1. The numerical integration command commonly used in Maple

$$evalf(int(f, x = a..b))$$

is quite different from the Gaussian quadrature we discussed in class. The default numerical method used in Maple is called Clenshaw-Curtis quadrature that explicitly and directly defines the abscissas in the interval $[-1, 1]$ to be these points:

$$x_k := \cos\left(\frac{k-1}{n-1}\pi\right), \quad k = 1, \ldots, n.$$

Such a scheme, as we pointed out in class, has less degree of precision than the Gaussian quadrature. The first part of this project asks you to experiment with Gaussian quadrature.

(a) Generate the Gaussian abscissas and weights.

i. Use the available library **orthopoly** in Maple to find out the first six orthogonal polynomials, $P_k(x)$, $k = 0, 1, \ldots, 5$.

ii. Use Maple or Matlab to find out the *roots* of these polynomials up to the 16th significant digits. (Note that these roots are the abscissas in the corresponding Gaussian quadrature.)

iii. Recall that

$$\alpha_i = \int_{-1}^{1} \ell_i(x)\, dx$$

where $\ell_i(x)$ is simply the Lagrange basis polynomial

$$\ell_i(x) = \prod_{j=1, j\neq i}^{n} \frac{x - x_j}{x_i - x_j},$$

use the Gaussian abscissas found above to calculate the corresponding Gaussian weights α_i.

(b) Apply the 5-point Gaussian quadrature to evaluate the following double integral:

$$\int_0^{\pi/4} \int_0^{\sin x} \frac{2y \sin x + (\cos x)^2}{\sqrt{1 - y^2}}\, dy\, dx$$

(c) Compare your numerical answer with that obtained from Maple or your calculator calculation.

2. The second of this project asks you to set up some theoretical schemes that transform an abstract integral equation into an algebraic equation. Consider numerical methods for solving the Fredholm integral equation of the second kind, i.e.,

$$\varphi(x) - \int_a^b K(x,y)\varphi(y)dy = f(x), \quad x \in [a,b] \tag{1}$$

with continuous kernel K and $\varphi \in C[a,b]$.

(a) (The Nyström method) Suppose that $\varphi(x)$ is represented by a set of discrete data $\varphi_n := [\varphi(x_0), \ldots, \varphi(x_n)]^T$ at quadrature points, i.e., abscissas, $x_0, \ldots, x_n \in [a,b]$. Describe how the solution to the integral equation (1) is approximated by the solution of a linear system

$$\varphi_n - A_n\varphi_n = f_n. \tag{2}$$

Be specific about entries in the matrix A_n.

(b) (The collocation method) Suppose that $\varphi(x)$ is approximated by the Lagrange interpolation polynomial

$$\varphi_n(x) = \sum_{k=0}^{n} \gamma_k \ell_k(x) \tag{3}$$

where $\ell_k(x)$, $k = 0, \ldots, n$, are the Lagrange factors

$$\ell_k(x) = \prod_{i=1, i\neq k}^{n} \frac{x - x_i}{x_k - x_i} \tag{4}$$

at a predesignated set of $n+1$ points $x_0, \ldots, x_n \in [a,b]$ (called the collocation points.) It is desired that the approximate solution $\varphi_n(x)$ satisfies (1) at all points $x_0, \ldots, x_n \in [a,b]$. Describe how the values γ_k should be determined from a linear system.