Mathematik, TU Chemnitz-Zwickau, Chemnitz, Germany, 1995. Available from http://www.tu-chemnitz.de/sfb393/spc95pr.html.

[5] S. Bittanti, P. Colaneri, and G. D. Nicolao, The periodic Riccati equations, in The Riccati Equation, S. Bittanti, A. Laub, and J. Willems, eds., 1991, 127-162.

[6] E. K.-W. Chu, H.-Y. Fan, W.-W. Lin, and C.-S. Wang, A structure-preserving doubling algorithm for periodic discrete-time algebraic Riccati equations, preprint, National Tsing Hua University, 2003.

[7] H. Faßbender, Symplectic Methods for the Symplectic Eigenvalue Problem, Kluwer Academic/Plenum Publishers, New York, 2000.

[8] J. J. Hench and A. J. Laub, Numerical solution of the discrete-time periodic Riccati equation, IEEE Trans. Auto. Control, 39(1994), 1197-1210.

[9] V. Mehrmann, The Autonomous Linear Quadratic Control Problem, Theory and Numerical Solution, Lecture Notes in Control and Information Sciences, 163, Springer-Verlag, Berlin, 1991.

[10] MATLAB, Control System Toolbox Reference, ver. 5.2, The MathWorks Inc., Mass., 2002. See also *http://www.mathworks.com/access/helpdesk/help/pdf_doc/control/reference.pdf*.

[11] T. Pappas, A. J. Laub, and N. R. Sandell, On the numerical solution of the discrete-time algebraic Riccati equation, IEEE Trans. Auto. Control, 25(1980), 631-641.

[12] J. Sreedhar and P. Van Dooren, Periodic descriptor systems: Solvability and conditionability, IEEE Trans. Auto. Control, 44(1999), 310-313.

[13] P. Van Dooren, A generalized eigenvalue approach for solving Riccati equations, SIAM J. Sci. Stat. Comput., 2(1981), 121-135.