THE RAYLEIGH QUOTIENT ITERATION: AN ANALYSIS OF LOCAL ORDER OF CONVERGENCE

HOMEWORK 3

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This paper examines the local order of convergence of the Rayleigh quotient (RQ). Let *A* be a real $n \times n$ symmetric matrix. For any nonzero $x \in R^n$, the RQ $\rho(x)$ of x is defined to be

$$\rho(x) \equiv \frac{\langle Ax, x \rangle}{\langle x, x \rangle} \tag{1}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in R^n. Particularly, if x is an eigenvector of *A*, then the RQ returns the eigenvalue corresponding to x. However, when the RQ is applied iteratively (such as in the Rayleigh Quotient Iteration), the RQ takes an approximated eigenvector into an approximation of its corresponding eigenvalue. In this implementation

$$\rho(x_k) = \frac{\langle Ax_k, x_k \rangle}{\langle x_k, x_k \rangle} \equiv \lambda_{k+1} \tag{2}$$

where x_k is the approximation to the eigenvector x at iteration *k*, and where λ_{k+1} is the resulting approximation to the corresponding eigenvalue λ (λ_{k+1} is then used to generate x_{k+1} and the process continues). The *global* rate of convergence for the RQ Iteration is cubic. In this paper, it is shown that, at each iteration in (2), an ε approximation to an eigenvector of *A* returns an $O\left(\|\varepsilon\|^2\right)$ approximation to the corresponding eigenvalue.

Consider an eigenvector $\hat{x}$ of *A* and its *k*th approximation x_k such that the error in x_k is $x_k - \hat{x} = \varepsilon$. To obtain λ_{k+1}, apply (2):

$$\rho(x_k) = \rho(\hat{x} + \varepsilon) = \frac{\langle A(\hat{x} + \varepsilon), \hat{x} + \varepsilon \rangle}{\langle \hat{x} + \varepsilon, \hat{x} + \varepsilon \rangle} \tag{3}$$

Recalling that $A = A^T$ and $A\hat{x} = \lambda \hat{x}$ and expanding (3), one finds

$$\lambda_{k+1} = \frac{\langle A\varepsilon, \varepsilon \rangle + \lambda \langle \hat{x}, \hat{x} \rangle + 2\lambda \langle \hat{x}, \varepsilon \rangle}{\langle \hat{x}, \hat{x} \rangle + 2\langle \hat{x}, \varepsilon \rangle + \langle \varepsilon, \varepsilon \rangle}. \tag{4}$$

Then the error in the λ_{k+1} approximation is

$$\lambda_{k+1} - \lambda = \frac{1}{\langle x_k, x_k \rangle}\left(\langle A\varepsilon, \varepsilon\rangle - \lambda\langle \varepsilon, \varepsilon\rangle\right) \tag{5}$$

Notice that the ε error in x_k results in an $\|\varepsilon\|^2$ error in λ_{k+1}.

Furthermore, one can obtain this result by considering the Taylor expansion of $\rho(x_k)$ about $x_k - \hat{x} = \varepsilon$:

$$\rho(x_k) = \rho(\hat{x}) + \langle \rho'(\hat{x}), \varepsilon\rangle + O\left(\|\varepsilon\|^2\right) \tag{6}$$

where $\rho'(x) = 2\dfrac{\langle x, x\rangle Ax - \langle Ax, x\rangle x}{\langle x, x\rangle^2}$ upon application of the Fréchet derivative. Note that $\rho'(\hat{x}) = \mathbf{0}$ so (6) becomes

$$\begin{aligned} \rho(x_k) - \rho(\hat{x}) &= O\left(\|\varepsilon\|^2\right) \\ \lambda_{k+1} - \lambda_k &= O\left(\|\varepsilon\|^2\right) \end{aligned} \tag{7}$$

In conclusion, at each iteration of the Rayleigh quotient, an ε approximation to an eigenvector of the system returns an $O(||\varepsilon||^2)$ approximation to the corresponding eigenvalue.