Symmetric Toeplitz Matrices with Two Prescribed Eigenpairs

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Abstract

The inverse problem of constructing a real symmetric Toeplitz matrix based on two prescribed eigenpairs is considered. Two new results are obtained. First we show that the dimension of the subspace of Toeplitz matrices with two generically prescribed eigenvectors is independent of the size of the problem, and in fact is either two, three or four, depending upon whether the eigenvectors are symmetric or skew-symmetric and whether n is even or odd. This result is quite notable in that when only one eigenvector is prescribed the dimension is known to be at least $[(n+1)/2]$. Taking into account the prescribed eigenvalues, we then show how each unit vector in the null subspace of a certain matrix uniquely determines a Toeplitz matrix that satisfies the prescribed eigenpairs constraint. The cases where two prescribed eigenpairs uniquely determine a Toeplitz matrix are explicitly characterized.

Key words. Toeplitz matrix, eigenvector, inverse problem

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1. Introduction.

A real $n \times n$ matrix $T = (t_{ij})$ is symmetric and Toeplitz if there exist real scalars $r_1, \ldots, r_n$ such that

$$t_{ij} = r_{|i-j|+1}$$

for all i and j. Clearly a symmetric Toeplitz matrix is uniquely determined by the entries of its first column. Thus we shall denote a symmetric Toeplitz matrix by $T(r)$ if its first column is given by the vector $r \in R^n$.

Due to their role in important applications like the trigonometric moment problem, the Szegö theory and the signal processing, many properties of Toeplitz matrices have been studied over the years. For example, efficient algorithms have been devised to solve a Toeplitz system of equations in $O(n^2)$ time. Brief discussion of algorithms and more references for solving Toeplitz systems can be found in [7, Section 4.7]. In this paper, we are more interested in the spectral properties of a symmetric Toeplitz matrix.

It is easy to see that if $Tv = \lambda v$ and λ is an eigenvalue of multiplicity one, then either $Ev = v$ or $Ev = -v$ where $E = (e_{ij}) \in R^{n \times n}$ is the exchange matrix defined by

$$e_{ij} = \begin{cases} 1, & \text{if } i + j = n + 1; \\ 0, & \text{otherwise.} \end{cases}$$

Accordingly, we shall call such an eigenvector either symmetric or skew-symmetric. For eigenvalues of multiplicity greater than one, the corresponding eigenspace has an orthonormal basis which splits as evenly as possible between symmetric and skew-symmetric eigenvectors [5, Theorem 8]. Thus it is sensible to say that the eigenvectors of a symmetric Toeplitz matrix can be split into two classes. More specifically, as any symmetric centrosymmetric matrix [2, Theorem 2], a symmetric Toeplitz matrix of order n has $\lceil n/2 \rceil$ symmetric and $\lfloor n/2 \rfloor$ skew-symmetric eigenvectors. For convenience, we shall use $\sigma^+(T)$ and $\sigma^-(T)$ to denote, respectively, the spectrum of eigenvalues corresponding to symmetric and skew-symmetric eigenvectors. Other spectral properties of Toeplitz matrices can be found in [2, 5, 9, 10] and the references contained therein.

The inverse Toeplitz eigenvalue problem (ITEP) has been an interesting yet difficult question studied in the literature. The problem is to find a vector $r \in R^n$ such that the Toeplitz matrix $T(r)$ has a prescribed real spectrum $\{\lambda_1, \ldots, \lambda_n\}$. At present, the ITEP remains unsolved when $n \geq 5$ [5]. Partial results and numerical algorithms for the ITEP can be found in, for example, [3, 6, 8, 11].

In [2, Theorem 3] it is claimed that any real $n \times n$ matrix which has a set of n real orthonormal eigenvectors, each being either symmetric or skew-symmetric, is both symmetric and centrosymmetric. Apparently it is another interesting and difficult problem to identify an orthogonal matrix so that its columns are eigenvectors of some Toeplitz matrix.

In [4] it is proved that being symmetric or skew-symmetric is sufficient for a single vector to be an eigenvector of a Toeplitz matrix. In fact, let

$$S_0(v) := \{r \in R^n | T(r)v = 0\} \tag{1}$$

denote the collection of (the first columns of) all symmetric Toeplitz matrices for which v is an eigenvector corresponding to the eigenvalue 0. It can be shown that

$S_0(v)$ is a linear subspace with dimension [4, Corollary 1]

(2) $$\dim(S_0(v)) = n - \pi(v)$$

where

(3) $$\pi(v) := \begin{cases} \lceil n/2 \rceil & \text{if } v \text{ is symmetric,} \\ \lfloor n/2 \rfloor & \text{if } v \text{ is skew-symmetric} \end{cases}$$

is called the index of v. Clearly $T(r)v = \lambda v$ if and only if $r - \lambda w \in S_0(v)$, where $w = [1, 0, \ldots, 0]^T$ is the first standard basis vector in R^n. Thus the set

(4) $$S(v) := \{r \in R^n | T(r)v = \lambda v \text{ for some } \lambda \in R\}$$

is precisely the direct sum $< w > \oplus S_0(v)$.

Suppose now $\{v^{(1)}, \ldots, v^{(k)}\}$, $k \geq 1$, is a set of real orthonormal vectors, each being symmetric or skew-symmetric. Then $\cap_{i=1}^k S(v^{(i)})$ contains all symmetric Toeplitz matrices for which each v_i is an eigenvector. Evidently, $w \in S(v^{(i)})$ for all i. So $\cap_{i=1}^k S(v^{(i)})$ is at least of dimension 1. An interesting question then is

PROBLEM 1. *Obtain a non-trivial lower bound on the dimension of* $\bigcap_{i=1}^k S(v^{(i)})$.

Toward this end, we show in this paper that for the case $k = 2$, the dimension of $\bigcap_{i=1}^2 S(v^{(i)})$ is almost always independent of of the size of the problem, and in fact is either two, three or four, depending upon whether the eigenvectors are symmetric or skew-symmetric.

In view of the ITEP, another interesting inverse problem is

PROBLEM 2. *Given a set of real orthonormal vectors,* $\{v^{(1)}, \ldots, v^{(k)}\}$, $k \geq 1$, *each symmetric or skew-symmetric, and a set of real numbers* $\{\lambda_1, \ldots, \lambda_k\}$, *find a symmetric Toeplitz matrix* T *(other than a scalar matrix) such that*

(5) $$Tv^{(i)} = \lambda_i v^{(i)},\ i = 1, \ldots, k.$$

We note in Problem 2 that T is required to be Toeplitz, thus the description of the given eigenpairs cannot be totally arbitrary. For instance, it is improper to request that all vectors be symmetric while $k > \lceil n/2 \rceil$. We recall a conjecture in [5] that a *universal* distribution of eigenvalues for Toeplitz matrices should be such that $\sigma^+(T)$ and $\sigma^-(T)$ interlace. Thus a Toeplitz matrix whose spectrum does not satisfy the interlaced distribution is perhaps more difficult to find [8]. On the other hand, as far as Problem 2 is concerned, there is a possibility that the remaining unspecified eigenpairs could make up the total spectrum so that the interlaced condition is eventually realized.

For the case $k = 2$, we show in this paper that in each direction in the subspace $\bigcap_{i=1}^2 S(v^{(i)})$ there is one and only one Toeplitz matrix for Problem 2. In particular, we show that if n is odd and if at least one of the given eigenvectors is symmetric, or if n is even and one eigenvector is symmetric and the other is skew-symmetric, then the Toeplitz matrix is uniquely determined.

2. An Example.

As we shall only consider the case $k = 2$ throughout the paper, it is more convenient to denote, henceforth, the eigenvectors $v^{(1)}$ and $v^{(2)}$ by u and v, respectively.

We begin our study of the set $S(u) \cap S(v)$ with the special case where $n = 3$. The example should shed some insights on higher dimensional case.

Due to the special eigenstructure of symmetric Toeplitz matrices, it is necessary that one of the two given eigenvectors, say u, must be symmetric. Denote $u = [u_1, u_2, u_1]^T$ where $2{u_1}^2 + u_2^2 = 1$. It can also be proved that the skew-symmetric vector $\hat{u} = [1/\sqrt{2}, 0, -1/\sqrt{2}]^T$ is a universal eigenvector for every symmetric Toeplitz matrix of order 3. Thus, given u, we imply from the orthogonality condition that the second prescribed eigenvector v must be either the second or the third column of the matrix

$$Q = \begin{bmatrix} u_1 & \frac{1}{\sqrt{2}} & -\frac{u_2}{\sqrt{2}} \\ u_2 & 0 & u_1\sqrt{2} \\ u_1 & -\frac{1}{\sqrt{2}} & -\frac{u_2}{\sqrt{2}} \end{bmatrix}. \tag{6}$$

In other words, one symmetric eigenvector completely determines all three orthonormal eigenvectors (up to a $\pm$ sign). It follows, from [4], that $\dim S(u) \cap S(v) = 2$. (Note that if the orthogonality condition is violated, then trivially $S(u) \cap S(v) = < w >$.)

Let $\Lambda = \operatorname{diag}\{\lambda_1, \lambda_2, \lambda_3\}$. We already know $Q\Lambda Q^T$ is a centrosymmetric matrix. It is not difficult to see that $Q\Lambda Q^T$ is Toeplitz if and only if

$$(3{u_1}^2 - 1)\lambda_1 + \frac{\lambda_2}{2} + (\frac{1}{2} - 3{u_1}^2)\lambda_3 = 0. \tag{7}$$

From (7), the following facts can easily be observed:

Lemma 2.1. *Let $2{u_1}^2 + u_2^2 = 1$. Then:*

1. *If λ_1 and λ_3 are given scalars, there is a unique symmetric Toeplitz matrix T such that*

$$T \begin{bmatrix} u_1 & -\frac{u_2}{\sqrt{2}} \\ u_2 & u_1\sqrt{2} \\ u_1 & -\frac{u_2}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} u_1 & -\frac{u_2}{\sqrt{2}} \\ u_2 & u_1\sqrt{2} \\ u_1 & -\frac{u_2}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_3 \end{bmatrix}. \tag{8}$$

2. *If $u_1 \neq \pm\sqrt{1/6}$ and λ_1 and λ_2 are given scalars, there is a unique symmetric Toeplitz matrix T such that*

$$T \begin{bmatrix} u_1 & \frac{1}{\sqrt{2}} \\ u_2 & 0 \\ u_1 & -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} u_1 & \frac{1}{\sqrt{2}} \\ u_2 & 0 \\ u_1 & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}. \tag{9}$$

3. *If $u_1 = \pm\sqrt{1/6}$ then there are infinitely many symmetric Toeplitz matrices T which satisfy (9) if $\lambda_1 = \lambda_2$; however, if $\lambda_1 \neq \lambda_2$, then (9) does not hold for any symmetric Toeplitz matrix T.*

3. General Consideration.

We now consider the case for general n. When v is an eigenvector, the idea of rewriting the matrix-vector product [4]

$$T(r)v = M(v)r \tag{10}$$

can be very useful. It is easy to see that

LEMMA 3.1. *The columns of $M(v)$ have the same symmetry as v has. That is, $EM(v) = \pm M(v)$ if and only if $Ev = \pm v$.*

Thus only the first $\sigma(v)$ rows of $M(v)$ need to be considered. For convenience, let $p := \lceil n/2 \rceil$ and let $N(v)$ denote the $p \times n$ submatrix of the first p rows of $M(v)$. It is easy to verify that $N(v)$ can be decomposed into blocks:

$$N(v) = [h(v), H(v), 0] + [0, L(v), 0] + [0, 0, U(v)] \tag{11}$$

where $h(v)$ is a $p \times 1$ column vector, $\tilde{H}(v) := [h(v), H(v)] = (\tilde{h}_{ij}(v))$ is the $p \times p$ Hankel matrix

$$\tilde{h}_{ij}(v) := v_{i+j-1}, \tag{12}$$

$\tilde{L}(v) := [0, L] = (\tilde{l}_{ij}(v))$ is the $p \times p$ lower triangular matrix

$$\tilde{l}_{ij}(v) := \begin{cases} v_{i-j+1}, & \text{if } 1 < j \leq i; \\ 0, & \text{otherwise,} \end{cases} \tag{13}$$

and $U(v) = (u_{ij}(v))$ is the $p \times (n-p)$ triangular matrix

$$u_{ij}(v) := \begin{cases} v_{p+i+j-1} & \text{if } i + j \leq n - p + 1; \\ 0 & \text{otherwise.} \end{cases} \tag{14}$$

We note that the last row of $N(v)$ is identically zero when n is odd and v is skew-symmetric. The rows of $N(u)$ and $N(v)$ will be used to construct a larger matrix.

Suppose

$$\begin{aligned} T(r)u &= \lambda_1 u, \\ T(r)v &= \lambda_2 v. \end{aligned} \tag{15}$$

Then the vector r must be such that the linear equations

$$\begin{aligned} N(u)(r - \lambda_1 w) &= 0, \\ N(v)(r - \lambda_2 w) &= 0 \end{aligned} \tag{16}$$

are satisfied. If we write

$$x := [r_1 - \lambda_2, r_1 - \lambda_1, r_2, \ldots r_n]^T, \tag{17}$$

then the system (16) is equivalent to

$$\tilde{M}(u, v)x = 0, \tag{18}$$

where $\tilde{M}(u,v)$ is the $(2p)\times(n+1)$ matrix defined by

$$\tilde{M}(u,v) := \begin{bmatrix} 0 & h(u) & H(u)+L(u) & U(u) \\ h(v) & 0 & H(v)+L(v) & U(v) \end{bmatrix}. \tag{19}$$

Given symmetric or skew-symmetric vectors u and v, a solution to (18) can be used to construct a Toeplitz matrix in the following way:

LEMMA 3.2. *Suppose $[x_0, x_1, \dots, x_n]^T$ is a solution to (18). For arbitrary real numbers λ_1 and α, define*

$$\begin{aligned} r_1 &:= \alpha x_1 + \lambda_1, \\ r_i &:= \alpha x_i, \ \text{for } i = 2, \dots, n \end{aligned} \tag{20}$$

and

$$\lambda_2 := \alpha(x_1 - x_0) + \lambda_1. \tag{21}$$

Then u and v are eigenvectors of the Toeplitz matrix $T(r)$. In other words, $S(u)\cap S(v)$ is the direct sum of the subspace spanned by w and the subspace obtained by deleting the first component from $\ker(\tilde{M})$.

On the other hand, suppose the two eigenvalues λ_1 and λ_2 are prescribed. Then the equation (21) implies that the constant α in (20) must be

$$\alpha = \frac{\lambda_1 - \lambda_2}{x_0 - x_1}, \tag{22}$$

provided $x_0 \neq x_1$. We conclude, therefore,

LEMMA 3.3. *Suppose x is a non-trivial solution of (18) satisfying $x_0 \neq x_1$. Then corresponding to the direction of x, there is a unique solution to Problem 2 when $k = 2$.*

The question now is to determine the null space of $\tilde{M}(u,v)$. It is convenient to use the abbreviated notation $\tilde{M} = \tilde{M}(u,v)$. It turns out that the dimension depends upon whether n is even or odd and whether the two eigenvectors are symmetric or skew-symmetric. In any case, we shall show that $\tilde{M}$ has a non-trivial null space. It is most interesting to note that the dimension does not depend upon the size of n. We discuss the different cases as follows:

Case 1. n is odd and both eigenvectors are symmetric.

When n is odd, the Hankel matrix $\tilde{H}(v)$ for a symmetric vector v takes the special form:

$$\tilde{H}(v) = \begin{bmatrix} v_1 & v_2 & \dots & v_{p-1} & v_p \\ v_2 & v_3 & & v_p & v_{p-1} \\ \vdots & & & & \vdots \\ v_{p-1} & v_p & \dots & v_3 & v_2 \\ v_p & v_{p-1} & \dots & v_2 & v_1 \end{bmatrix}. \tag{23}$$

The corresponding $U(v)$ becomes

$$U(v) = EL(v) = \begin{bmatrix} v_{p-1} & v_{p-2} & \dots & v_2 & v_1 \\ v_{p-2} & & \dots & v_1 & 0 \\ \vdots & & & & \vdots \\ v_1 & 0 & \dots & & 0 \\ 0 & & \dots & & 0 \end{bmatrix}. \tag{24}$$

It is useful to illustrate the basic structure of $\tilde{M}$ with a simple example when both u and v are symmetric. When $n = 5$, we have

$$\tilde{M} = \begin{bmatrix} 0 & u_1 & u_2 & u_3 & u_2 & u_1 \\ 0 & u_2 & u_3 + u_1 & u_2 & u_1 & 0 \\ 0 & u_3 & 2u_2 & 2\,u_1 & 0 & 0 \\ v_1 & 0 & v_2 & v_3 & v_2 & v_1 \\ v_2 & 0 & v_3 + v_1 & v_2 & v_1 & 0 \\ v_3 & 0 & 2v_2 & 2v_1 & 0 & 0 \end{bmatrix}. \tag{25}$$

The matrix $\tilde{M}$ in general is a square matrix of order $n + 1$. The determinant of $\tilde{M}$ is an algebraic expression involving independent variables $v_1, \ldots, v_p, u_1, \ldots, u_p$. It would not be too surprising if $\det(\tilde{M}) \neq 0$ for generic u and v (See the Appendix.) Nevertheless, under the additional condition that u and v are perpendicular to each other, we will show by elementary row operations that $\tilde{M}$ is in fact rank deficient.

For any symmetric vector v, let the $p \times p$ upper triangular matrix $G(v)$ be defined by

$$G(v) := \begin{bmatrix} 2v_1 & 2v_2 & \ldots & 2v_{p-1} & v_p \\ 0 & 2v_1 & & 2v_{p-2} & v_{p-1} \\ \vdots & 0 & \ddots & \vdots & \vdots \\ 0 & & & 2v_1 & v_2 \\ 0 & 0 & 0 & 0 & v_1 \end{bmatrix}. \tag{26}$$

We also define the $2p \times 2p$ matrix

$$\tilde{G} = \tilde{G}(u, v) := \begin{bmatrix} I & 0 \\ -G(v) & G(u) \end{bmatrix}, \tag{27}$$

which in fact is the accumulation of a sequence of elementary row operations. We remark here that the ordering of u and v is immaterial. If $u_1 = 0$, then the roles of u and v may as well be switched. The extremely rare case when both $u_1 = v_1 = 0$ can be reduced to a lower dimensional problem. Without loss of generality, therefore, we may assume $u_1 \neq 0$ and, hence, the matrix $\tilde{G}(u, v)$ is a non-singular matrix. It follows that the product $\tilde{W} := \tilde{G}\tilde{M}$ has the same rank as $\tilde{M}$.

We claim that

LEMMA 3.4. *Suppose that n is odd and that the two symmetric vectors u and v are orthogonal. Then the matrix $\tilde{W}$ is rank deficient. In fact,*

$$p + 1 \leq \mathrm{rank}(\tilde{\mathrm{W}}) \leq \mathrm{n}. \tag{28}$$

Proof. The proof is tedious but straightforward. For $i = 1, \ldots, p$, the i^{th} component of $G(u)h(v)$ is given by

$$2 \sum_{\substack{s=1 \\ t-s=i-1}}^{p-i} u_s v_t + u_{p-i+1} v_p. \tag{29}$$

The first component is trivially seen to be

$$2\sum_{s=1}^{p-1} u_s v_s + u_p v_p \tag{30}$$

which is zero because u and v are perpendicular to each other. For the same reason, the first component of $-G(v)h(u)$ is zero.

The product $G(u)U(v)$ has the same triangular structure as $U(v)$. On the other hand, the $(i,j)^{th}$ component of $G(u)U(v)$ with $i+j \le p$ is given by

$$2 \sum_{s+t=p-i-j+2} u_s v_t. \tag{31}$$

It is important to note that the summation (31) is a symmetric function of u and v. It follows that the $p \times (n-p)$ block

$$-G(v)U(u) + G(u)U(v) \tag{32}$$

is identically zero.

For $i = 1, \ldots, p$ and $j = 1, \ldots, p-1$, the $(i,j)^{th}$ component of $G(u)H(v)$ is given by

$$2 \sum_{\substack{s=1 \\ t-s=i+j-1}}^{p-i-j+1} u_s v_t + 2 \sum_{\substack{s=p-i-j+2 \\ t+s=2p-i-j+1}}^{p-i} u_s v_t + u_{p-i+1} v_{p-j}, \tag{33}$$

if $j < p-i+1$; or

$$2 \sum_{\substack{s=1 \\ t+s=2p-i-j+1}}^{p-i} u_s v_t + u_{p-i+1} v_{p-j}, \tag{34}$$

if $j \ge p-i+1$. The $(i,j)^{th}$ component of $G(u)L(v)$ is given by

$$2 \sum_{\substack{s=1 \\ t-s=i-j-1}}^{p-i} u_s v_t + u_{p-i+1} v_{p-j}, \tag{35}$$

if $j \le i-1$; or

$$2 \sum_{\substack{s=j-i+2 \\ t-s=i-j-1}}^{p-i} u_s v_t + u_{p-i+1} v_{p-j}, \tag{36}$$

if $j > i-1$. Using (33) and (36), it follows that the $(1,j)^{th}$ component of $G(u)(H(v)+L(v))$ is given by

$$2 \sum_{\substack{s=1 \\ t-s=j}}^{p-j} u_s v_t + 2 \sum_{\substack{s=p-j+1 \\ t+s=2p-j}}^{p-1} u_s v_t + 2 \sum_{\substack{t=1 \\ s-t=j}}^{p-j} u_s v_t. \tag{37}$$

The first and the last summations in (37) are symmetric to each other. The second summation in (37) is a symmetric function of u and v. Over all, (37) is a symmetric function of u and v which will be completely canceled by its counterpart in $-G(v)(H(u)+L(u))$.

By now we have proved that the $(p+1)^{th}$ row of $\tilde{W}$ is identically zero. If follows that the null space of $\tilde{M}$ is at least of dimension one.

Using (34), (35) and (36), it is further observed that the $(i,p-1)^{th}$ component of $G(u)(H(v)+L(v))$ is given by

$$2 \sum_{\substack{s=1 \\ t+s=p+2-i}}^{p} u_s v_t \tag{38}$$

which, once again, is a symmetric function of u and v, and will be completely canceled by its counterpart in $-G(v)(H(u)+L(u))$. The zero structure of $\tilde{W}$ clearly indicates that (28) is true. □

The following example for the case $n=5$ illustrates the typical structure of $\tilde{W}$:

$$\tilde{W} = \begin{bmatrix} 0 & u_1 & u_2 & u_3 & u_2 & u_1 \\ 0 & u_2 & u_3+u_1 & u_2 & u_1 & 0 \\ 0 & u_3 & 2u_2 & 2u_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 2v_2u_1+v_3u_2 & -2v_1u_2-v_2u_3 & 2v_3u_1-2v_1u_3 & 0 & 0 & 0 \\ v_3u_1 & -v_1u_3 & 2v_2u_1-2v_1u_2 & 0 & 0 & 0 \end{bmatrix}. \tag{39}$$

Let W denote the lower left $(p-1)\times p$ submatrix of $\tilde{W}$. That is, W is the matrix obtained by deleting the first row and the last column of

$$[G(u)h(v), -G(v)h(u), -G(v)(H(u)+L(u))+G(u)(H(v)+L(v))]. \tag{40}$$

The rank of $\tilde{W}$ can be less than n if and only if W is rank deficient, which will be true if and only if values of u_i and v_i are such that $\det(WW^T)=0$. We note that $\det(WW^T)$ is a polynomial in the independent variables u_i and v_i. We note also that $\det(WW^T)$ is not identically zero (See the Appendix for a proof.) Thus $\mathrm{rank}(\tilde{\mathrm{W}}) < \mathrm{n}$ if and only if u_i and v_i come from a codimension one surface. We conclude, therefore, that for almost all u and v satisfying $u^Tv=0$ the matrix $\tilde{W}$ is of rank n. Unfortunately, even for the case $n=5$ (see (39)), it is fairly complicated to express the rank deficiency of W in terms of components of u and v. At present we cannot provide a further characterization of the set where W is rank deficient. Given the fact that the orthogonality of u and v has already been used to prove the rank deficiency of $\tilde{W}$, it is conceivably true that the orthogonality condition cannot be used again to reduce the rank of W.

In conclusion, we have proved the following theorem:

THEOREM 3.5. *Suppose that n is odd and that u and v are two symmetric vectors satisfying $u^Tv=0$. Then*

1. *The dimension of $S(u)\cap S(v)$ is at least two.*
2. *For almost all u and v, the dimension of $S(u)\cap S(v)$ is exactly two.*
3. *For almost all u and v and for any values of λ_1 and λ_2, there exists a unique symmetric Toeplitz matrix T satisfying $Tu=\lambda_1 u$ and $Tv=\lambda_2 v$.*

Case 2. n is odd and both eigenvectors are skew-symmetric.

When n is odd, the Hankel matrix $\tilde{H}(v)$ for a skew-symmetric vector v takes the form:

$$\tilde{H}(v) = \begin{bmatrix} v_1 & v_2 & \dots & v_{p-1} & 0 \\ v_2 & v_3 & & 0 & -v_{p-1} \\ \vdots & & & & \vdots \\ v_{p-1} & 0 & \dots & -v_3 & -v_2 \\ 0 & -v_{p-1} & \dots & -v_2 & -v_1 \end{bmatrix}. \tag{41}$$

The corresponding $U(v)$ becomes

$$U(v) = -EL(v) = \begin{bmatrix} -v_{p-1} & -v_{p-2} & \dots & -v_2 & -v_1 \\ -v_{p-2} & & \dots & -v_1 & 0 \\ \vdots & & & & \vdots \\ -v_1 & 0 & \dots & & 0 \\ 0 & 0 & \dots & & 0 \end{bmatrix}. \tag{42}$$

It follows that the last row of $N(v)$ is identically zero. For the equation (18), it is now obvious that the kernel of $\tilde{M}$ is of dimension at least two. In fact, we show that

LEMMA 3.6. *Suppose that n is odd and that the skew-symmetric vectors u and v are perpendicular. Then*

$$p \leq \mathrm{rank}(\tilde{\mathrm{W}}) \leq \mathrm{n} - 2. \tag{43}$$

Indeed, for almost all u and v, $\mathrm{rank}(\tilde{\mathrm{W}}) = \mathrm{n} - 2$.

Proof. The proof is very similar to that of Lemma 3.4. So we simply outline a recipe for constructing the transformation matrix that does the elimination. The details of justification are omitted.

It suffices to consider the $2(p-1) \times (n+1)$ submatrix $\hat{M}$ obtained by deleting the p^{th} and the $2p^{th}$ rows of $\tilde{M}$. For a skew-symmetric vector v, define the $(p-1) \times (p-1)$ matrix $G(v)$ by

$$G(v) := \begin{bmatrix} -v_1 & -v_2 & \dots & -v_{p-2} & -v_{p-1} \\ 0 & v_1 & & v_{p-3} & v_{p-2} \\ \vdots & 0 & \ddots & \vdots & \vdots \\ 0 & & & v_1 & v_2 \\ 0 & 0 & 0 & 0 & v_1 \end{bmatrix}. \tag{44}$$

Then construct the $2(p-1) \times 2(p-1)$ transformation matrix $\tilde{G}(u, v)$ in the same way as is defined in (27). It can be proved now that the p^{th} row of the product $\hat{W} := \tilde{G}\hat{M}$ is identically zero. Furthermore, the lower right $(p-1) \times (p-1)$ submatrix of $\hat{W}$ is also identically zero. The assertion follows from these observations. $\square$

As an example, for $n = 5$, the matrix $\hat{M}$ takes the form

$$\hat{M} = \begin{bmatrix} 0 & u_1 & u_2 & 0 & -u_2 & -u_1 \\ 0 & u_2 & u_1 & -u_2 & -u_1 & 0 \\ v_1 & 0 & v_2 & 0 & -v_2 & -v_1 \\ v_2 & 0 & v_1 & -v_2 & -v_1 & 0 \end{bmatrix} \tag{45}$$

and after the transformation the matrix $\hat{W}$ looks like

$$(46) \qquad \hat{W} = \begin{bmatrix} 0 & u_1 & u_2 & 0 & -u_2 & -u_1 \\ 0 & u_2 & u_1 & -u_2 & -u_1 & 0 \\ -v_1u_1 - v_2u_2 & v_1u_1 + v_2u_2 & 0 & 0 & 0 & 0 \\ u_1v_2 & -v_1u_2 & 0 & v_1u_2 - u_1v_2 & 0 & 0 \end{bmatrix}.$$

The third row of $\hat{W}$ is identically zero because $u^Tv = 0$.

We conclude this case by the following theorem:

THEOREM 3.7. *Suppose that $n \geq 5$ is odd and that u and v are two skew-symmetric vectors satisfying $u^Tv = 0$. Then*

1. *The dimension of $S(u) \cap S(v)$ is at least four.*
2. *For almost all u and v, the dimension of $S(u) \cap S(v)$ is exactly four.*
3. *For almost all u and v and for any values of λ_1 and λ_2, the symmetric Toeplitz matrices T satisfying $Tu = \lambda_1 u$ and $Tv = \lambda_2 v$ form a 2-dimensional manifold.*

Proof. By Lemma 3.6, the dimension of $\ker(\tilde{W})$ is almost always three. The first two assertions then follow from Lemma 3.2. The last assertion follows from Lemma 3.3. □

Case 3. n is odd, one eigenvector is symmetric and the other is skew-symmetric.

A symmetric vector is always orthogonal to a skew-symmetric vector regardless of what the values of the components are. Thus, unlike the previous two cases, the orthogonality condition $u^Tv = 0$ no longer helps to reduce the rank of $\tilde{M}$. As $\tilde{M}$ does contain an identically zero row, we should have the same conclusion as in Theorem 3.5. That is,

THEOREM 3.8. *Suppose that n is odd and that u and v are symmetric and skew-symmetric vectors, respectively. Then*

1. *The dimension of $S(u) \cap S(v)$ is at least two.*
2. *For almost all u and v, the dimension of $S(u) \cap S(v)$ is exactly two.*
3. *For almost all u and v and for any values of λ_1 and λ_2, there exists a unique symmetric Toeplitz matrix T satisfying $Tu = \lambda_1 u$ and $Tv = \lambda_2 v$.*

Case 4. n is even and both eigenvectors are symmetric.

When n is even, the Hankel matrix $\tilde{H}(v)$ for a symmetric vector v takes the special form:

$$(47) \qquad \tilde{H}(v) = \begin{bmatrix} v_1 & v_2 & \dots & v_{p-1} & v_p \\ v_2 & v_3 & & v_p & v_p \\ \vdots & & & & \vdots \\ v_{p-1} & v_p & \dots & v_4 & v_3 \\ v_p & v_p & \dots & v_3 & v_2 \end{bmatrix}.$$

Once again, we define

$$(48) \qquad G(v) := \begin{bmatrix} v_1 & v_2 & \dots & v_{p-1} & v_p \\ 0 & v_1 & & v_{p-2} & v_{p-1} \\ \vdots & 0 & \ddots & \vdots & \vdots \\ 0 & & & v_1 & v_2 \\ 0 & 0 & 0 & 0 & v_1 \end{bmatrix}$$

and construct $\tilde{G}(u, v)$ according to (27). If the two symmetric vectors u and v are orthogonal, then it can be shown that the rank of the $n \times (n+1)$ matrix $\tilde{W} := \tilde{G}\tilde{M}$ satisfies

$$p + 1 \leq \operatorname{rank}(\tilde{W}) \leq n - 1 \tag{49}$$

and for almost all u and v, the dimension of $\ker(\tilde{M})$ is exactly two. So we have

THEOREM 3.9. *Suppose that n is even and that u and v are two symmetric vectors satisfying $u^T v = 0$. Then*

1. *The dimension of $S(u) \cap S(v)$ is at least three.*
2. *For almost all u and v, the dimension of $S(u) \cap S(v)$ is exactly three.*
3. *For almost all u and v and for any values of λ_1 and λ_2, the symmetric Toeplitz matrices T satisfying $Tu = \lambda_1 u$ and $Tv = \lambda_2 v$ form a 1-dimensional manifold.*

Case 5. n is even and both eigenvectors are skew-symmetric.

When n is even, the Hankel matrix $\tilde{H}(v)$ for a skew-symmetric vector v takes the special form:

$$\tilde{H}(v) = \begin{bmatrix} v_1 & v_2 & \dots & v_{p-1} & v_p \\ v_2 & v_3 & & v_p & -v_p \\ \vdots & & & & \vdots \\ v_{p-1} & v_p & \dots & -v_4 & -v_3 \\ v_p & -v_p & \dots & -v_3 & -v_2 \end{bmatrix}. \tag{50}$$

To construct the transformation matrix $\tilde{G}(u, v)$, the matrix $G(v)$ for a skew-symmetric vector v is defined in exactly the same way as (48). The conclusion is as follows:

THEOREM 3.10. *Suppose that n is even and that u and v are two skew-symmetric vectors satisfying $u^T v = 0$. Then*

1. *The dimension of $S(u) \cap S(v)$ is at least three.*
2. *For almost all u and v, the dimension of $S(u) \cap S(v)$ is exactly three.*
3. *For almost all u and v and for any values of λ_1 and λ_2, the symmetric Toeplitz matrices T satisfying $Tu = \lambda_1 u$ and $Tv = \lambda_2 v$ form a 1-dimensional manifold.*

Case 6. n is even, one eigenvector is symmetric and the other is skew-symmetric.

Just like Case 3, the orthogonality condition does not help to reduce the rank of $\tilde{M}$. The $n \times (n+1)$ matrix $\tilde{M}$ in general is of full rank. The conclusion, therefore, is similar to Theorem 3.8:

THEOREM 3.11. *Suppose that n is even and that u and v are symmetric and skew-symmetric vectors, respectively. Then*

1. *The dimension of $S(u) \cap S(v)$ is at least two.*
2. *For almost all u and v, the dimension of $S(u) \cap S(v)$ is exactly two.*
3. *For almost all u and v and for any values of λ_1 and λ_2, there exists a unique symmetric Toeplitz matrix T satisfying $Tu = \lambda_1 u$ and $Tv = \lambda_2 v$.*

4. Conclusion.

We have shown by symbolic computation that the dimension of the subspace $S(u) \cap S(v)$ of Toeplitz matrices with two generically prescribed eigenvectors u and v is independent of the size of the problem. We have further shown that the dimension is either two, three or four, depending upon whether the eigenvectors are symmetric or skew-symmetric. All the cases are justified to the extent that the transformation matrices that result in the desired elimination are fully described in terms of the components of u and v. Only one proof (Lemma 3.4) is detailed, but the rest can be done in a very similar way.

Our result extends that in [4] where only one eigenvector is prescribed. On the other hand, our discovery that the dimension is independent of the size of the problem is quite a surprising and remarkable fact.

We also have studied the inverse problem of constructing a Toeplitz matrix from two prescribed eigenpairs. We have shown that in almost every direction of $\ker(\tilde{M})$, there is one and only one Toeplitz matrix with the prescribed eigenpairs. In particular, it is shown that if n is odd and if at least one of the given eigenvectors is symmetric, or if n is even and one eigenvector is symmetric and the other is skew-symmetric, then the Toeplitz matrix is unique.

5. Appendix.

In the proof of Theorem 3.5 we need to show that $\det(WW^T)$ is not identically zero. This can be done by simply showing that $\det(WW^T) \neq 0$ for a certain u and v. In particular, choose

$$\begin{aligned} u_i &= \begin{cases} 1, & \text{if } i = 1, \\ 0, & \text{if } 1 < i \leq p; \end{cases} \\ v_i &= \begin{cases} 1, & \text{if } i = p, \\ 0, & \text{if } 1 \leq i < p. \end{cases} \end{aligned}$$

Then it is easy to see that the $(p-1) \times p$ submatrix W is given by

$$\begin{bmatrix} 0 & 0 & 0 & 0 & \dots & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & \dots & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & & 2 & 0 & 0 \\ \vdots & \vdots & & & & & \vdots & \vdots \\ 0 & 0 & 0 & 2 & & & 0 & \\ 0 & 0 & 2 & 0 & \dots & & 0 & 0 \\ 1 & 0 & \dots & & \dots & & 0 & 0 \end{bmatrix}.$$

Obviously W is of full rank $p-1$.

With the same choice of u and v, it is also easy to see that

$$x = \left[0, -\sqrt{.5}, 0, \dots, 0, \sqrt{.5}\right]^T$$

is a solution to (18). Specifically, we have proved that x_0 cannot be identical to x_1 for all u and v.

Similar arguments can be deduced for the proof of other cases.

REFERENCES

[1] S. L. Campbell and C. D. Meyer, Generalized Inverses of Linear Transformations, Pitman, London, 1979.

[2] A. Cantoni and P. Butler, Eigenvalues and eigenvectors of symmetric centrosymmetric matrices, Linear Alg. Appl., 13(1976), 275-288.

[3] M. Chu, Matrix differential equations: A continuous realization process for linear algebra problems, Nonlinear Anal., TMA, 18(1992), 1125-1146.

[4] G. Cybenko, On the eigenstructure of Toeplitz matrices, IEEE Trans. Acoust., Speech, Signal Process., ASSP-32(1984), 918-920.

[5] P. Delsarte and Y. Genin, Spectral properties of finite Toeplitz matrices, Proceedings of the 1983 International Symposium of Mathematical Theory of Networks and Systems, Beer-Sheva, Israel, 194-213.

[6] S. Friedland, J. Nocedal and M. L. Overton, The formulation and analysis of numerical methods for inverse eigenvalue problems, SIAM J. Numer. Anal. 24(1987), 634-667.

[7] G. H. Golub and C. F. Van Loan, Matrix Computations, 2nd ed., John Hopkins University Press, Baltimore, 1989.

[8] D. P. Laurie, A numerical approach to the inverse Toeplitz eigenproblem, SIAM J. Sci. Stat. Comput., 9(1988), 401-405.

[9] J. Makhoul, On the eigenvectors of symmetric Toeplitz matrices, IEEE Trans. Acoust., Speech, Signal Process., ASSP-29(1981), 868-872.

[10] W. F. Trench, Numerical solution of the eigenvalue problem for Hermitian Toeplitz matrices, SIAM J. Matrix Anal. Appl., 10(1989), 135-146.

[11] W. F. Trench, Spectral evolution of a one-parameter extension of a real symmetric Toeplitz matrix, SIAM J. Matrix Theory Appl., to appear.